Exercises related to chapter 1: eigenvalues, eigenvectors, matrix norms, plus some revision exercises based on previous modules

1. Determine the eigenvalues and eigenvectors of the following matrices.

$$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Solution

Let

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}.$$

The characteristic equation is

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)+2 = \lambda^2-5\lambda+6 = (\lambda-2)(\lambda-3) = 0.$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

For the eigenvector associated with λ_1 consider the matrix

$$A - \lambda_1 I = A - 2I = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

By design this is a singular matrix and for a non-trivial solution of $(A-2I)\underline{v} = \underline{0}$ we only need to consider one of the equations to get $v_1 + v_2 = 0$. Hence for the eigenvector we can take the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For the eigenvector associated with λ_2 consider the matrix

$$A - \lambda_2 I = A - 3I = \begin{pmatrix} -2 & -1 \\ 2 & 1 \end{pmatrix}.$$

$(A-3I)\underline{v} = \underline{0}$ requires that $2v_1 + v_2 = 0$ and we can take

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

as an eigenvector.

Consider now the other 2×2 matrix which is symmetric and now let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

The characteristic equation is

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = \lambda^2 - 5\lambda = \lambda(\lambda-5) = 0.$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$. For the eigenvector $\underline{v}$ associated with $\lambda_1 = 0$ we need $v_1 + 2v_2 = 0$ and we can take

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

As A is symmetric the eigenvector associated with $\lambda_2 = 5$ is orthogonal to this vector and we can take

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

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2. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and let } \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

By first computing $A\underline{v}$ determine all the eigenvalues and eigenvectors of A.

By using your results about the eigenvalues and eigenvectors of matrix A determine the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

Solution

$$A\underline{v} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2\underline{v}$$

and hence 2 is an eigenvalue and $\underline{v}$ is an eigenvector.

As A is a 3×3 matrix the characteristic polynomial is a cubic and the above indicates that $(2-\lambda)$ is a factor. To get the characteristic polynomial we do the following. Expanding the determinant about row 1 gives

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix} \\ &= (-\lambda)(\lambda^2-1) - (-\lambda-1) + (1+\lambda) \\ &= -\lambda^3 + \lambda + 2(1+\lambda) = -\lambda^3 + 3\lambda + 2. \end{aligned}$$

As $2-\lambda$ is a factor and by considering the constant term and the coefficient of λ^3 we get a factorization of the form

$$\det(A-\lambda I) = -\lambda^3 + 3\lambda + 2 = (2-\lambda)(1+\alpha\lambda+\lambda^2).$$

The parameter α can be obtained by equating the coefficents of λ^2 to give

$$0 = 2-\alpha, \quad \text{i.e. } \alpha = 2.$$

Thus

$$\det(A-\lambda I) = (2-\lambda)(1+2\lambda+\lambda^2) = (2-\lambda)(1+\lambda)^2.$$

The other eigenvalues are -1 and -1 which is a repeated eigenvalue. As we have a real symmetric matrix there is a complete set of eigenvectors and thus there is a two-dimensional eigenspace associated with the eigenvalue -1 and all vectors in this space are orthogonal to the vector $\underline{v}$ associated with the eigenvalue $\lambda_1 = 2$. The question does not ask for orthogonal vectors to be given and hence it is sufficient here to note that

$$A-(-1)I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and two linearly independent eigenvectors are

$$\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

If $\underline{v}_3$ is replaced by

$$\underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

then $\underline{v}$, $\underline{v}_2$ and $\underline{v}_3$ are orthogonal to each other.

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3. Let

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

Determine all the eigenvalues and eigenvectors of this matrix.

Prove by induction that

$$A^n = \begin{pmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix}, \quad n = 1, 2, \ldots$$

Hence or otherwise determine $\lim_{n\to\infty} A^n$ when $|\alpha| < 1$.

Solution

The eigenvalues of triangular matrices are the diagonal entries and hence α (repeated twice) is the eigenvalue of A.

$$A - \alpha I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $(A - \lambda I)\underline{v} = \underline{0}$ implies that $v_2 = 0$. The only eigenvector of A has direction

$$\underline{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

If we let $n = 1$ in the formula

$$\begin{pmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix}$$

we get the matrix A and hence the formula is true when $n = 1$.

The induction hypothesis to use here is to assume that the result is true for $n = m \geq 1$.

Now consider the case $n = m + 1$. We do not know that the formula is true yet but we do know that we can write

$$A^{m+1} = AA^m$$

and we can use the hypothesis to replace the A^m term. Hence

$$\begin{aligned} A^{m+1} &= \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \alpha^m & m\alpha^{m-1} \\ 0 & \alpha^m \end{pmatrix} = \begin{pmatrix} \alpha^{m+1} & \alpha(m\alpha^{m-1}) + \alpha^m \\ 0 & \alpha^{m+1} \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{m+1} & (m+1)\alpha^m \\ 0 & \alpha^{m+1} \end{pmatrix}. \end{aligned}$$

The result is thus true when $n = m + 1$ and by induction it it true for $n = 1, 2, \ldots$.

