Exercises related to chapters 4: Finite differences and the 2-pt BVP

1. *This question was in the May 2019* MA2815 *exam.*

   Let $u(x)$ denote an infinitely continuously differentiable function on an interval containing $[0, 3h]$, where $h > 0$ is small for all the Taylor expansions that you use to be valid. By using appropriate Taylor expansions determine constants b_1, b_2, b_3, c_1, c_2 and c_3 in the following.

$$\frac{u(h)-u(0)}{h} = b_1 u'(0) + b_2 h u''(0) + b_3 h^2 u'''(0) + \cdots$$
$$\frac{-u(3h)+9u(h)-8u(0)}{6h} = c_1 u'(0) + c_2 h u''(0) + c_3 h^2 u'''(0) + \cdots$$

Solution

$$u(h) = u(0) + u'(0)h + \frac{u''(0)}{2}h^2 + \frac{u'''(0)}{6}h^3 + \cdots$$
$$u(3h) = u(0) + u'(0)(3h) + \frac{u''(0)}{2}(3h)^2 + \frac{u'''(0)}{6}(3h)^3 + \cdots$$

Thus
$$\frac{u(h)-u(0)}{h} = u'(0) + \frac{u''(0)}{2}h + \frac{u'''(0)}{6}h^2 + \cdots$$
$b_1 = 1$, $b_2 = 1/2$ and $b_3 = 1/6$.

$$-u(3h) + 9u(h) = 8u(0) + u'(0)(6h) + \frac{u'''(0)}{6}(-27+9)h^3 + \cdots$$
$$= 8u(0) + u'(0)(6h) + u'''(0)(-3h^3) + \cdots .$$

Thus
$$\frac{-u(3h)+9u(h)-8u(0)}{6h} = u'(0) - \frac{u'''(0)}{2}h^2 + \cdots$$
$c_1 = 1$, $c_2 = 0$ and $c_3 = -1/2$.

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2. *This question was in the May 2018* MA2815 *exam.*

   Let $h > 0$ be small and let $u(x)$ denote an infinitely differentiable continuous function defined on an interval which contains $[-2h, 2h]$ and assume that the Maclaurin expansion of $u(x)$ converges in this domain. Show that

$$u(h) - u(-h) = 2hu'(0) + \frac{h^3}{3}u'''(0) + \frac{h^5}{60}u^{(5)}(0) + \mathcal{O}(h^7)$$

   and determine c_1, c_3, c_5 in the following expressions.

$$-u(2h) + 8u(h) - 8u(-h) + u(-2h) = c_1 h u'(0) + c_3 h^3 u'''(0) + + c_5 h^5 u^{(5)}(0) + \mathcal{O}(h^7) .$$

Solution

Let $u_0 = u(0)$, $u_0' = u'(0)$ etc. in the following.

$$\begin{aligned} u(h) &= u_0 + hu_0' + \frac{h^2}{2}u_0'' + \frac{h^3}{6}u_0''' + \frac{h^4}{24}u_0'''' + \frac{h^5}{120}u_0^{(5)} + \frac{h^6}{720}u_0^{(6)} + \mathcal{O}(h^7) \\ u(-h) &= u_0 - hu_0' + \frac{h^2}{2}u_0'' - \frac{h^3}{6}u_0''' + \frac{h^4}{24}u_0'''' - \frac{h^5}{120}u_0^{(5)} + \frac{h^6}{720}u_0^{(6)} + \mathcal{O}(h^7). \end{aligned}$$

When we subtract the even index terms cancel and we have

$$u(h) - u(-h) = 2hu_0' + \frac{h^3}{3}u_0''' + \frac{h^5}{60}u_0^{(5)} + \mathcal{O}(h^7).$$

Replacing h by $2h$ in the above gives

$$u(2h) - u(-2h) = 4hu_0' + \frac{8h^3}{3}u_0''' + \frac{32h^5}{60}u_0^{(5)} + \mathcal{O}(h^7).$$

$$-u(2h) + 8u(h) - 8u(-h) + u(-2h) = 8(u(h) - u(-h)) - (u(2h) - u(-2h)).$$

Thus $c_1 = 12$, $c_3 = 0$ and

$$c_5 = -\frac{24}{60} = -\frac{2}{5}.$$

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3. *This question was in the May 2015* `MA2815` *exam.*

