Exercises on Fourier series

1. *This question was in the May 2019* `MA2815` *exam.*

   Let $f : \mathbb{R} \to \mathbb{R}$ denote a 2π-periodic function which is piecewise continuous. The Fourier series for this function is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) ,$$

   where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,\mathrm{d}x, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,\mathrm{d}x.$$

   Let f_1 and f_2 be 2π-periodic function defined on $(-\pi, \pi]$ as follows.

$$f_1(x) = \begin{cases} 1, & \text{if } |x| \le \pi/2, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi, \end{cases}$$

$$f_2(x) = \begin{cases} 1, & \text{if } 0 \le x \le \pi/2, \\ -1, & \text{if } -\pi/2 \le x < 0, \\ 0, & \text{if } -\pi < x < -\pi/2 \text{ or } \pi/2 < x \le \pi. \end{cases}$$

   (a) Sketch $f_1(x)$ on the interval $(-\pi, 3\pi)$.

   (b) Show that the Fourier series for $f_1(x)$ is

$$\frac{1}{2} + \frac{2}{\pi}\left(\cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} + \cdots + (-1)^{m+1}\frac{\cos((2m-1)x)}{2m-1} + \cdots\right).$$

   (c) Determine the Fourier series for $f_2(x)$ giving the general formula for the a_n coefficients and giving the values of b_1, b_2, b_3, b_4 and b_5.

   (d) State for what values of $x \in (-\pi, \pi)$ the Fourier series for $f_1(x)$ is the same as $f_1(x)$ and for what values, if any, they differ.

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2. *This question was in the May 2017* `MA2815` *exam.*

   Consider the function $f : [-\pi, \pi) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } -\pi \le x < -\frac{\pi}{3}, \\ 0, & \text{if } -\frac{\pi}{3} \le x \le \frac{\pi}{3}, \\ -1, & \text{if } \frac{\pi}{3} < x < \pi. \end{cases}$$

   Denote by $g : \mathbb{R} \to \mathbb{R}$ the 2π-periodic extension of f to $\mathbb{R}$.

   (a) Sketch $g(x)$ over the interval $x \in [-3\pi, 3\pi]$ indicating carefully the key values on both axes.

   The Fourier series of f is given by

$$S(x) = \lim_{N\to\infty} S_N(x) \quad \text{where} \quad S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\Big(a_n \cos(nx) + b_n \sin(nx)\Big)$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

(b) Give those values of x for which $S(x) = f(x)$ and those values of x for which $S(x) \neq f(x)$.

(c) Determine a_0, general expressions for every a_n and b_n and show that the Fourier series S of f is

$$S(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - \cos(\frac{n\pi}{3})}{n}\sin(nx)\,.$$

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3. *This question was in the May 2018* `MA2815` *exam.*

Let $g : \mathbb{R} \to \mathbb{R}$ denote a 2π-periodic function which is piecewise continuous. The Fourier series for this function is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)\,,$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,\mathrm{d}x\,, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(nx)\,\mathrm{d}x\,.$$

Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ denote the 2π-periodic functions given on $(-\pi, \pi]$ by

$$f_1(x) = \begin{cases} 1\,, & -\pi < x \le -\pi/2\,, \\ 0\,, & -\pi/2 < x < \pi/2\,, \\ 1\,, & \pi/2 \le x \le \pi\,, \end{cases} \quad \text{and} \quad f_2(x) = -\frac{x}{2} + \int_0^x f_1(t)\,\mathrm{d}t\,.$$

(a) Obtain the Fourier coefficients of f_1 in their simplest form.

(b) Show that $f_2(x)$ can be written in the form

$$f_2(x) = \begin{cases} \dfrac{x+\pi}{2}\,, & -\pi < x \le -\pi/2\,, \\ -\frac{x}{2}\,, & -\pi/2 < x < \pi/2\,, \\ \dfrac{x-\pi}{2}\,, & \pi/2 \le x \le \pi\,. \end{cases}$$

and sketch $f_2(x)$ on the interval $-\pi \le x \le \pi$.

(c) Obtain the Fourier coefficients of f_2 in their simplest form.

(d) For each of f_1 and f_2 state the set of points, if any, where the value of the Fourier series is not the same as the value of the function it represents.

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4. *This question was in the May 2016* `MA2815` *exam.*

   (a) Consider the function $g\colon [0, 2\pi] \to \mathbb{R}$ defined by

   $$g(x) = \begin{cases} 2, & \text{if } 0 \le x < \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} < x \le 2\pi, \\[2ex] 1, & \text{if } \dfrac{\pi}{2} \le x \le \dfrac{3\pi}{2}. \end{cases}$$

   Denote by $f\colon \mathbb{R} \to \mathbb{R}$ the 2π-periodic extension of g over $\mathbb{R}$.
   Sketch $f(x)$ over the interval $x \in [-2\pi, 2\pi]$ indicating carefully the key values on the axis.