For the limit as $n \to \infty$ we have that the diagonal entries $\alpha^n \to 0$ as $|\alpha| < 1$. For the $1, 2$ entry it is a standard result that

$$n\alpha^{n-1} \to 0 \quad \text{as } n \to \infty$$

when $|\alpha| < 1$. This follows by letting $a_n = n\alpha^{n-1}$ and considering the ratios

$$\left| \frac{a_{n+1}}{a_n} \right| = (1 + 1/n)\, |\alpha| \to |\alpha| < 1 \quad \text{as } n \to \infty.$$

For large n ratios are less than 1 and getting closer and closer to $|\alpha|$ and in particular the sequence (a_n) is eventually converging to 0 faster than any sequence (β^n) for all $\beta \in (|\alpha|, 1)$.

All the entries are tending to 0 as $n \to \infty$ and hence $A^n \to$ zero matrix as $n \to \infty$.

4. If A is a diagonalisable matrix and the spectral radius $\rho(A)$ is less than 1 then explain why $\lim_{n\to\infty} A^n =$ zero matrix.

   The result that $A^n \to 0$ as $n \to \infty$ if and only if the spectral radius is less than 1 is actually true for all square matrices (i.e. not just diagonalisable matrices) but the proof in the non-diagonalisable case is longer.

Solution

When A is a $m \times m$ diagonalisable matrix there are m linearly independent eigenvectors $\underline{v}_1, \ldots, \underline{v}_m$ and when we let $V = (\underline{v}_1, \ldots, \underline{v}_m)$ we have

$$AV = VD,$$

where $D = \mathrm{diag}\{\lambda_1, \ldots, \lambda_m\}$ with λ_i being the eigenvalue associated with $\underline{v}_i$. As the columns of V are linearly independent the matrix is invertible and we have

$$A = VDV^{-1}$$

and

$$A^n = VD^nV^{-1}, \quad D^n = \mathrm{diag}\{\lambda_1^n, \ldots, \lambda_m^n\}.$$

When $\rho(A) < 1$ we have $|\lambda_i| < 1$ and $\lambda_i^n \to 0$ as $n \to \infty$. Thus $D^n \to$zero matrix as $n \to \infty$ and $A^n \to$zero matrix as $n \to \infty$.

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5. Let A be an invertible matrix and let $\underline{x}$ be such that $\|\underline{x}\| = 1$. Show that for the matrix norm induced by the vector norm we have

$$\frac{1}{\|A^{-1}\|} \le \|A\underline{x}\| \le \|A\|.$$

   [Hint: For the lower bound consider vectors of the form $A^{-1}\underline{y}$ with $\|\underline{y}\| = 1$.]

Solution

The upper bound is a consequence of the definition of $\|A\|$.

Let $\underline{x}$ have unit norm and let

$$\underline{y} = \frac{A\underline{x}}{\|A\underline{x}\|}$$

which also has unit norm. Thus

$$A^{-1}\underline{y} = \frac{\underline{x}}{\|A\underline{x}\|}.$$

The vector $\underline{x}$ has the same direction as $A^{-1}\underline{y}$ and as it is a unit vector we can write is as

$$\underline{x} = \frac{A^{-1}\underline{y}}{\|A^{-1}\underline{y}\|}.$$

The reason for all this detail here is just to confirm that for all unit vectors $\underline{x}$ there is a corresponding unit vector $\underline{y}$ and for all unit vectors $\underline{y}$ there is a corresponding unit vector $\underline{x}$. Using this representation

$$\|A\underline{x}\| = \frac{1}{\|A^{-1}\underline{y}\|}\|AA^{-1}\underline{y}\| = \frac{1}{\|A^{-1}\underline{y}\|}\|\underline{y}\|.$$

As $\|\underline{y}\| = 1$

$$\|A\underline{x}\| = \frac{1}{\|A^{-1}\underline{y}\|} \ge \frac{1}{\|A^{-1}\|}$$

where the last inequality is a consequence of the definition of the norm of A^{-1}.

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6. *This was a question in the MA2815 paper in May 2019 exam and was worth in total 5 marks of the 100 marks on the 3 hour exam.*

   Let $\underline{x} = (x_i)$ denote a real $n \times 1$ column vector. Define $\|\underline{x}\|_\infty$.

