Exercises related to chapter 3: $\underline{u}' = A\underline{u}$, $\underline{u}(0) = \underline{u}_0$.

1. *This was in the May 2019* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = -1, \quad u_2(0) = 7.$$

   Suppose the 2×2 matrix above is replaced by

$$\begin{pmatrix} -2 & \alpha \\ \alpha & -2 \end{pmatrix},$$

   where $\alpha \geq 0$. For what values of α will the solution $\underline{u}(x) \to \underline{0}$ as $x \to \infty$ for all starting vectors $\underline{u}(0)$?

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2. *This was in the May 2018* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 6, \quad u_2(0) = -8.$$

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3. *This was in the May 2017* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 3 & 7 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 14, \quad u_2(0) = -10.$$

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4. *This was in the May 2016* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations with the given initial conditions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -8 & 3 \\ -18 & 7 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 6, \quad u_2(0) = -7.$$

   Suppose that the same ordinary differential equations are considered with the initial conditions replaced by $u_1(0) = \alpha$ and $u_2(0) = -7$. Determine the value of α such that the solution $\underline{u}(x) \to \underline{0}$ as $x \to \infty$.

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5. *This was in the May 2015* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -17 & 4 \\ -6 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 10, \quad u_2(0) = 15.$$

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6. *This was in the May 2014* `MA2815` *exam paper*

   In the following $\underline{u} = (u_i)$ is a column vector of length 2 with each component $u_i = u_i(x)$. Find the solution of the following system of ordinary differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 7, \quad u_2(0) = 2.$$

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7. Consider the following system of differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1(0) = 0, \quad u_2(0) = 2.$$

   Show that the solution is $u_1(x) = 2\mathrm{e}^{-x}\sin(2x)$, $u_2(x) = 2\mathrm{e}^{-x}\cos(2x)$.

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8. Let A denote a $n \times n$ matrix with linearly independent eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$ with $A\underline{v}_i = \lambda_i \underline{v}_i$, let $V = (\underline{v}_1, \ldots, \underline{v}_n)$ and let $D = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_n\}$. Explain why the expression for the exponential matrix

$$\exp(xA) = V\exp(xD)V^{-1}$$

   is unchanged if V is replaced by the matrix

$$\widetilde{V} = (\alpha_1\underline{v}_1, \ldots, \alpha_n\underline{v}_n)$$

   for all non-zero values of $\alpha_1, \ldots, \alpha_n$.

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9. Let the matrix A be as in question 8 and let $\underline{u} = \underline{u}(x)$ denote the solution to the system

$$\underline{u}' = A\underline{u}, \quad \underline{u}(0) = \underline{u}_0.$$

   What properties must A have to ensure that $\underline{u}(x) \to \underline{0}$ as $x \to \infty$ for all initial values $\underline{u}_0$?
