Exercises on Fourier series

1. *This question was in the May 2019* `MA2815` *exam.*

   Let $f : \mathbb{R} \to \mathbb{R}$ denote a 2π-periodic function which is piecewise continuous. The Fourier series for this function is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) ,$$

   where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,\mathrm{d}x, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,\mathrm{d}x.$$

   Let f_1 and f_2 be 2π-periodic function defined on $(-\pi, \pi]$ as follows.

$$f_1(x) = \begin{cases} 1, & \text{if } |x| \le \pi/2, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi, \end{cases}$$

$$f_2(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi/2, \\ -1, & \text{if } -\pi/2 \le x < 0, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi. \end{cases}$$

   (a) Sketch $f_1(x)$ on the interval $(-\pi, 3\pi)$.

   (b) Show that the Fourier series for $f_1(x)$ is

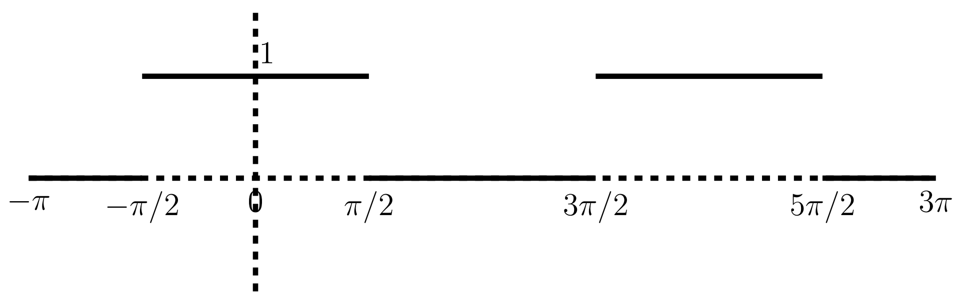
$$\frac{1}{2} + \frac{2}{\pi} \left(\cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} + \cdots + (-1)^{m+1} \frac{\cos((2m-1)x)}{2m-1} + \cdots \right).$$

   (c) Determine the Fourier series for $f_2(x)$ giving the general formula for the a_n coefficients and giving the values of b_1, b_2, b_3, b_4 and b_5.

   (d) State for what values of $x \in (-\pi, \pi)$ the Fourier series for $f_1(x)$ is the same as $f_1(x)$ and for what values, if any, they differ.

Solution

(a) $(-\pi, 3\pi)$ has length 4π and is 2-periods. A sketch on this interval is as follows.



(b) $f_1(x)$ is even on the interval $(-\pi,\pi)$ and thus the Fourier series has no sine terms. .

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \mathrm{d}x = 1 \quad \text{and hence } \frac{a_0}{2} = \frac{1}{2}.$$

For $n \ge 1$ we have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos(nx)\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi/2} \cos(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi/2} = \frac{2}{\pi}\frac{\sin(n\pi/2)}{n}. \end{aligned}$$

When n is even $\sin(n\pi/2)=0$. When $n=2m-1$, $n\pi/2 = m\pi-\pi/2$ and

$$\sin(m\pi-\pi/2) = -\cos(m\pi) = (-1)^{m+1}.$$

Thus the Fourier series for $f_1(x)$ is the given expression.

(c) $f_2(x)$ is odd on the interval $(-\pi,\pi)$ and thus the Fourier series has no cosine terms and $a_n=0$ for all n.

$$b_n = \frac{2}{\pi}\int_0^{\pi/2}\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi/2} = \frac{2}{n\pi}\left(-\cos(n\pi/2)+1\right).$$

$\cos(n\pi/2)$ takes values 0, -1, 0, 1 and 0 as $n=1,\ldots,5$. Thus

$$b_1 = \frac{2}{\pi}, \quad b_2 = \frac{4}{2\pi} = b_1, \quad b_3 = \frac{2}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{2}{5\pi}.$$

(d) $f_1(x)$ is the same as the Fourier series for $f_1(x)$ at the points of continuity and it is not the same as the points of discontinuity which are the points $\pm\pi/2$ in the interval $(-\pi,\pi)$.

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2. *This question was in the May 2017* MA2815 *exam.*

Consider the function $f : [-\pi,\pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } -\pi \le x < -\frac{\pi}{3}, \\ 0, & \text{if } -\frac{\pi}{3} \le x \le \frac{\pi}{3}, \\ -1, & \text{if } \frac{\pi}{3} < x < \pi. \end{cases}$$

Denote by $g : \mathbb{R} \to \mathbb{R}$ the 2π-periodic extension of f to $\mathbb{R}$.

(a) Sketch $g(x)$ over the interval $x \in [-3\pi, 3\pi]$ indicating carefully the key values on both axes.