   Let $A = (a_{ij})$ be an $n \times n$ real matrix. The matrix norm induced by the ∞-vector norm is given by

$$\|A\|_\infty = \max\left\{\|A\underline{x}\|_\infty : \ \|\underline{x}\|_\infty = 1\right\} = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

   In the case of the 3×3 matrix A given by

$$A = \begin{pmatrix} -4 & 3 & 2 \\ -1 & -1 & 8 \\ 1 & 1 & 10 \end{pmatrix}$$

   determine $\|A\|_\infty$.

   Give any vector $\underline{x}$ with $\|\underline{x}\|_\infty = 1$ such that $\|A\underline{x}\|_\infty = \|A\|_\infty$ and indicate whether or not the vector $\underline{x}$ given is an eigenvector of the matrix A.

Solution

$$\|\underline{x}\|_\infty = \max\left\{|x_i| : \ 1 \le i \le n\right\}.$$

The row sums of the absolute values are respectively 9, 10 and 12 and hence $\|A\|_\infty = 12$.

The largest row sum of absolute values occurs on row 3 and by considering the signs of the 3 entries we take $\underline{x} = (1, 1, 1)^T$ to give

$$A\underline{x} = \begin{pmatrix} -4 & 3 & 2 \\ -1 & -1 & 8 \\ 1 & 1 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 12 \end{pmatrix}.$$

$\|A\underline{x}\|_\infty = 12$ but $A\underline{x}$ does not have the same direction as $\underline{x}$ and hence $\underline{x}$ is not an eigenvector.

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7. *This was a question in the MA2815 paper in May 2018 exam and was worth in total 2 marks of the 70 marks on the 3 hour exam.*

Let D be the following 3×3 matrix.

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

If $\underline{x}$ is a 3×1 vector then give the components of $D\underline{x}$ and $D^{-1}\underline{x}$ and give the ∞-matrix norms $\|D\|_\infty$ and $\|D^{-1}\|_\infty$.

Solution

$$D^{-1} = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/5 \end{pmatrix}, \quad D\underline{x} = \begin{pmatrix} -3x_1 \\ 2x_2 \\ -5x_3 \end{pmatrix} \quad \text{and} \quad D^{-1}\underline{x} = \begin{pmatrix} -x_1/3 \\ x_2/2 \\ -x_3/5 \end{pmatrix}.$$

$$\|D\|_\infty = 5 \quad \text{and} \quad \|D^{-1}\|_\infty = 1/2.$$

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8. *This was a question in the MA2815 paper in May 2017 exam and was worth in total 4 marks of the 70 marks on the 3 hour exam.*

Let $\underline{x} = (x_i)$ denote a $n \times 1$ real column vector. Define the 1-norm $\|\underline{x}\|_1$.

Let $A = (a_{ij})$ denote a $n \times n$ real matrix. The matrix 1-norm induced by the vector 1-norm is given by

$$\|A\|_1 = \max\{\|A\underline{x}\|_1 : \ \|\underline{x}\|_1 = 1\} = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

Determine $\|A\|_1$ in the case of the 3×3 matrix A given by

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 2 & -5 & 0 \\ 1 & 3 & -4 \end{pmatrix}.$$

For this 3×3 matrix give any vector $\underline{x}$ with $\|\underline{x}\|_1 = 1$ such that $\|A\underline{x}\|_1 = \|A\|_1$.

Solution

The vector 1-norm is

$$\|\underline{x}\|_1 = |x_1| + \cdots + |x_n|.$$

The column sums of absolute values are 7, 9 and 5. Hence

$$\|A\|_1 = 9.$$

If we take the base vector $\underline{x} = \underline{e}_2$, i.e. the 2nd column of the identity matrix I, then

$$A\underline{e}_2 = \text{2nd column of } A = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}.$$

The 1-norm of this vector is 9.

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9. *This was a question in the MA2815 paper in April/May 2015 exam and was worth in total 5 marks of the 70 marks on the 3 hour exam.*

   Let $\underline{x} = (x_i)$ denote a real $n \times 1$ column vector. Define $\|\underline{x}\|_\infty$.

   Let $A = (a_{ij})$ be an $n \times n$ real matrix. The matrix norm induced by the ∞-vector norm is given by

$$\|A\|_\infty = \max\left\{\|A\underline{x}\|_\infty : \ \|\underline{x}\|_\infty = 1\right\} = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

   In the case of the 3×3 matrix A given by

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -5 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

   determine $\|A\|_\infty$.

   For this matrix give a vector $\underline{x}$ with $\|\underline{x}\|_\infty = 1$ such that $\|A\underline{x}\|_\infty = \|A\|_\infty$.

Solution

$$\|\underline{x}\|_\infty = \max\left\{|x_i| : \ 1 \le i \le n\right\}.$$

The row sums of absoulte values are 6, 7 and 6. Thus $\|A\|_\infty = 7$.