   (b) The Fourier series of f is given by

   $$\begin{aligned} S(x) &= \lim_{N\to\infty} S_N(x) \\ \text{where} \quad S_N(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} \Big(a_n \cos(nx) + b_n \sin(nx) \Big) \end{aligned}$$

   with

   $$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,\mathrm{d}x \qquad \text{and} \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,\mathrm{d}x.$$

   Determine a_0, general expressions for every a_n and b_n, and show that the Fourier series S of f is

   $$S(x) = \frac{3}{2} + \frac{2}{\pi} \left(\sum_{n=0}^{\infty} (-1)^n \frac{\cos((2n+1)x)}{2n+1} \right).$$

   (c) For what values of x do we have $S(x) \ne f(x)$ on $[-\pi, \pi]$?

   (d) Explain why the Fourier series suggests that,

   $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$$

   and use the Leibniz alternating series test to test this series for convergence.
   Before 2017/8 the study block `MA2730` *also contributed to the* `MA2815` *May exam. The use the Leibniz alternating series test would not be a* `MA2815` *paper now.*

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5. *This question was in the May 2014* `MA2815` *exam.*

   Sketch the graph of the periodic function

   $$f(x) = \begin{cases} -2x^2, & -\pi/2 \le x < 0, \\ 2x^2, & 0 \le x < \pi/2, \end{cases} \qquad f(x+\pi) = f(x),$$

   find its full-range Fourier series on $[-\pi/2, \pi/2]$ and the sine Fourier series on $[0, \pi/2]$.

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6. Let $f(x)$ denote the 2π-periodic function defined on $(-\pi, \pi]$ by

   $$f(x) = \begin{cases} x, & |x| < \pi/2, \\ 0, & x \in (-\pi, -\pi/2) \cup (\pi/2, \pi]. \end{cases}$$

   Construct the Fourier series representation.

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7. Let $f_1(x) = |x|$ and $f_2(x) = 3x^2$. Show that for $|x| \le \pi$ we have

$$\begin{aligned} f_1(x) &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \cdots + \frac{\cos((2n+1)x)}{(2n+1)^2} + \cdots\right), \\ f_2(x) &= \pi^2 + 12\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx). \end{aligned}$$

By appropriately integrating these expressions show that for $0 \le x \le \pi$

$$\begin{aligned} x(\pi - x) &= \frac{8}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3^3} + \frac{\sin((2n+1)x)}{(2n+1)^3} + \cdots\right), \\ x(\pi - x)(\pi + x) &= 12\left(\sin(x) - \frac{\sin(2x)}{2^3} + \cdots + (-1)^{n+1}\frac{\sin(nx)}{n^3} + \cdots\right) \end{aligned}$$

which are valid for $0 \le x \le \pi$.

By making use of the appropriate results above show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and

$$\left(1 + \frac{1}{3^3}\right) - \left(\frac{1}{5^3} + \frac{1}{7^3}\right) + \left(\frac{1}{9^3} + \frac{1}{11^3}\right) - \left(\frac{1}{13^3} + \frac{1}{15^3}\right) + \cdots = \frac{3\pi^3\sqrt{2}}{128}.$$

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8. Show that

$$\begin{aligned} x\sin(x) &= 1 - \frac{1}{2}\cos(x) \\ &\quad -2\left(\frac{\cos(2x)}{2^2 - 1} - \frac{\cos(3x)}{3^2 - 1} + \cdots + \frac{(-1)^n\cos(nx)}{n^2 - 1} + \cdots\right) \end{aligned}$$

which is valid for $|x| \le \pi$.

By using the above result, or otherwise, show that

$$\begin{aligned} x\cos(x) &= -\frac{1}{2}\sin(x) \\ &\quad +2\left(\frac{2\sin(2x)}{2^2 - 1} - \frac{3\sin(3x)}{3^2 - 1} + \cdots + \frac{(-1)^n n\sin(nx)}{n^2 - 1} + \cdots\right) \end{aligned}$$

which is valid for $|x| < \pi$.

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9. Obtain the half range cosine series valid on $(0, \pi)$ for the function

$$f(x) = \frac{x^2 - 2\pi x + 2\pi^2/3}{4}.$$

Hence give the function whose Fourier series on $(-\pi, \pi)$ is

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$