The Fourier series of f is given by

$$S(x) = \lim_{N\to\infty} S_N(x) \quad \text{where} \quad S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big(a_n\cos(nx) + b_n\sin(nx)\Big)$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

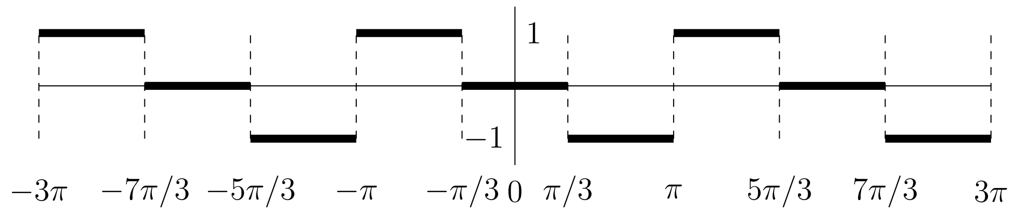
(b) Give those values of x for which $S(x) = f(x)$ and those values of x for which $S(x) \neq f(x)$.

(c) Determine a_0, general expressions for every a_n and b_n and show that the Fourier series S of f is

$$S(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - \cos(\frac{n\pi}{3})}{n} \sin(nx) \,.$$

Solution

(a) A sketch of the piecewise constant function is shown below.



(b) $S(x) = f(x)$ at all the points of continuity, i.e. it holds in $-\pi < x < -\frac{\pi}{3}$, in $-\frac{\pi}{3} < x < \frac{\pi}{3}$, and in $\frac{\pi}{3} < x < \pi$.
$S(x) \neq f(x)$ at the points of discontinuity which in $[-\pi, \pi]$ are at $-\pi$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$ and π.

(c) $f(x)$ is an odd function, therefore $a_0 = 0$ and $a_n = 0$ and only sine terms are involved.

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx \\ &= \frac{2}{\pi} \int_{\frac{\pi}{3}}^{\pi} (-1) \sin(nx)\, dx \\ &= \frac{2}{\pi}(-1) \left[\frac{-\cos(nx)}{n} \right]_{\frac{\pi}{3}}^{\pi} \\ &= \frac{2}{\pi}\frac{1}{n} \left(\cos(n\pi) - \cos(n\frac{\pi}{3}) \right) \\ &= \frac{2}{\pi}\frac{1}{n} \left((-1)^n - \cos(n\frac{\pi}{3}) \right) \end{aligned}$$

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3. *This question was in the May 2018* MA2815 *exam.*

Let $g : \mathbb{R} \to \mathbb{R}$ denote a 2π-periodic function which is piecewise continuous. The Fourier series for this function is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right) \,,$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx)\, \mathrm{d}x \,, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx)\, \mathrm{d}x \,.$$

Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ denote the 2π-periodic functions given on $(-\pi, \pi]$ by

$$f_1(x) = \begin{cases} 1\,, & -\pi < x \le -\pi/2\,, \\ 0\,, & -\pi/2 < x < \pi/2\,, \\ 1\,, & \pi/2 \le x \le \pi\,, \end{cases} \quad \text{and} \quad f_2(x) = -\frac{x}{2} + \int_0^x f_1(t)\,\mathrm{d}t\,.$$

(a) Obtain the Fourier coefficients of f_1 in their simplest form.

(b) Show that $f_2(x)$ can be written in the form

$$f_2(x) = \begin{cases} \dfrac{x+\pi}{2}\,, & -\pi < x \le -\pi/2\,, \\ -\dfrac{x}{2}\,, & -\pi/2 < x < \pi/2\,, \\ \dfrac{x-\pi}{2}\,, & \pi/2 \le x \le \pi\,. \end{cases}$$

and sketch $f_2(x)$ on the interval $-\pi \le x \le \pi$.

(c) Obtain the Fourier coefficients of f_2 in their simplest form.

(d) For each of f_1 and f_2 state the set of points, if any, where the value of the Fourier series is not the same as the value of the function it represents.

Solution

(a) At all the points of continuity of $f_1(x)$ we have $f_1(-x) = f_1(x)$ and thus the Fourier series only involves cosine terms, i.e. $b_n = 0$ for all n.

$$\pi a_0 = 2\int_0^{\pi} \mathrm{d}x = 2\int_{\pi/2}^{\pi} \mathrm{d}x = \pi, \quad a_0 = 1.$$

For $n \ge 1$,

$$\pi a_n = 2\int_{\pi/2}^{\pi} \cos(nx)\,\mathrm{d}x = \frac{2}{n}\left[\sin(nx)\right]_{\pi/2}^{\pi} = -\frac{2}{n}\sin(n\pi/2).$$