Let $u(x)$ denote a 4 times continuously differentiable function on an interval containing $[0, 2h]$ where $h > 0$. The left hand sides in the following are finite difference approximations to $u'(0)$ using $u(2h)$, $u(h/2)$ and $u(0)$. Assuming that h is small determine the constants b_2, b_3, c_2 and c_3 in these expressions.

$$\begin{aligned} \frac{2(u(h/2) - u(0))}{h} &= u'(0) + b_2 hu''(0) + b_3 h^2 u'''(0) + \mathcal{O}(h^3), \\ \frac{-u(2h) + 16u(h/2) - 15u(0)}{6h} &= u'(0) + c_2 hu''(0) + c_3 h^2 u'''(0) + \mathcal{O}(h^3). \end{aligned}$$

Solution

By a Taylor expansion about 0 we have

$$u(k) - u(0) = ku'(0) + \frac{k^2}{2}u''(0) + \frac{k^3}{6}u'''(0) + \mathcal{O}(k^4).$$

When $k = h/2$ we get

$$\begin{aligned} \frac{u(h/2) - u(0)}{h/2} &= u'(0) + \frac{(h/2)}{2}u''(0) + \frac{(h/2)^2}{6}u'''(0) + \mathcal{O}(h^3) \\ &= u'(0) + b_1 hu''(0) + b_2 h^2 u'''(0) + \mathcal{O}(h^3) \end{aligned}$$

with $b_1 = 1/4$ and $b_2 = 1/24$.

When $k = 2h$ we have

$$u(2h) - u(0) = 2hu'(0) + \frac{(2h)^2}{2}u''(0) + \frac{(2h)^3}{6}u'''(0) + \mathcal{O}(h^4).$$

Combining the two expansions gives

$$\begin{aligned} 16(u(h/2)-u(0))-(u(2h)-u(0)) &= 6hu'(0)+h^3\left(\frac{16}{48}-\frac{8}{6}\right)u'''(0)+\mathcal{O}(h^4) \\ &= 6hu'(0)-h^3u'''(0)+\mathcal{O}(h^4). \end{aligned}$$

Thus

$$\frac{-u(2h)+16u(h/2)-15u(0)}{6h}=u'(0)+c_2hu''(0)+c_3h^2u'''(0)+\mathcal{O}(h^3).$$

with $c_2=0$ and $c_3=-1/6$.

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4. *This question was in the May 2016* `MA2815` *exam.*

   Let $u(x)$ denote a 4 times continuously differentiable function on an interval containing $[0,2h]$ where $h>0$. The left hand side in the following are finite difference approximations to $u'(0)$ using $u(2h)$, $u(h)$ and $u(0)$. Assuming that h is small determine the constants b_2, b_3, c_2 and c_3 in the following expressions:

$$\begin{aligned} \frac{u(h)-u(0))}{h} &= u'(0)+b_2hu''(0)+b_3h^2u'''(0)+\mathcal{O}(h^3), \\ \frac{-u(2h)+4u(h)-3u(0)}{2h} &= u'(0)+c_2hu''(0)+c_3h^2u'''(0)+\mathcal{O}(h^3). \end{aligned}$$

Solution

Taylor expansions about 0 are

$$\begin{aligned} u(h) &= u(0)+hu'(0)+\frac{h^2}{2}u''(0)+\frac{h^3}{6}u'''(0)+\mathcal{O}(h^4), \\ u(2h) &= u(0)+2hu'(0)+\frac{4h^2}{2}u''(0)+\frac{8h^3}{6}u'''(0)+\mathcal{O}(h^4). \end{aligned}$$

Thus

$$\frac{u(h)-u(0)}{h}=u'(0)+\frac{h}{2}u''(0)+\frac{h^2}{6}u'''(0)+\mathcal{O}(h^3).$$

$b_2=1/2$ and $b_3=1/6$.

$$4u(h)-u(2h)-3u(0)=2hu'(0)-\frac{4h^3}{6}u'''(0)+\mathcal{O}(h^4).$$

Dividing this by $2h$ leads to $c_2=0$ and $c_3=-1/3$.

