Exercises related to chapter 2: Gauss elimination, LU factorizations $\cdots$

1. Suppose that we have the following factorization of a matrix A.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Use this factorization, or otherwise, to determine the 4th and 3rd columns of A^{-1}. Give $\det(A)$.

Solution

The 4th column of A^{-1} is described by $\underline{x} = A^{-1}\underline{e}_4$ where, as usual, $\underline{e}_4$ denotes the 4th column of the 4×4 identity matrix. Hence $\underline{x}$ is the solution to the linear system

$$A\underline{x} = \underline{e}_4.$$

As we have a factorization $A = LU$ we have

$$A\underline{x} = L(U\underline{x}) = \underline{e}_4.$$

The method to obtain $\underline{x}$ is to first solve $L\underline{y} = \underline{e}_4$ by forward substitution and then to solve $U\underline{x} = \underline{y}$ by backward substitution.

Solving $L\underline{y} = \underline{e}_4$ immediately gives $\underline{y} = \underline{e}_4$. (The inverse of a unit lower triangular matrix is also unit lower triangular and thus the last column of L and L^{-1} is always the last base vector.)

Solving $U\underline{x} = \underline{y} = \underline{e}_4$ involves the following.

$$\begin{aligned} 4x_4 &= 1, \quad x_4 = 1/4. \\ 3x_3 &= -3x_4, \quad x_3 = -1/4. \\ 2x_2 &= -2x_3, \quad x_2 = +1/4. \\ x_1 &= -x_2, \quad x_1 = -1/4. \end{aligned}$$

Let now $\underline{x}$ denote the 3rd column of A^{-1}, i.e.

$$\underline{x} = A^{-1}\underline{e}_3, \quad \text{which we re-write as} \quad A\underline{x} = L(U\underline{x}) = \underline{e}_3.$$

As earlier the technique is to solve $L\underline{y} = \underline{e}_3$ by forward substitution and then to solve $U\underline{x} = \underline{y}$ by backward substitution.

To solve $L\underline{y} = \underline{e}_3$ we immediately have $y_1 = y_2 = 0$, $y_3 = 1$. Using the last row of L then gives $y_4 = 1$. For the system involving U we have the following.

$$\begin{aligned} 4x_4 &= 1, \quad x_4 = 1/4. \\ 3x_3 &= 1 - 3x_4 = 1/4, \quad x_3 = 1/12. \\ 2x_2 &= -2x_3, \quad x_2 = -1/12. \\ x_1 &= -x_2, \quad x_1 = 1/12. \end{aligned}$$

From the properties of determinants we have

$$\det(A) = \det(L)\det(U) = \det(U) = 24$$

as the determinant of a triangular matrix is the product of the diagonal entries.

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2. *This question was in the May 2019* MA2815 *exam paper.*

   Let
$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix}.$$
   Determine the first column of the inverse L^{-1} using the forward substitution technique.

Solution

Let $\underline{e}_1$ denote the first base vector and let $\underline{x}$ denote the first column of L^{-1} which is described by
$$\underline{x} = L^{-1}\underline{e}_1, \quad \text{i.e. } L\underline{x} = \underline{e}_1.$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$
By forward substitution we have
$$\begin{aligned} & x_1 = 1, \\ & 2 + x_2 = 0, \quad \text{gives } x_2 = -2, \\ & 3(-2) + x_3 = 0, \quad \text{gives } x_3 = 6, \\ & 4(6) + x_4 = 0, \quad \text{gives } x_4 = -24. \end{aligned}$$

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3. *This question was in the May 2019* MA2815 *exam paper.*

   Consider the following 3×3 matrices.
$$A = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & -1 \\ 8 & 5 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -1 & -3 \\ -3 & -3 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 3 & 1 & -4 \\ 9 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$
   In each case either determine the LU factorization involving a unit lower triangular matrix L and an upper triangular matrix U or indicate that no such factorization exists. If a factorization does not exist then you need to give a reason. For each matrix which has an LU factorization give the determinant.