The values of $\sin(n\pi/2)$ are respectively 1, 0, -1, 0, 1 etc. Thus $a_{2m} = 0$ and

$$a_{2m-1} = \frac{2(-1)^m}{\pi(2m-1)}.$$

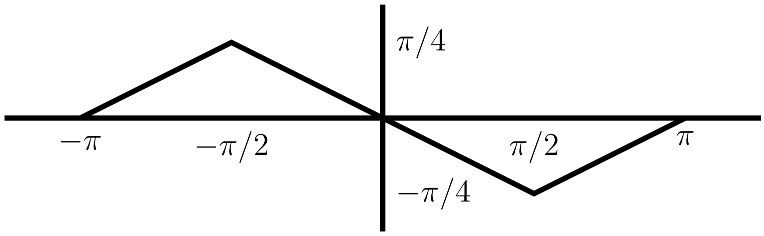
(b) For $|x| < \pi/2$, $f_2(x) = -x/2$.
For $x \ge \pi/2$,

$$f_2(x) = -\frac{x}{2} + \int_{\pi/2}^{x} \mathrm{d}t = -\frac{x}{2} + x - \pi/2 = \frac{x-\pi}{2}.$$

For $x \le -\pi/2$,

$$f_2(x) = -\frac{x}{2} + \int_{-\pi/2}^{x} \mathrm{d}t = -\frac{x}{2} + x - (-\pi/2) = \frac{x+\pi}{2}.$$

$f_2(-\pi) = 0$, $f_2(-\pi/2) = \pi/4$, $f_2(\pi/2) = -\pi/4$ and $f_2(\pi) = 0$. A sketch of $f_2(x)$ on $[-\pi, \pi]$ is given below.



(c) We can obtain the Fourier coefficients of f_2 from f_1 by term-by-term integration of the series for f_1.

At most points

$$f_1(x) - \frac{1}{2} = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m-1} \cos(2m-1)x.$$

$$\begin{aligned} f_2(x) = \int_0^x f_1(t)\,\mathrm{d}t - \frac{x}{2} &= \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)^2} \left[\sin(2m-1)t\right]_0^x \\ &= \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)^2} \sin(2m-1)x. \end{aligned}$$

(d) There are no points where the Fourier series for f_2 differs from that of f_2.

The Fourier series for $f_1(x)$ differs from $f_1(x)$ at the points of discontinuity which are $\pm\pi/2$.

4. *This question was in the May 2016* `MA2815` *exam.*

   (a) Consider the function $g\colon [0,2\pi] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 2, & \text{if } 0 \le x < \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} < x \le 2\pi, \\[2ex] 1, & \text{if } \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}. \end{cases}$$

   Denote by $f\colon \mathbb{R} \to \mathbb{R}$ the 2π-periodic extension of g over $\mathbb{R}$.
   Sketch $f(x)$ over the interval $x \in [-2\pi, 2\pi]$ indicating carefully the key values on the axis.

   (b) The Fourier series of f is given by

$$\begin{aligned} S(x) &= \lim_{N\to\infty} S_N(x) \\ \text{where} \quad S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \end{aligned}$$

   with

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,\mathrm{d}x \qquad \text{and} \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,\mathrm{d}x.$$

   Determine a_0, general expressions for every a_n and b_n, and show that the Fourier series S of f is

$$S(x) = \frac{3}{2} + \frac{2}{\pi} \left(\sum_{n=0}^{\infty} (-1)^n \frac{\cos((2n+1)x)}{2n+1} \right).$$

   (c) For what values of x do we have $S(x) \ne f(x)$ on $[-\pi, \pi]$?

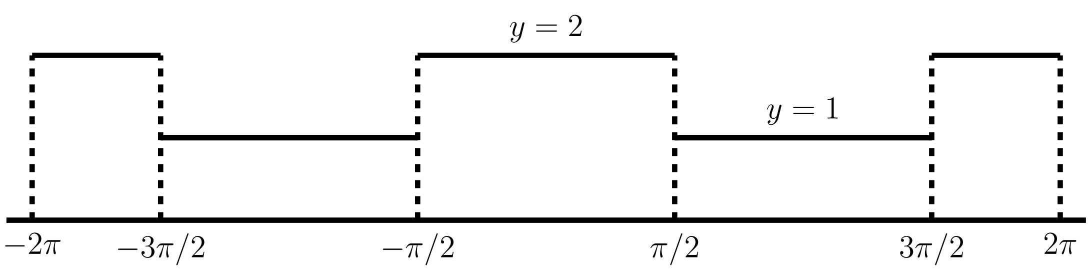
   (d) Explain why the Fourier series suggests that,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

   and use the Leibniz alternating series test to test this series for convergence.
   Before 2017/8 the study block `MA2730` *also contributed to the* `MA2815` *May exam. The use the Leibniz alternating series test would not be a* `MA2815` *paper now.*