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5. *This question was in the May 2017* `MA2815` *exam.*

   Let $u(x)$ denote a four times continuously differentiable function on an interval containing $-h$, 0 and $2h$ where $h>0$ is small for all the Taylor expansions that you use to be valid. By using appropriate Taylor expansions determine constants c_1, c_2 and c_3 in the following.

$$2u(-h)-3u(0)+u(2h)=c_1hu'(0)+c_2h^2u''(0)+c_3h^3u'''(0)+\cdots$$

   By using this relation, or otherwise, give a suitable finite difference approximation for $u''(0)$ using the values $u(-h)$, $u(0)$ and $u(2h)$.

Solution

$$\begin{aligned} u(2h) &= u(0) + 2hu'(0) + \frac{4h^2}{2}u''(0) + \frac{8h^3}{6}u'''(0) + \cdots \\ u(-h) &= u(0) - hu'(0) + \frac{h^2}{2}u''(0) - \frac{h^3}{6}u'''(0) + \cdots . \\ 2u(-h) - 3u(0) + u(2h) &= 3h^2u''(0) + \frac{6}{6}h^3u'''(0) + \cdots . \end{aligned}$$

Hence $c_1 = 0$, $c_2 = 3$ and $c_3 = 1$.

A suitable approximation to $u''(0)$ is given by

$$\frac{2u(-h) - 3u(0) + u(2h)}{3h^2}.$$

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6. *The following is about the local truncation error with Numerov's method wheich is one of the schemes in the* `MA2895` *assignment.*

Consider the following two-point boundary value problem

$$u''(x) = q(x)u(x) + r(x), \quad a < x < b, \quad u(a) = g_1, \quad u(b) = g_2,$$

where $q(x)$ and $r(x)$ are functions defined in $[a, b]$, $q(x) \ge 0$ on $[a, b]$ and these functions are such that $u(x)$ is at least 6-times continuously differentiable.

Given an integer $N > 1$, let $h = (b - a)/N$ and let

$$x_i = a + ih, \quad i = 0, 1, \ldots, N$$

denote mesh point in a uniform mesh. Further let $u_i = u(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$ for $i = 0, 1, \ldots, N$ and define

$$L_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{(q_{i-1}u_{i-1} + r_{i-1}) + 10(q_iu_i + r_i) + (q_{i+1}u_{i+1} + r_{i+1})}{12}.$$

Given that u satisfies the differential equation show that

$$L_i = \mathcal{O}(h^4).$$

Solution

The first thing to note is that as $u''(x) = q(x)u(x) + r(x)$ for all $x \in [a, b]$ it holds in particular at all the mesh points and the expression for L_i can be written as

$$L_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{u''_{i+1} + 10u''_i + u''_{i-1}}{12}.$$

Now adding the Taylor's expansions of $u_{i+1} = u(x_i + h)$ and $u_{i-1} = u(x_i - h)$ gives

$$u_{i+1} + u_{i-1} = 2u_i + h^2u''_i + \frac{h^4}{12}u''''_i + \frac{h^6}{360}u^{(6)}_i + \cdots$$

Similarly if we replace u by u'' we have

$$u''_{i+1} + u''_{i-1} = 2u''_i + h^2 u''''_i + \frac{h^4}{12} u^{(6)}_i + \cdots .$$

The parts we need for the expression for L_i are

$$\begin{aligned} \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} &= u''_i + \frac{h^2}{12} u''''_i + \frac{h^4}{360} u^{(6)}_i + \cdots , \\ \frac{u''_{i+1} + 12u''_i + u''_{i-1}}{12} &= u''_i + \frac{h^2}{12} u''''_i + \frac{h^4}{144} u^{(6)}_i + \cdots . \end{aligned}$$

Subtracting the second of these expressions from the first gves

$$L_i = \mathcal{O}(h^4).$$

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7. As in the previous question let $a = x_0 < x_1 < \cdots < x_N = b$ be points in a uniform mesh with mesh spacing $h = (b-a)/N$ and let $u_i = u(x_i)$ for $i = 0, 1, \ldots, N$. Also assume that $u(x)$ has as many continuous derivatives as is needed in Taylor series expansions. Let $2 \le i \le N-2$. By considering the expansions about x_i of $u_{i+2} - 2u_i + u_{i-2}$ and $u_{i+1} - 2u_i + u_{i-1}$ show that we have a representation of the form

$$\frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2} = u''(x_i) + ch^4 u^{(6)}(x_i) + \mathcal{O}(h^6)$$

and give the constant c.

