Exercises related to chapters 4: Finite differences and the 2-pt BVP

1. *This question was in the May 2019* `MA2815` *exam.*

   Let $u(x)$ denote an infinitely continuously differentiable function on an interval containing $[0, 3h]$, where $h > 0$ is small for all the Taylor expansions that you use to be valid. By using appropriate Taylor expansions determine constants b_1, b_2, b_3, c_1, c_2 and c_3 in the following.

$$\begin{aligned} \frac{u(h) - u(0)}{h} &= b_1 u'(0) + b_2 h u''(0) + b_3 h^2 u'''(0) + \cdots \\ \frac{-u(3h) + 9u(h) - 8u(0)}{6h} &= c_1 u'(0) + c_2 h u''(0) + c_3 h^2 u'''(0) + \cdots \end{aligned}$$

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2. *This question was in the May 2018* `MA2815` *exam.*

   Let $h > 0$ be small and let $u(x)$ denote an infinitely differentiable continuous function defined on an interval which contains $[-2h, 2h]$ and assume that the Maclaurin expansion of $u(x)$ converges in this domain. Show that

$$u(h) - u(-h) = 2hu'(0) + \frac{h^3}{3} u'''(0) + \frac{h^5}{60} u^{(5)}(0) + \mathcal{O}(h^7)$$

   and determine c_1, c_3, c_5 in the following expressions.

$$\begin{aligned} -u(2h) + 8u(h) - 8u(-h) + u(-2h) &= c_1 h u'(0) + c_3 h^3 u'''(0) + \\ &\qquad + c_5 h^5 u^{(5)}(0) + \mathcal{O}(h^7)\,. \end{aligned}$$

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3. *This question was in the May 2015* `MA2815` *exam.*

   Let $u(x)$ denote a 4 times continuously differentiable function on an interval containing $[0, 2h]$ where $h > 0$. The left hand sides in the following are finite difference approximations to $u'(0)$ using $u(2h)$, $u(h/2)$ and $u(0)$. Assuming that h is small determine the constants b_2, b_3, c_2 and c_3 in these expressions.

$$\begin{aligned} \frac{2(u(h/2) - u(0))}{h} &= u'(0) + b_2 h u''(0) + b_3 h^2 u'''(0) + \mathcal{O}(h^3), \\ \frac{-u(2h) + 16u(h/2) - 15u(0)}{6h} &= u'(0) + c_2 h u''(0) + c_3 h^2 u'''(0) + \mathcal{O}(h^3). \end{aligned}$$

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4. *This question was in the May 2016* `MA2815` *exam.*

   Let $u(x)$ denote a 4 times continuously differentiable function on an interval containing $[0, 2h]$ where $h > 0$. The left hand side in the following are finite difference approximations to $u'(0)$ using $u(2h)$, $u(h)$ and $u(0)$. Assuming that h is small determine the constants b_2, b_3, c_2 and c_3 in the following expressions:

$$\begin{aligned} \frac{u(h) - u(0))}{h} &= u'(0) + b_2 h u''(0) + b_3 h^2 u'''(0) + \mathcal{O}(h^3), \\ \frac{-u(2h) + 4u(h) - 3u(0)}{2h} &= u'(0) + c_2 h u''(0) + c_3 h^2 u'''(0) + \mathcal{O}(h^3). \end{aligned}$$

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5. *This question was in the May 2017* `MA2815` *exam.*

   Let $u(x)$ denote a four times continuously differentiable function on an interval containing $-h$, 0 and $2h$ where $h > 0$ is small for all the Taylor expansions that you use to be valid. By using appropriate Taylor expansions determine constants c_1, c_2 and c_3 in the following.

   $$2u(-h) - 3u(0) + u(2h) = c_1 h u'(0) + c_2 h^2 u''(0) + c_3 h^3 u'''(0) + \cdots$$

   By using this relation, or otherwise, give a suitable finite difference approximation for $u''(0)$ using the values $u(-h)$, $u(0)$ and $u(2h)$.

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6. *The following is about the local truncation error with Numerov's method wheich is one of the schemes in the* `MA2895` *assignment.*

   Consider the following two-point boundary value problem

   $$u''(x) = q(x)u(x) + r(x), \quad a < x < b, \quad u(a) = g_1, \quad u(b) = g_2,$$

   where $q(x)$ and $r(x)$ are functions defined in $[a, b]$, $q(x) \geq 0$ on $[a, b]$ and these functions are such that $u(x)$ is at least 6-times continuously differentiable.

   Given an integer $N > 1$, let $h = (b - a)/N$ and let

   $$x_i = a + ih, \quad i = 0, 1, \ldots, N$$

   denote mesh point in a uniform mesh. Further let $u_i = u(x_i)$, $q_i = q(x_i)$ and $r_i = r(x_i)$ for $i = 0, 1, \ldots, N$ and define

   $$L_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{(q_{i-1}u_{i-1} + r_{i-1}) + 10(q_i u_i + r_i) + (q_{i+1}u_{i+1} + r_{i+1})}{12}.$$

   Given that u satisfies the differential equation show that

   $$L_i = \mathcal{O}(h^4).$$

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7. As in the previous question let $a = x_0 < x_1 < \cdots < x_N = b$ be points in a uniform mesh with mesh spacing $h = (b - a)/N$ and let $u_i = u(x_i)$ for $i = 0, 1, \ldots, N$. Also assume that $u(x)$ has as many continuous derivatives as is needed in Taylor series expansions. Let $2 \leq i \leq N - 2$. By considering the expansions about x_i of $u_{i+2} - 2u_i + u_{i-2}$ and $u_{i+1} - 2u_i + u_{i-1}$ show that we have a representation of the form

   $$\frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12h^2} = u''(x_i) + ch^4 u^{(6)}(x_i) + \mathcal{O}(h^6)$$

   and give the constant c.

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8. In the lectures the 3-point central difference approximation to the second derivative is given involving equally spaced points. This question involves a 3-point approximation using points which are not equally spaced. Let $h_1 > 0$ and $h_2 > 0$ be sufficiently small that the Taylor expansions of a function $u(x)$ about $x = a$ are valid. Show that

   $$\frac{2u(a + h_1)}{h_1(h_1 + h_2)} - \frac{2u(a)}{h_1 h_2} + \frac{2u(a - h_2)}{h_2(h_1 + h_2)}$$
   $$= u''(a) + (h_1 - h_2)\frac{u'''(a)}{3} + \left(\frac{h_1^3 + h_2^3}{h_1 + h_2}\right)\frac{u''''(a)}{12} + \cdots.$$