Solution

(a) For a sketch of $y = f(x)$ you can have the following,



(b) The function $f(x)$ is even and hence the Fourier series $S(x)$ only involves cosine terms is given by

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{with } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\,\mathrm{d}x.$$

$f(x)$ is defined differently on $[0, \pi/2)$ and $[\pi/2, \pi]$ and thus we have

$$\frac{\pi a_n}{2} = \int_0^{\pi/2} 2\cos(nx)\,\mathrm{d}x + \int_{\pi/2}^{\pi} \cos(nx)\,\mathrm{d}x.$$

When $n = 0$ we have

$$\frac{\pi a_0}{2} = \pi + \frac{\pi}{2} \quad \text{giving } \frac{a_0}{2} = \frac{3}{2}.$$

When $n \ge 1$ we have

$$\begin{aligned} \frac{\pi a_n}{2} &= \left[2\frac{\sin(nx)}{n} \right]_0^{\pi/2} + \left[\frac{\sin(nx)}{n} \right]_{\pi/2}^{\pi} \\ &= \frac{1}{n}(2\sin(n\pi/2) - \sin(n\pi/2)) = \frac{1}{n}\sin(n\pi/2). \end{aligned}$$

If n is even then we have $a_n = 0$. If $n = 2m + 1$, with $m = 0, 1, 2, \ldots$, then

$$\sin((2m + 1)\pi/2) = \sin((m + 1/2)\pi) = \cos(m\pi) = (-1)^m.$$

Hence

$$\frac{\pi a_{2m+1}}{2} = \frac{(-1)^m}{2m + 1}$$

which re-arranges to

$$a_{2m+1} = \left(\frac{2}{\pi}\right) \frac{(-1)^m}{2m + 1}.$$

(c) The function $f(x)$ is piecewise continuously differentiable and agrees with $S(x)$ at the points of continuity. There are two points of dis-continuity in $[-\pi, \pi]$ and these are at $\pm\pi/2$. At these points the Fourier series $S(x)$ is $3/2$ which is not the same as $f(x)$ as $f(\pm\pi/2) = 1$.
As $S(0) = f(0)$ we have

$$2 = \frac{3}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m + 1}$$

which rearranges to

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} \frac{(-1)^m}{2m + 1}.$$

If we have the substitution $n = m + 1$ so that $m = n - 1$ then we have

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n - 1}.$$

As

$$(-1)^{n-1} = (-1)^{n+1}$$

we get the result in the form given in the question.

(d) The coefficients in the series for $n = 1, 2, \ldots$ are of the form

$$(-1)^{n+1} c_n, \quad \text{where } c_n = \frac{1}{2n-1}.$$

The sequence of magnitudes c_n decrease to 0 and we satisfy the conditions of the alternating series test and hence the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} c_n$$

converges.

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5. *This question was in the May 2014* `MA2815` *exam.*

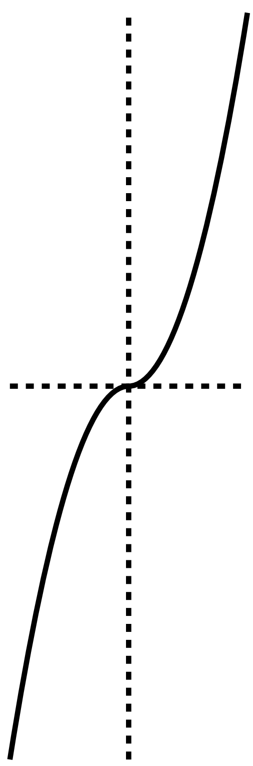
Sketch the graph of the periodic function

$$f(x) = \begin{cases} -2x^2, & -\pi/2 \le x < 0, \\ 2x^2, & 0 \le x < \pi/2, \end{cases} \qquad f(x+\pi) = f(x),$$

find its full-range Fourier series on $[-\pi/2, \pi/2]$ and the sine Fourier series on $[0, \pi/2]$.

Solution

On $[-\pi/2, \pi/2]$ the function is defined in two parts and it is an odd function. The two points join at 0 and the slope at the join is also continuous and is 0 (i.e. the derivative at 0 is 0). The π-periodic extension of $f(x)$ is not continuous at $\pm\pi/2$. For a sketch we can have the following.



