Exercises related to chapter 3: $\underline{u}' = A\underline{u}$, $\underline{u}(0) = \underline{u}_0$.

1. *This was in the May 2019* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = -1, \quad u_2(0) = 7.$$

   Suppose the 2×2 matrix above is replaced by

$$\begin{pmatrix} -2 & \alpha \\ \alpha & -2 \end{pmatrix},$$

   where $\alpha \ge 0$. For what values of α will the solution $\underline{u}(x) \to \underline{0}$ as $x \to \infty$ for all starting vectors $\underline{u}(0)$?

Solution

Let

$$A = \begin{pmatrix} 5 & -2 \\ 3 & -2 \end{pmatrix}.$$

$$\det(A - \lambda I) = (5-\lambda)(-2-\lambda) + 6 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4).$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

$$A - \lambda_1 I = \begin{pmatrix} 6 & -2 \\ 3 & -1 \end{pmatrix} \quad \text{and} \quad A - \lambda_2 I = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}.$$

For the eigenvectors we can take

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\underline{u}(x) = c_1 \mathrm{e}^{\lambda_1 x}\underline{v}_1 + c_2 \mathrm{e}^{\lambda_2 x}\underline{v}_2.$$

To satisfy the initial conditions

$$\underline{u}(0) = c_1\underline{v}_1 + c_2\underline{v}_2 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}.$$

Using basic Gauss elimination

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 10 \end{pmatrix}$$

Thus $c_2 = -2$ and $c_1 = 3$.

Let now

$$A = \begin{pmatrix} -2 & \alpha \\ \alpha & -2 \end{pmatrix}.$$

The eigenvalues of A now satisfy

$$\det(A - \lambda I) = (-2-\lambda)^2 - \alpha^2 = (-2-\alpha-\lambda)(-2+\alpha-\lambda) = 0.$$

Let $\lambda_1 = -2-\alpha$ and $\lambda_2 = -2+\alpha$. We have $\lambda_1 \le \lambda_2$ and $\lambda_2 < 0$ if $\alpha < 2$. When this is the case $\underline{u}(x) = c_1\mathrm{e}^{\lambda_1 x}\underline{v}_1 + c_2\mathrm{e}^{\lambda_2 x}\underline{v}_2 \to \underline{0}$ as $x \to \infty$ for all c_1 and c_2.

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2. *This was in the May 2018* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 6, \quad u_2(0) = -8.$$

Solution

We need the eigenvalues and eigenvectors of the 2×2 matrix

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}.$$

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21 = (\lambda+7)(\lambda-3).$$

The eigenvalues are $\lambda_1 = -7$ and $\lambda_2 = 3$.

$$A - \lambda_1 I = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}, \quad A - \lambda_2 I = \begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix},$$

By inspection we can take as eigenvectors

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{and} \quad \underline{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The general solution is

$$\underline{u}(x) = c_1 \mathrm{e}^{\lambda_1 x}\underline{v}_1 + c_2 \mathrm{e}^{\lambda_2 x}\underline{v}_2.$$

By inspection

$$\underline{u}(0) = c_1\underline{v}_1 + c_2\underline{v}_2 = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -8 \end{pmatrix}.$$

Adding 3 times the first equation to the second equation gives $10c_2 = 10$ and thus $c_2 = 1$ and then we get $c_1 = 6 - 3 = 3$. The solution is

$$\underline{u}(x) = 3\mathrm{e}^{-7x}\begin{pmatrix} 1 \\ -3 \end{pmatrix} + \mathrm{e}^{3x}\begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

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3. *This was in the May 2017* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 14, \quad u_2(0) = -10.$$

Solution

We need the eigenvalues and eigenvectors of the 2×2 matrix

$$A = \begin{pmatrix} 3 & 7 \\ 5 & 5 \end{pmatrix}.$$

$$\det(A-\lambda I) = (3-\lambda)(5-\lambda) - 35 = \lambda^2 - 8\lambda - 20 = (\lambda - 10)(\lambda + 2).$$

The eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 10$.

$$A - \lambda_1 I = \begin{pmatrix} 5 & 7 \\ 5 & 7 \end{pmatrix}, \quad A - \lambda_2 I = \begin{pmatrix} -7 & 7 \\ 5 & -5 \end{pmatrix}$$