Solution

We use basic Gauss elimination to get the LU factorization.
$$A = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & -1 \\ 8 & 5 & 7 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 3 \\ 0 & 3 & 5 \\ 0 & 9 & 19 \end{pmatrix} \to \begin{pmatrix} -2 & 1 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{pmatrix} = U$$
The vector of multipliers are
$$\underline{m}_1 = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \quad \text{giving } L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & 3 & 1 \end{pmatrix}.$$

$\det(A) = \det(U) = -24$.

$$B = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -1 & -3 \\ -3 & -3 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = U.$$

The vector of multipliers are

$$\underline{m}_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{giving } L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

$\det(A) = \det(U) = 0$.

$$C = \begin{pmatrix} 3 & 1 & -4 \\ 9 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 3 & 1 & -4 \\ 0 & 0 & 12 \\ 0 & -2/3 & 2/3 \end{pmatrix}$$

Basic Gauss elimination cannot continue as the $2, 2$ entry is 0. The 2×2 principal submatrix is singular. The matrix C does not have a LU factorization.

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4. *This question was in the May 2018* `MA2815` *exam paper.* Let

$$A_1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 4 \\ 1 & 1 & -2 \end{pmatrix}.$$

The matrices differ in the order of the rows. For each matrix either obtain the LU factorization, where L is a unit lower triangular matrix and U is an upper triangular matrix, or explain why the matrix does not have a LU factorization.

Solution

To have a LU factorization every principal sub-matrix needs to be non-singular. A_1 does not have a LU factorization as the 2×2 principal sub-matrix is singular.

When a LU factorization exists we can obtain it by using basic Gauss elimination.

$$A_2 = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & -2 & 4 \\ 0 & -1 & 3 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} -1 & -2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

with the multipliers being

$$\underline{m}_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$A_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$A_3 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 4 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{with multipliers } \underline{m}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

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5. *This question was in the May 2017* `MA2815` *exam paper.* Let

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -3 & 3 & -1 \\ -3 & -1 & 6 \end{pmatrix}.$$

Determine the unit lower triangular matrix L and the upper triangular matrix U such that $A = LU$. Using this factorization find the second column of A^{-1}.

Solution

Basic Gauss elimination gives the LU factorization. The sequence of matrices is as follows.

$$\begin{pmatrix} 3 & -1 & -1 \\ -3 & 3 & -1 \\ -3 & -1 & 6 \end{pmatrix} \to \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{pmatrix} \to \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix} = U$$

with the multipliers being

$$\underline{m}_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \text{so that } L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}.$$

The second column of the inverse is $\underline{x} = A^{-1}\underline{e}_2$ so that $A\underline{x} = LU\underline{x} = \underline{e}_2$. Let $\underline{y} = U\underline{x}$. We solve $L\underline{y} = \underline{e}_2$ by forward substitution followed by $U\underline{x} = \underline{y}$ by backward substitution.

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad \text{gives } y_1 = 0,\ y_2 = 1,\ y_3 = 1.$$

$$\begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

gives

$$x_3 = 1/3, \quad x_2 = (1 + 2/3)/2 = 5/6, \quad x_1 = (x_2 + x_3)/3 = 7/18.$$

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6. *This question was in the May 2016* `MA2815` *exam paper.*

Consider the following three 3×3 matrices which differ in the order of the rows.

$$A_1 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \\ 3 & 6 & 1 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 3 & 6 & 1 \end{pmatrix}.$$

Determine which of these matrices has a LU factorization where L denotes a unit lower triangular matrix and U denotes an upper triangular matrix. If a matrix does not have a factorization then you must give a reason. If a matrix does have a factorization then you need to determine L and U.

Give the absolute value of the determinant of A_2, i.e. give $|\det(A_2)|$.

Solution

An $n \times n$ matrix has a LU factorization if and only if the principal submatries of order up to $n-1$ are non-singular.

The 2×2 principal sub-matrix of A_1 has determinant of 0 and hence A_1 does not have a LU factorization.

The $1, 1$ entry of A_2 is 0 and hence A_2 does not have a LU factorization.

We can attempt to get the LU factorization of A_3 by using basic Gauss elimination. The sequence of matrices are as follows.