Note that the period is π in this questions and thus if you consider the general case of a function with period $2L$ then $L = \pi/2$ and in the formulas $\pi/L = 2$. As it is an odd function the series only involves sine terms and the Fourier series $S(x)$ is given by

$$S(x) = \sum_{n=1}^{\infty} b_n \sin(2nx),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{nx\pi}{L}\right)\,\mathrm{d}x = \frac{4}{\pi}\int_0^{\pi/2} 2x^2\sin(2nx)\,\mathrm{d}x.$$

Two integration by parts steps are needed and the first integration by parts gives

$$\begin{aligned}
\pi b_n = 8\int_0^{\pi/2} x^2\sin(2nx)\,\mathrm{d}x &= \left[-8x^2\left(\frac{\cos(2nx)}{2n}\right)\right]_0^{\pi/2} + \int_0^{\pi/2} 16x\frac{\cos(2nx)}{2n}\,\mathrm{d}x \\
&= -8(\pi/2)^2\left(\frac{\cos(n\pi)}{2n}\right) + \int_0^{\pi/2} 16x\frac{\cos(2nx)}{2n}\,\mathrm{d}x \\
&= \pi^2\frac{(-1)^{n+1}}{n} + \int_0^{\pi/2} 16x\frac{\cos(2nx)}{2n}\,\mathrm{d}x.
\end{aligned}$$

The next integration by parts gives

$$\begin{aligned}
\pi b_n &= \pi^2\frac{(-1)^{n+1}}{n} + \left[16x\left(\frac{\sin(2nx)}{(2n)^2}\right)\right]_0^{\pi/2} - \int_0^{\pi/2} 16\left(\frac{\sin(2nx)}{(2n)^2}\right)\mathrm{d}x \\
&= \pi^2\frac{(-1)^{n+1}}{n} + \left[16\left(\frac{\cos(2nx)}{(2n)^3}\right)\right]_0^{\pi/2} \\
&= \pi^2\frac{(-1)^{n+1}}{n} + 2\left(\frac{(-1)^n - 1}{n^3}\right).
\end{aligned}$$

Finally

$$b_n = \pi\frac{(-1)^{n+1}}{n} + \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^3}\right).$$

As $f(x)$ is an odd function the full range Fourier series just found on $(-\pi/2, \pi/2)$ is also the sine Fourier series on $(0, \pi/2)$.

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6. Let $f(x)$ denote the 2π-periodic function defined on $(-\pi, \pi]$ by

$$f(x) = \begin{cases} x, & |x| < \pi/2, \\ 0, & x \in (-\pi, -\pi/2) \cup (\pi/2, \pi]. \end{cases}$$

Construct the Fourier series representation.

Solution

The function is an odd function on $(-\pi, \pi)$ which means that the Fourier series only involves sine terms. Also the function has discontinuities at $\pm\pi/2$.

The Fourier series is

$$S(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x.$$

In this case by using $f(x) = 0$ on $(-\pi/2, \pi]$ and integration by parts we have

$$\begin{aligned}\frac{\pi b_n}{2} &= \int_0^{\pi/2} x\sin(nx)\,\mathrm{d}x \\ &= \left[-x\frac{\cos(nx)}{n}\right]_0^{\pi/2} + \int_0^{\pi/2}\frac{\cos(nx)}{n}\,\mathrm{d}x.\end{aligned}$$

Now

$$\left[-x\frac{\cos(nx)}{n}\right]_0^{\pi/2} = -\frac{\pi}{2}\left(\frac{\cos(n\pi/2)}{n}\right).$$

This is 0 when n is odd. For the other term to consider we have

$$\int_0^{\pi/2}\frac{\cos(nx)}{n}\,\mathrm{d}x = \left[\frac{\sin(nx)}{n^2}\right]_0^{\pi/2} = \frac{\sin(n\pi/2)}{n^2}.$$

This is 0 when n is even. Putting all parts together we have the following. When $n = 2m-1$, $m = 1, 2, \dots$ (i.e. n is odd)

$$\begin{aligned}b_{2m-1} &= \frac{2}{\pi}\left(\frac{\sin((2m-1)\pi/2)}{(2m-1)^2}\right) = \frac{2}{\pi}\left(\frac{\sin((2m-1)\pi/2)}{(2m-1)^2}\right) \\ &= = \frac{2}{\pi}\left(\frac{-\cos(m\pi)}{(2m-1)^2}\right) = \frac{2}{\pi}\frac{(-1)^{m+1}}{(2m-1)^2}.\end{aligned}$$

When $n = 2m$, $m = 1, 2, \dots$ (i.e. n is even)

$$b_{2m} = -\frac{\cos(m\pi)}{2m} = \frac{(-1)^{m-1}}{2m}.$$

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7. Let $f_1(x) = |x|$ and $f_2(x) = 3x^2$. Show that for $|x| \le \pi$ we have

$$\begin{aligned}f_1(x) &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos((2n+1)x)}{(2n+1)^2} + \cdots\right), \\ f_2(x) &= \pi^2 + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).\end{aligned}$$

By appropriately integrating these expressions show that for $0 \le x \le \pi$

$$\begin{aligned}x(\pi - x) &= \frac{8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin((2n+1)x)}{(2n+1)^3} + \cdots\right), \\ x(\pi - x)(\pi + x) &= 12\left(\sin(x) - \frac{\sin(2x)}{2^3} + \cdots + (-1)^{n+1}\frac{\sin(nx)}{n^3} + \cdots\right)\end{aligned}$$

which are valid for $0 \le x \le \pi$.