By inspection we can take as eigenvectors

$$\underline{v}_1 = \begin{pmatrix} 7 \\ -5 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution is

$$\underline{u}(x) = c_1 \mathrm{e}^{\lambda_1 x} \underline{v}_1 + c_2 \mathrm{e}^{\lambda_2 x} \underline{v}_2.$$

By inpection

$$\underline{u}(0) = \begin{pmatrix} 7 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 14 \\ -10 \end{pmatrix} \quad \text{implies } c_1 = 2, \quad c_2 = 0.$$

The solution is

$$\underline{u}(x) = 2\mathrm{e}^{-2x} \underline{v}_1.$$

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4. *This was in the May 2016* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations with the given initial conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -8 & 3 \\ -18 & 7 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 6, \quad u_2(0) = -7.$$

   Suppose that the same ordinary differential equations are considered with the initial conditions replaced by $u_1(0) = \alpha$ and $u_2(0) = -7$. Determine the value of α such that the solution $\underline{u}(x) \to \underline{0}$ as $x \to \infty$.

Solution

Let A denote the 2×2 matrix. An eigenvalue λ of A satisfies

$$0 = \det(A - \lambda I) = (-8 - \lambda)(7 - \lambda) - (-54) = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1).$$

The eigenvalues are hence $\lambda_1 = -2$ and $\lambda_2 = 1$.

Let $\underline{v}_1$ and $\underline{v}_2$ denote eigenvectors corresponding to λ_1 and λ_2 respectively.

$$A - \lambda_1 I = A + 2I = \begin{pmatrix} -6 & 3 \\ -18 & 9 \end{pmatrix} \quad \text{and we can take } \underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$A - \lambda_2 I = A - I = \begin{pmatrix} -9 & 3 \\ -18 & 6 \end{pmatrix} \quad \text{and we can take } \underline{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The general solution of the system of differential equations is

$$\underline{u}(x) = c_1 \mathrm{e}^{\lambda_1 x} \underline{v}_1 + c_2 \mathrm{e}^{\lambda_2 x} \underline{v}_2.$$

To satisfy the initial condition we need

$$\underline{u}(0) = c_1\underline{v}_1 + c_2\underline{v}_2 = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}.$$

Solve by basic Gauss elimination.

$$\begin{pmatrix} 1 & 1 & 6 \\ 2 & 3 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 6 \\ 0 & 1 & -19 \end{pmatrix}.$$

Hence $c_2 = -19$ and $c_1 = 25$.

The general solution $\underline{u}(x) \to \underline{0}$ as $x \to \infty$ if and only if $\underline{u}(x)$ is of the form $c_1 \mathrm{e}^{-2x}$, i.e. the initial condition is such that $c_2 = 0$.

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -7 \end{pmatrix}.$$

Hence $\alpha = -7/2$.

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5. *This was in the May 2015* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -17 & 4 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 10, \quad u_2(0) = 15.$$

Solution

Let

$$A = \begin{pmatrix} -17 & 4 \\ -6 & -3 \end{pmatrix}.$$

The characteristic equation of A is

$$\begin{aligned} |A - \lambda I| &= (-17-\lambda)(-3-\lambda) + 24 \\ &= \lambda^2 + 20\lambda + (51+24) = \lambda^2 + 20\lambda + 75 = (\lambda+5)(\lambda+15) = 0. \end{aligned}$$

The eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -15$.

$$A - \lambda_1 I = \begin{pmatrix} -12 & 4 \\ -6 & 2 \end{pmatrix}.$$

For an eigenvector we can take

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

$$A - \lambda_2 I = \begin{pmatrix} -2 & 4 \\ -6 & 12 \end{pmatrix}.$$

For an eigenvector we can take

$$\underline{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution of the system is

$$\underline{u}(x) = c_1 e^{\lambda_1 x} \underline{v}_1 + c_2 e^{\lambda_2 x} \underline{v}_2.$$

To satisfy the initial condition we need

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 15 \end{pmatrix}.$$

By Gauss elimination we get

$$\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 10 \\ -15 \end{pmatrix}, \quad c_2 = 3, \quad c_1 = 4.$$

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6. *This was in the May 2014* `MA2815` *exam paper*