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 3 & 6 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{pmatrix}$$

with the vector of multipliers being

$$\underline{m}_1 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

The matrix after one step is already in upper triangular form and thus

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{pmatrix}.$$

Swapping rows changes the sign of a determinant but not the magnitude and thus the magnitude of the determinant all 3 matrices is the same. Thus by properties of determinants

$$|\det(A_2)| = |\det(A_3)| = |\det(U)| = 11.$$

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7. *This question was in the May 2015* `MA2815` *exam paper.*

Suppose that we have the following factorization of a matrix A.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Use this factorization to find the third column of A^{-1}.

Solution

The 3rd column of A^{-1} is

$$\underline{x} = A^{-1}\underline{e}_3, \quad \text{i.e. } A\underline{x} = \underline{e}_3$$

where $\underline{e}_3$ is the usual base vector. As $A = LU$ we can solve $L\underline{y} = \underline{e}_3$ followed by $U\underline{x} = \underline{y}$.

$$L\underline{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{gives } \underline{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$U\underline{x} = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This gives

$$\begin{aligned} x_3 &= 1, \\ 2x_2 &= x_3, \quad x_2 = 1/2, \\ 3x_1 &= x_2 + x_3 = 3/2, \quad x_1 = 1/2. \end{aligned}$$

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8. Solve the following linear systems $A\underline{x} = \underline{b}$ and determine a factorization of the form $PA = LU$ where P is a permutation matrix, L is unit lower triangular matrix and U is an upper triangular matrix. In your answer you need to state the matrix PA as well as L and U.

$$\text{(i)} \quad A = \begin{pmatrix} 0 & 3 & 1 \\ -2 & 1 & -1 \\ 1 & 10 & 3 \end{pmatrix}, \underline{b} = \begin{pmatrix} -4 \\ -8 \\ -12 \end{pmatrix}, \qquad \text{(ii)} \quad A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 2 \\ 2 & -4 & -7 \end{pmatrix}, \underline{b} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}.$$

Solution

(a) Basic Gauss elimination does not work here as the $1,1$ entry is 0 and to proceed with Gauss elimination we need to swap row 1 with one of the other rows. For the ease of the hand calculations we swap with row 3 and we do the workings with the right hand side vector from the start. The Gauss elimination is then as follows.

$$\begin{pmatrix} 1 & 10 & 3 & -12 \\ -2 & 1 & -1 & -8 \\ 0 & 3 & 1 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 10 & 3 & -12 \\ 0 & 21 & 5 & -32 \\ 0 & 3 & 1 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 10 & 3 & -12 \\ 0 & 21 & 5 & -32 \\ 0 & 0 & 2/7 & 4/7 \end{pmatrix}.$$

The multipliers in these steps give the vectors

$$\underline{m}_1 = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 1/7 \end{pmatrix}.$$

To get $\underline{x}$ we use backward substitution to give

$$\begin{aligned} x_3 &= 2. \\ 21x_2 &= -32 - 5x_3 = -42, \quad x_2 = -2. \\ x_1 &= -12 - 10x_2 - 3x_3 = -12 + 20 - 6 = 2. \end{aligned}$$

For the factorization

$$PA = \begin{pmatrix} 1 & 10 & 3 \\ -2 & 1 & -1 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 1/7 & 1 \end{pmatrix} \begin{pmatrix} 1 & 10 & 3 \\ 0 & 21 & 5 \\ 0 & 0 & 2/7 \end{pmatrix} = LU.$$

(b) Basic Gauss elimination does not work here as the $1,1$ entry is 0 and to proceed with Gauss elimination we need to swap row 1 with one of the other rows. For the ease of the hand calculations we swap with row 2 and we do the workings with the right hand side vector from the start. The Gauss elimination is then as follows.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 1 & -5 \\ 2 & -4 & -7 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 1 & -5 \\ 0 & -6 & -11 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & -14 & 14 \end{pmatrix}.$$

The multipliers in these steps give the vectors

$$\underline{m}_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \quad \underline{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

To get $\underline{x}$ we use backward substitution to give

$$\begin{aligned} -14x_3 &= 14, \quad x_3 = -1. \\ -2x_2 &= -5 - x_3 = -4, \quad x_2 = 2. \\ x_1 &= 1 - x_2 - 2x_3 = 1. \end{aligned}$$

For the factorization

$$PA = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 2 & -4 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -14 \end{pmatrix} = LU.$$