By making use of the appropriate results above show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\left(1+\frac{1}{3^3}\right) - \left(\frac{1}{5^3}+\frac{1}{7^3}\right) + \left(\frac{1}{9^3}+\frac{1}{11^3}\right) - \left(\frac{1}{13^3}+\frac{1}{15^3}\right) + \cdots = \frac{3\pi^3\sqrt{2}}{128}.$$

Solution

Both functions f_1 and f_2 are even and hence the Fourier series for both functions only involve cosine terms. In both cases integration by parts is needed to get the entries in the series. The 2π-periodic extensions of both functions are continuous and the functions are piecewise continuously differentiable which is sufficient conditions for the Fourier series to be the same as the functions for all points in $[-\pi, \pi]$.

The series for f_1.

The series is

$$f_1(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} f_1(x)\cos(nx)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\mathrm{d}x.$$

When $n = 0$ we have

$$a_0 = \frac{2}{\pi}\frac{\pi^2}{2} = \pi \quad \text{and thus } \frac{a_0}{2} = \frac{\pi}{2}.$$

For $n \ge 1$ we use integration by parts to give

$$\begin{aligned}
\frac{a_n\pi}{2} &= \left[x\frac{\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n}\,\mathrm{d}x \\
&= -\int_0^{\pi} \frac{\sin(nx)}{n}\,\mathrm{d}x \\
&= \left[\frac{\cos(nx)}{n^2}\right]_0^{\pi} = \left[\frac{\cos(n\pi)-1}{n^2}\right] \\
&= \frac{-1+(-1)^n}{n^2}.
\end{aligned}$$

When n is even $a_n = 0$ and when $n = 2m+1$, $m \ge 0$ we have

$$\frac{a_{2m+1}\pi}{2} = -\frac{2}{(2m+1)^2} \quad \text{giving } a_{2m+1} = -\frac{4}{\pi(2m+1)^2}.$$

The series for f_2.

The series is

$$f_2(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} f_2(x)\cos(nx)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} 3x^2\cos(nx)\mathrm{d}x.$$

When $n = 0$ we have

$$a_0 = \frac{2}{\pi}\pi^3 = 2\pi^2 \quad \text{and thus } \frac{a_0}{2} = \pi^2.$$

For $n \ge 1$ we use integration by parts to give

$$\begin{aligned}\frac{a_n\pi}{2} &= \left[3x^2\frac{\sin(nx)}{n}\right]_0^{\pi} - \int_0^{\pi} 6x\frac{\sin(nx)}{n}\,\mathrm{d}x \\ &= -\int_0^{\pi} 6x\frac{\sin(nx)}{n}\,\mathrm{d}x.\end{aligned}$$

Integration by parts again gives

$$\begin{aligned}\frac{a_n\pi}{2} &= \left[6x\frac{\cos(nx)}{n^2}\right]_0^{\pi} + \int_0^{\pi} 6\frac{\cos(nx)}{n^2}\,\mathrm{d}x \\ &= \frac{6\pi(-1)^n}{n^2} \quad \text{giving } a_n = 12\frac{(-1)^n}{n^2}.\end{aligned}$$

The next part of the questions mentions integrating the Fourier series for f_1 and f_2 and for his note that for $n \ge 1$

$$\int_0^x \cos(nt)\,\mathrm{d}t = \frac{\sin(x)}{n}.$$

and for the constant term in the series we have

$$\int_0^x \frac{a_0}{2}\,\mathrm{d}t = \frac{a_0 x}{2}.$$

Integrating the series for f_1.

Now for $x \ge 0$

$$\int_0^x f_1(t)\,\mathrm{d}t = \frac{x^2}{2}$$

Equating this with the term-by-term integration of the Fourier series gives for $x \ge 0$

$$\frac{x^2}{2} = \frac{\pi x}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3}.$$

By multiplying by -1 we have

$$\frac{\pi x - x^2}{2} = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3}$$

and further re-arrangement gives

$$x(\pi - x) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3}.$$

This establishes the result in the question. As a comment as to what changes if we wish to consider $x \in [-\pi, 0]$ the first thing to note is that the right hand side is an odd function and we have

$$\int_0^x f_1(t)\,\mathrm{d}t = -\frac{x^2}{2}$$

leading to

$$x(\pi + x) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{\sin((2n+1)x)}{(2n+1)^3}.$$

The function

$$g_1(x) = \begin{cases} x(\pi - x), & \text{if } x \ge 0, \\ x(\pi + x), & \text{if } x < 0 \end{cases}$$

is continuously differentiable at $x = 0$ with $g_1(0) = 0$ and $g'(0) = \pi$ but the second derivative is not continuous at $x = 0$.