In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations:

$$\frac{d}{dx} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 7, \quad u_2(0) = 2.$$

Solution

Let

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}$$

The characteristic equation of A is

$$\det(A - \lambda I) = (2 - \lambda)(-2 - \lambda) - 12 = \lambda^2 - 16.$$

Thus $\lambda_1 = -4$ and $\lambda_2 = 4$ are the eigenvalues.

$$A + 4I = \begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}.$$

By inspection an eigenvector of A corresponding to λ_1 is $(1, -2)^T$.

$$A - 4I = \begin{pmatrix} -2 & 3 \\ 4 & -6 \end{pmatrix}.$$

By inspection an eigenvector of A corresponding to λ_2 is $(3, 2)^T$.

The general solution is of the form

$$\underline{u}(x) = c_1 e^{-4x} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{4x} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

To satisfy the initial condition we need

$$\begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}.$$

Adding 2× the first row to the second row gives

$$\begin{pmatrix} 1 & 3 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 16 \end{pmatrix}.$$

Thus $c_2 = 2$ and $c_1 = 1$.

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7. Consider the following system of differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}-1 & 2\\-2 & -1\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}, \quad u_1(0)=0, \quad u_2(0)=2.$$

Show that the solution is $u_1(x) = 2\mathrm{e}^{-x}\sin(2x)$, $u_2(x) = 2\mathrm{e}^{-x}\cos(2x)$.

Solution

To use the techniques given in the lectures we need the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix}-1 & 2\\-2 & -1\end{pmatrix}$$

and by forming the characteristic equation we have

$$|A-\lambda I| = (-1-\lambda)^2+4 = 0.$$

The eigenvalues are the complex conjugate pair

$$\lambda_1 = -1-2i, \quad \lambda_2 = -1+2i.$$

As we have a real matrix with a complex conjugate pair of eigenvalues we have complex eigenvectors and we can take one to be the complex conjugate of the other. In the case of λ_1 we form

$$A-\lambda_1 I = \begin{pmatrix}-1-(-1-2i) & 2\\-2 & -1-(-1-2i)\end{pmatrix} = \begin{pmatrix}2i & 2\\-2 & 2i\end{pmatrix}.$$

$(A-\lambda_1 I)\underline{v}_1 = \underline{0}$ if we take

$$\underline{v}_1 = \begin{pmatrix}i\\1\end{pmatrix}.$$

In the case of λ_2 we similarly form

$$A-\lambda_2 I = \begin{pmatrix}-2i & 2\\-2 & -2i\end{pmatrix}.$$

In this case we satisfy $(A-\lambda_2 I)\underline{v}_2 = \underline{0}$ if we take

$$\underline{v}_2 = \begin{pmatrix}-i\\1\end{pmatrix}$$

which is the complex conjugate of $\underline{v}_1$ given above.

The general solution is of the form

$$\underline{u}(x) = c_1\mathrm{e}^{\lambda_1 x}\underline{v}_1 + c_2\mathrm{e}^{\lambda_2 x}\underline{v}_2.$$

To satisfy the initial condition we need c_1 and c_2 to satisfy

$$\begin{pmatrix}i & -i\\1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}.$$

If we multiply the first equation by i and add to the second equation then we get $2c_2 = 2$ so that $c_1 = c_2 = 1$.

Now properties of the complex exponential give

$$e^{\lambda_1 x} = e^{-x}e^{-2ix}, \quad e^{\lambda_2 x} = e^{-x}e^{2ix}$$

and thus

$$\underline{u}(x) = e^{-x}\left(e^{-2ix}\begin{pmatrix} i \\ 1 \end{pmatrix} + e^{2ix}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right) = 2e^{-x}\begin{pmatrix} \sin(2x) \\ \cos(2x) \end{pmatrix}.$$

As a final point, to just verify that the expressions given are the solution we can quickly note that they satisfy the initial condition and to also verify that they satisfy the differential equations we can use the product rule of differentiation. For the details,

$$u_1' = 2e^{-x}(2\cos(2x) - \sin(2x)) = 2u_2 - u_1$$

and

$$u_2' = 2e^{-x}(-2\sin(2x) - \cos(2x)) = -2u_1 - u_2.$$

The solution to the differential equations with given initial condition is unique and thus the candidate solution is the unique solution.

