Exercises related to chapter 2: Gauss elimination, *LU* factorizations ⋯

1. Suppose that we have the following factorization of a matrix A.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Use this factorization, or otherwise, to determine the 4th and 3rd columns of A^{-1}. Give $\det(A)$.

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2. *This question was in the May 2019* `MA2815` *exam paper.*

Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix}.$$

Determine the first column of the inverse L^{-1} using the forward substitution technique.

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3. *This question was in the May 2019* `MA2815` *exam paper.*

Consider the following 3×3 matrices.

$$A = \begin{pmatrix} -2 & 1 & 3 \\ 4 & 1 & -1 \\ 8 & 5 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -1 & -3 \\ -3 & -3 & 1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 3 & 1 & -4 \\ 9 & 3 & 0 \\ 2 & 0 & -2 \end{pmatrix}.$$

In each case either determine the LU factorization involving a unit lower triangular matrix L and an upper triangular matrix U or indicate that no such factorization exists. If a factorization does not exist then you need to give a reason. For each matrix which has an LU factorization give the determinant.

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4. *This question was in the May 2018* `MA2815` *exam paper.* Let

$$A_1 = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -2 & 4 \\ 1 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -2 & 4 \\ 1 & 1 & -2 \end{pmatrix}.$$

The matrices differ in the order of the rows. For each matrix either obtain the LU factorization, where L is a unit lower triangular matrix and U is an upper triangular matrix, or explain why the matrix does not have a LU factorization.

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5. *This question was in the May 2017* `MA2815` *exam paper.* Let

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -3 & 3 & -1 \\ -3 & -1 & 6 \end{pmatrix}.$$

Determine the unit lower triangular matrix L and the upper triangular matrix U such that $A = LU$. Using this factorization find the second column of A^{-1}.

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6. *This question was in the May 2016* `MA2815` *exam paper.*

   Consider the following three 3×3 matrices which differ in the order of the rows.

$$A_1 = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 4 \\ 3 & 6 & 1 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 3 & 6 & 1 \end{pmatrix}.$$

   Determine which of these matrices has a LU factorization where L denotes a unit lower triangular matrix and U denotes an upper triangular matrix. If a matrix does not have a factorization then you must give a reason. If a matrix does have a factorization then you need to determine L and U.

   Give the absolute value of the determinant of A_2, i.e. give $|\det(A_2)|$.

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7. *This question was in the May 2015* `MA2815` *exam paper.*

   Suppose that we have the following factorization of a matrix A.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

   Use this factorization to find the third column of A^{-1}.

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8. Solve the following linear systems $A\underline{x} = \underline{b}$ and determine a factorization of the form $PA = LU$ where P is a permutation matrix, L is unit lower triangular matrix and U is an upper triangular matrix. In your answer you need to state the matrix PA as well as L and U.

$$\text{(i)} \quad A = \begin{pmatrix} 0 & 3 & 1 \\ -2 & 1 & -1 \\ 1 & 10 & 3 \end{pmatrix}, \underline{b} = \begin{pmatrix} -4 \\ -8 \\ -12 \end{pmatrix}, \quad \text{(ii)} \quad A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 2 \\ 2 & -4 & -7 \end{pmatrix}, \underline{b} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}.$$

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9. (The following result is just stated in the notes.) Let

$$M_k = I - \underline{m}_k \underline{e}_k^T, \quad \text{where } \underline{m}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{pmatrix}$$

   which is a Gauss transformation matrix of size $n \times n$. Prove by induction that

$$M_1^{-1} \cdots M_r^{-1} = I + \underline{m}_1 \underline{e}_1^T + \cdots + \underline{m}_r \underline{e}_r^T, \quad r = 1, \ldots, n-1.$$

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10. Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

   i.e. L is a unit lower triangular matrix with each entry below the diagonal being equal to -1. Determine the first column of L^{-1}. If you can spot the pattern in the answer to the previous part then give L^{-1} and further determine $\|L\|_\infty$ and $\|L^{-1}\|_\infty$.