Integrating the series for f_2.

$$\int_0^x f_2(t)\,\mathrm{d}t = \int_0^x 3t^2\,\mathrm{d}t = x^3.$$

Thus

$$x^3 = \pi^2 x + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nx).$$

Now

$$\pi^2 x - x^3 = x(\pi^2 - x^2) = x(\pi - x)(\pi + x).$$

Hence

$$x(\pi - x)(\pi + x) = 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin(nx).$$

As the series for $f_2(x)$ is valid for all $x \in [-\pi, \pi]$ we can substitute $x = \pi$ in the expression. Now

$$f_2(\pi) = 3\pi^2$$

and by using the Fourier series we must have

$$3\pi^2 = \pi^2 + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi) = \pi^2 + 12\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Re-arranging gives

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

It is a little harder to immediately decide what to do for the final part although it does involve choosing an appropriate value of x in the formula

$$x(\pi - x) = \frac{8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin((2n+1)x)}{(2n+1)^3} + \cdots\right).$$

As there is a $\sqrt{2}$ in the answer this suggests that we try $x = \pi/4$ as

$$\sin(\pi/4) = \frac{1}{\sqrt{2}}, \quad \sin(3\pi/4) = \frac{1}{\sqrt{2}}, \quad \sin(5\pi/4) = -\frac{1}{\sqrt{2}}, \quad \sin(7\pi/4) = -\frac{1}{\sqrt{2}}.$$

Now on the left hand side $x(\pi - x)$ gives

$$\frac{\pi}{4}\pi - \frac{\pi}{4} = \frac{3\pi^2}{16}.$$

Hence

$$\frac{3\pi^2}{16} = \left(\frac{8}{\pi}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\left(1 + \frac{1}{3^3}\right) - \left(\frac{1}{5^3} + \frac{1}{7^3}\right) + \left(\frac{1}{9^3} + \frac{1}{11^3}\right) - \left(\frac{1}{13^3} + \frac{1}{15^3}\right) + \cdots\right)$$

which re-arranges to give the expression in the question.

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8. Show that

$$\begin{aligned} x\sin(x) &= 1 - \frac{1}{2}\cos(x) \\ &\quad -2\left(\frac{\cos(2x)}{2^2 - 1} - \frac{\cos(3x)}{3^2 - 1} + \cdots + \frac{(-1)^n\cos(nx)}{n^2 - 1} + \cdots\right) \end{aligned}$$

which is valid for $|x| \le \pi$.

By using the above result, or otherwise, show that

$$\begin{aligned} x\cos(x) &= -\frac{1}{2}\sin(x) \\ &\quad +2\left(\frac{2\sin(2x)}{2^2 - 1} - \frac{3\sin(3x)}{3^2 - 1} + \cdots + \frac{(-1)^n n\sin(nx)}{n^2 - 1} + \cdots\right) \end{aligned}$$

which is valid for $|x| < \pi$.

Solution

The function $f(x) = x\sin(x)$ is even and the 2π-periodic extension of $f(x)$ defined on $[-\pi, \pi]$ is a continuous function which is piecewise continuously differentiable. This is a sufficient condition for the Fourier series to be the same as $f(x)$ on $[-\pi, \pi]$.

As the function is even the Fourier series only involves cosine terms and it is of the form

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n\cos(nx), \quad a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x.$$

Unlike earlier examples we need integration by parts to get all the coefficients, i.e. in the case of a_0 as well as a_n for $n \ge 1$. In the case of a_0 we have

$$\begin{aligned} \frac{\pi}{2}a_0 &= \int_0^\pi x\sin(x)\,\mathrm{d}x = [-x\cos(x)]_0^\pi + \int_0^\pi \cos(x)\,\mathrm{d}x \\ &= -\pi\cos(\pi) = \pi \quad \text{giving } \frac{a_0}{2} = 1. \end{aligned}$$

For $n \ge 1$,

$$\frac{\pi}{2} a_n = \int_0^\pi x \sin(x) \cos(nx)\,\mathrm{d}x.$$

As in an earlier question we use trig. identities to re-express $\sin(x)\cos(nx)$. We have

$$\sin((n+1)x) - \sin((n-1)x) = 2\cos(nx)\sin(x)$$

and thus

$$\frac{\pi}{2} a_n = \int_0^\pi x \left(\frac{\sin((n+1)x) - \sin((n-1)x)}{2} \right) \mathrm{d}x,$$

i.e.