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8. Let A denote a $n \times n$ matrix with linearly independent eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ with $A\underline{v}_i = \lambda_i \underline{v}_i$, let $V = (\underline{v}_1, \ldots, \underline{v}_n)$ and let $D = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$. Explain why the expression for the exponential matrix

$$\exp(xA) = V\exp(xD)V^{-1}$$

is unchanged if V is replaced by the matrix

$$\widetilde{V} = (\alpha_1\underline{v}_1, \ldots, \alpha_n\underline{v}_n)$$

for all non-zero values of $\alpha_1, \ldots, \alpha_n$.

Solution

The main thing to note to answer this questions is that we can write $\widetilde{V}$ as

$$\widetilde{V} = (\alpha_1\underline{v}_1, \ldots, \alpha_n\underline{v}_n) = (\underline{v}_1, \ldots, \underline{v}_n)\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} = V\widetilde{D}$$

where

$$\widetilde{D} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}.$$

Thus with V replaced by $\widetilde{V}$ in the expression for the exponential matrix we have

$$V\widetilde{D}\exp(xD)\widetilde{D}^{-1}V^{-1}.$$

Now $\widetilde{D}$, $\exp(xD)$ and $\widetilde{D}^{-1}$ are all diagonal matrices and diagonal matrices commute and thus

$$\widetilde{D}\exp(xD)\widetilde{D}^{-1} = (\widetilde{D}\widetilde{D}^{-1})\exp(xD) = \exp(xD).$$

Hence the exponential matrix is unchanged if we replace V by $\widetilde{V}$.

In the context of solving systems of ODEs note that the columns of both V and $\widetilde{V}$ are the eigenvectors of A and this result confirms that it does not matter how you choose to scale the eigenvectors in order to solve the ODEs. If you do the calculations with different scalings then you change the intermediate calculations but you get the same solution to the problem.

As a related comment, which was not part of this question, if you put the eigenvectors in a different order then this corresponds to having a matrix

$$\widetilde{V} = VP \quad \text{and} \quad \widetilde{V}^{-1} = P^{-1}V^{-1},$$

where P is a permutation matrix which is a matrix which you obtain from the identity matrix by re-arranging the columns (or the rows). The entries of D would also be in the order corresponding to $\widetilde{V}$ and we would now have

$$\widetilde{D} = P^{-1}DP = PDP^{-1}.$$

Now

$$\widetilde{V}\exp(x\widetilde{D})\widetilde{V}^{-1} = VP\exp(x\widetilde{D})P^{-1}V^{-1}.$$

From the result above about re-arranging diagonal matrices we have

$$P\exp(x\widetilde{D})P^{-1} = \exp(xD).$$

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9. Let the matrix A be as in question 8 and let $\underline{u} = \underline{u}(x)$ denote the solution to the system

$$\underline{u}' = A\underline{u}, \quad \underline{u}(0) = \underline{u}_0.$$

What properties must A have to ensure that $\underline{u}(x) \to \underline{0}$ as $x \to \infty$ for all initial values $\underline{u}_0$?

Solution

When A is diagonalisable the general solution to the equations is of the form

$$\underline{u}(x) = c_1 e^{\lambda_1 x}\underline{v}_1 + \cdots + c_n e^{\lambda_n x}\underline{v}_n,$$

with $\underline{v}_k \neq 0$ and $A\underline{v}_k = \lambda_k \underline{v}_k$ for $k = 1, 2, \ldots, n$. In this case the solution $\underline{u} \to \underline{0}$ as $x \to \infty$ for all possible $c_1, \ldots, c_n$ if and only if

$$\left|e^{\lambda_k x}\right| \to 0 \quad \text{as } x \to \infty \quad \text{for } k = 1, 2, \ldots, n.$$

When λ_k is real this requires that $\lambda_k < 0$. When $\lambda_k = \alpha_k + i\beta_k$ is not real we have

$$\left|e^{\lambda_k x}\right| = \left|e^{(\alpha_k + i\beta_k)x}\right| = e^{\alpha_k x} \to 0$$

provided $\alpha_k < 0$. In all cases the solution tends to $\underline{0}$ as $x \to \infty$ provided the real part of each eigenvalue is negative.