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9. (The following result is just stated in the notes.) Let

$$M_k = I - \underline{m}_k \underline{e}_k^T, \quad \text{where } \underline{m}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix}$$

which is a Gauss transformation matrix of size $n \times n$. Prove by induction that

$$M_1^{-1} \cdots M_r^{-1} = I + \underline{m}_1 \underline{e}_1^T + \cdots + \underline{m}_r \underline{e}_r^T, \quad r = 1, \ldots, n-1.$$

Solution

Before the induction proof is started we note the identity

$$(I - \underline{m}_k \underline{e}_k^T)(I + \underline{m}_k \underline{e}_k^T) = I - \underline{m}_k (\underline{e}_k^T \underline{m}_k) \underline{e}_k^T = I$$

because the $\underline{e}_k^T \underline{m}_k$ is the kth entry of $\underline{m}_k$ and this is 0. Thus

$$M_k^{-1} = I + \underline{m}_k \underline{e}_k^T.$$

The base case in the induction proof is when the number of terms in the product is just one and as indicated above $M_1 = I + \underline{m}_1 \underline{e}_1^T$. Thus the result is true in the base case.

The induction hypothesis is that we suppose that

$$M_1^{-1} \cdots M_r^{-1} = I + \underline{m}_1 \underline{e}_1^T + \cdots + \underline{m}_r \underline{e}_r^T$$

for some $1 \le r \le n-2$.

To complete the proof we need to show that the result is true when we have $k = r+1$ terms. We write the product of this number of terms as

$$(M_1^{-1} \cdots M_r^{-1}) M_{r+1}^{-1} = (I + \underline{m}_1 \underline{e}_1^T + \cdots + \underline{m}_r \underline{e}_r^T)(I + \underline{m}_{r+1} \underline{e}_{r+1}^T),$$

where we have used the induction hypothesis to replace the first part and the expression for M_{r+1}^{-1} for the last part.

$$M_1^{-1}\cdots M_{r+1}^{-1} = (I + \underline{m}_1\underline{e}_1^T + \cdots + \underline{m}_r\underline{e}_r^T) + (\underline{m}_1\underline{e}_1^T + \cdots + \underline{m}_r\underline{e}_r^T)\underline{m}_{r+1}\underline{e}_{r+1}^T.$$

The result follows because the first $r+1$ entries of $\underline{m}_{r+1}$ are zero which implies in particular that

$$\underline{e}_1^T\underline{m}_{r+1} = \cdots = \underline{e}_r^T\underline{m}_{r+1} = 0.$$

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10. Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

i.e. L is a unit lower triangular matrix with each entry below the diagonal being equal to -1. Determine the first column of L^{-1}. If you can spot the pattern in the answer to the previous part then give L^{-1} and further determine $\|L\|_\infty$ and $\|L^{-1}\|_\infty$.

Solution

The first column of L^{-1} is $\underline{x} = L^{-1}\underline{e}_1$ so that $L\underline{x} = \underline{e}_1$ and in full this linear system is as follows.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Forward substitution starts with $x_1 = 1$. As the right hand side is 0 after the first position we have

$$x_{i+1} = x_1 + x_2 + \cdots + x_i, \quad i = 1,2,3,4$$

i.e. each entry is the sum of the previous entries. Hence $x_2 = 1$, $x_3 = 2$, $x_4 = 4$ and $x_5 = 8$.

It can be shown that the inverse of a unit lower triangular matrix is also unit lower triangular and hence if now $\underline{x} = L^{-1}\underline{e}_i$ we get $x_i = 1$ and then the entries $x_{i+1},\ldots,x_n$ are $1, 2, \ldots 2^{n-i-1}$. The inverse matrix is

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 & 0 \\ 8 & 4 & 2 & 1 & 1 \end{pmatrix}.$$

The ∞-norm is the maximum row sum of absolute values and in the case of both L and L^{-1} the maximum occurs on the last row to give

$$\|L\|_\infty = 5, \quad \|L^{-1}\|_\infty = 16 = 2^4.$$

This can be generalised to the $n \times n$ case (i.e. with all entries below the diagonal being -1) giving

$$\|L\|_\infty = n, \quad \|L^{-1}\|_\infty = 2^{n-1}.$$

Thus we have an example of a sequence of square matrices of larger and larger size with the condition number growing rapidly but with each matrix having determinant equal to 1.