$$\pi a_n = \int_0^\pi x \left(\sin((n+1)x) - \sin((n-1)x) \right) \mathrm{d}x,$$

It helps here to first get an expression for the following.

$$\begin{aligned} \int_0^\pi x \sin(mx)\,\mathrm{d}x &= \left[-x \frac{\cos(mx)}{m} \right]_0^\pi + \int_0^\pi \frac{\cos(mx)}{m}\,\mathrm{d}x \\ &= -\pi \frac{\cos(m\pi)}{m} = \pi \frac{(-1)^{m+1}}{m}. \end{aligned}$$

If we return the problem of obtaining a_n note that we should consider separately the case $n = 1$ when we have

$$\pi a_1 = \frac{\pi}{1+1}, \quad \text{i.e. } a_1 = \frac{1}{2}.$$

For $n \ge 2$ we have

$$\begin{aligned} \pi a_n &= \pi \left(\frac{(-1)^{n+2}}{n+1} - \frac{(-1)^n}{n-1} \right) \\ &= -(-1)^n \pi \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = -(-1)^n \pi \left(\frac{2}{n^2-1} \right) \end{aligned}$$

and thus

$$a_n = \frac{-2(-1)^n}{n^2-1}.$$

As $f(-\pi) = f(\pi)$ it is valid to differentiate term-by-term to get the Fourier series for $f'(x)$ in this example which is valid in $|x| < \pi$. Now

$$f'(x) = x\cos(x) + \sin(x).$$

If we compare with the term-by-term differentiation of the series we have

$$x\cos(x) + \sin(x) = 1 + \frac{1}{2}\sin(x) - \sum_{n=2}^\infty n a_n \sin(nx)$$

and hence

$$x\cos(x) = 1 - \frac{1}{2}\sin(x) - \sum_{n=2}^\infty n a_n \sin(nx).$$

9. Obtain the half range cosine series valid on $(0, \pi)$ for the function
$$f(x) = \frac{x^2 - 2\pi x + 2\pi^2/3}{4}.$$
Hence give the function whose Fourier series on $(-\pi, \pi)$ is
$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

Solution

The coefficients in the half range series are given by
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \, \mathrm{d}x.$$

$$a_0 = \frac{2}{\pi} \left(\frac{1}{4}\right) \left(\frac{\pi^3}{3} - 2\pi\frac{\pi^2}{2} + \frac{2\pi^3}{3}\right) = 0.$$

For $n \ge 1$ integration by parts two times gives
$$\begin{aligned}
\frac{\pi a_n}{2} &= \int_0^{\pi} f(x) \cos(nx) \, \mathrm{d}x = \left[f(x) \frac{\sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} f'(x) \frac{\sin(nx)}{n} \, \mathrm{d}x \\
&= -\int_0^{\pi} f'(x) \frac{\sin(nx)}{n} \, \mathrm{d}x \\
&= \left[f'(x) \frac{\cos(nx)}{n^2} \right]_0^{\pi} - \int_0^{\pi} f''(x) \frac{\cos(nx)}{n^2} \, \mathrm{d}x.
\end{aligned}$$

Now $f'(x) = x/2 - \pi/2$ so that $f'(0) = -\pi/2$ and $f'(\pi) = 0$ and $f''(x) = 1/2$ is constant.
$$\int_0^{\pi} \cos(nx) \, \mathrm{d}x = \left[\frac{\sin(nx)}{n} \right]_0^{\pi} = 0$$
and
$$f'(\pi) \cos(n\pi) - f'(0) = \frac{\pi}{2}.$$
Thus
$$\frac{\pi a_n}{2} = \frac{\pi}{2n^2} \quad \text{and } a_n = \frac{1}{n^2}.$$
The half range function extended to $(-\pi, \pi)$ is
$$f(x) = \frac{x^2 - 2\pi|x| + 2\pi^2/3}{4} = \sum_1^{\infty} \frac{\cos(nx)}{n^2}.$$

We get the same limit as we approach π as we do when we approach π and hence the 2π periodic extension is a continuous function of $\mathbb{R}$.

It is valid to differentiate term-by-term to give
$$f'(x) = -\sum_1^{\infty} \frac{\sin(nx)}{n} = \begin{cases} \dfrac{2x - 2\pi}{4}, & \text{if } 0 < x < \pi, \\ \dfrac{2x + 2\pi}{4}, & \text{if } -\pi < x < 0 \end{cases}$$

(Note that the derivative is not defined at $x = 0$.)

From the above it follows that

$$\sum_{1}^{\infty} \frac{\sin(nx)}{n} = -f'(x).$$