Solution

As in the answer to exercise 6 we have

$$u_{i+1} + u_{i-1} = 2\left(u_i + \frac{h^2}{2} u''_i + \frac{h^4}{24} u''''_i + \frac{h^6}{720} u^{(6)}_i + \mathcal{O}(h^8) \right).$$

If we replace h by $2h$ then we have

$$u_{i+2} + u_{i-2} = 2\left(u_i + \frac{4h^2}{2} u''_i + \frac{16h^4}{24} u''''_i + \frac{64h^6}{720} u^{(6)}_i + \mathcal{O}(h^8) \right).$$

We combine these as

$$16(u_{i+1} + u_{i-1} - 2u_i) - (u_{i+2} + u_{i-2} - 2u_i) = 12h^2 u''_i - \frac{48}{360} h^4 u^{(6)}_i + \mathcal{O}(h^8).$$

Thus

$$\frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2} = u''(x_i) + ch^4 u^{(6)}(x_i) + \mathcal{O}(h^6)$$

with $c = -4/360 = -1/90$.

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8. In the lectures the 3-point central difference approximation to the second derivative is given involving equally spaced points. This question involves a 3-point approximation using points which are not equally spaced. Let $h_1 > 0$ and $h_2 > 0$ be

sufficiently small that the Taylor expansions of a function $u(x)$ about $x = a$ are valid. Show that

$$\frac{2u(a+h_1)}{h_1(h_1+h_2)} - \frac{2u(a)}{h_1h_2} + \frac{2u(a-h_2)}{h_2(h_1+h_2)}$$
$$= u''(a) + (h_1-h_2)\frac{u'''(a)}{3} + \left(\frac{h_1^3+h_2^3}{h_1+h_2}\right)\frac{u''''(a)}{12} + \cdots .$$

Solution

Taylor expansions gives

$$\begin{aligned}
u(a+h_1) &= u(a) + h_1u'(a) + \frac{h_1^2}{2}u''(a) + \frac{h_1^3}{6}u'''(a) + \frac{h_1^4}{24}u'''(a) + \cdots \\
u(a-h_2) &= u(a) - h_2u'(a) + \frac{h_2^2}{2}u''(a) - \frac{h_2^3}{6}u'''(a) + \frac{h_2^4}{24}u'''(a) + \cdots .
\end{aligned}$$

If we form $h_2u(a+h_1) + h_1u(a-h_2)$ then we eliminate the $u'(a)$ term and we have

$$h_2u(a+h_1) + h_1u(a-h_2) = (h_1+h_2)u(a)$$
$$+(h_2h_1^2 + h_1h_2^2)\frac{u''(a)}{2} + (h_2h_1^3 - h_1h_2^3)\frac{u'''(a)}{6} + (h_2h_1^4 + h_1h_2^4)\frac{u''''(a)}{24} + \cdots$$

To simplify a little note that

$$\begin{aligned}
h_1^2h_2 + h_1h_2^2 &= h_1h_2(h_1+h_2), \\
h_1^3h_2 - h_1h_2^3 &= h_1h_2(h_1^2-h_2^2) = h_1h_2(h_1+h_2)(h_1-h_2), \\
h_1^4h_2 + h_1h_2^4 &= h_1h_2(h_1^3+h_2^3).
\end{aligned}$$

Thus

$$h_2u(a+h_1) + h_1u(a-h_2) - (h_1+h_2)u(a)$$
$$= h_1h_2(h_1+h_2)\frac{u''(a)}{2} + h_1h_2(h_1+h_2)(h_1-h_2)\frac{u'''(a)}{6}$$
$$+h_1h_2(h_1^3+h_2^3)\frac{u''''(a)}{24} + \cdots$$

If we multiply through by $2/(h_1h_2(h_1+h_2))$, then we get the result to prove stated in the question.

Note that the above reduces to the equally spaced version when $h_1 = h_2 = h$.