The largest row sum of absolute values occurs on the 2nd row.

$$(A\underline{x})_2 = (\text{2nd row of } A) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 - 5x_2 + x_3.$$

This quantity is $\|A\|_\infty = 7$ with $x_1 = 1$, $x_2 = -1$ and $x_3 = 1$ and this vector $\underline{x}$ is such that $\|\underline{x}\|_\infty = 1$.

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10. The ∞ vector norm of $\underline{x} \in \mathbb{R}^n$ is defined by

$$\|\underline{x}\|_\infty = \max\left\{|x_i| : \ 1 \le i \le n\right\}.$$

   Let $A = (a_{ij})$ denote a $n \times n$ matrix and let $\underline{x}$ denote a $n \times 1$ real column vector. Show that if $\|\underline{x}\|_\infty = 1$ then

$$|(A\underline{x})_i| \le \sum_{j=1}^{n} |a_{ij}|.$$

   Further determine any vector $\underline{x}$ such that

$$|(A\underline{x})_i| = \sum_{j=1}^{n} |a_{ij}|.$$

Hence prove the result

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}| = \text{maximum } \textbf{row} \text{ sum of absolute values.}$$

Solution

If $\|\underline{x}\|_\infty = 1$ then $|x_i| \le 1$ and at least one of the components has magnitude 1. Using this propery and the triangle inequality gives

$$|(A\underline{x})_i| = \left|\sum_{j=1}^{n} a_{ij}x_j\right| \le \sum_{j=1}^{n} |a_{ij}x_j| \le \sum_{j=1}^{n} |a_{ij}|.$$

For

$$(A\underline{x})_i = \sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} |a_{ij}|$$

we need

$$a_{ij}x_j = |a_{ij}|, \quad \text{for } j = 1, \ldots, n.$$

Thus for $\underline{x} = (x_j)$ we can take

$$x_j = \begin{cases} 1, & \text{if } a_{ij} = 0, \\ \dfrac{|a_{ij}|}{a_{ij}}, & \text{otherwise.} \end{cases}$$

From the earlier part we have

$$\|A\|_\infty = \max\left\{|(A\underline{x})_i| : \ 1 \le i \le n\right\} \le \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

For each row we have shown that we can construct a vector which gives the row sum of the absolute values. Thus in particular we can do this for any row which gives the maximum of the row sums and hence there exists a vector $\underline{x}$ with $\|\underline{x}\|_\infty = 1$ and

$$\|A\underline{x}\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

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11. Let λ be an eigenvalue of an $n \times n$ matrix A and let $\|A\|$ denote any matrix norm induced by a vector norm. Show that

$$|\lambda| \le \|A\|.$$

By using the results $\|A\|_1 = \|A^T\|_\infty$ and $\|A\|_2^2 = \rho(A^T A)$ show that

$$\|A\|_2^2 \le \|A\|_1 \|A\|_\infty.$$

Solution

A vector $\underline{v} \neq \underline{0}$ is an eigenvector of A with corresponding eigenvalue λ if $A\underline{v} = \lambda\underline{v}$. If we take $\underline{v}$ to have unit norm, i.e. $\|\underline{v}\| = 1$, then

$$|\lambda| = \|\lambda\underline{v}\| = \|A\underline{v}\|.$$

Now for the matrix norm induced by the vector norm we have

$$\|A\| = \max\left\{\|A\underline{x}\| : \ \|x\| = 1\right\}.$$

Thus as $\|\underline{v}\| = 1$ we have

$$|\lambda| = \|\lambda\underline{v}\| = \|A\underline{v}\| \leq \|A\|.$$

The above result holds for all the eigenvalues and in the case of using the ∞-norm we have

$$\|A\|_2^2 = \rho(A^T A) \leq \|A^T A\|_\infty.$$

The multiplicative property of the norm gives

$$\|A^T A\|_\infty \leq \|A^T\|_\infty \|A\|_\infty.$$

The results follows as $\|A\|_1 = \|A^T\|_\infty$.

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12. When the finite difference method is considered later in the module the explanation of the method will involve Taylor expansions about various points. Based on what you have done already about Maclaurin expansions determine the Maclaurin expansions of the following giving all non-zero terms up to the one involving x^6 in your answer.

    (a) $2(\cosh(x) - 1)$.

    (b) $\sinh(x)$.

    (c) $32(\cosh(x) - 1) - 2(\cosh(2x) - 1)$.

    (d) $8\sinh(x) - \sinh(2x)$.

    Please note that

$$\cosh(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}.$$

Solution

First note the standard series for e^x and e^{-x} which are

$$\begin{aligned}
\mathrm{e}^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots \\
\mathrm{e}^{-x} &= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + \frac{x^6}{720} + \cdots
\end{aligned}$$

Thus

$$2\cosh(x) - 2 = \mathrm{e}^x + \mathrm{e}^{-x} - 2 = x^2 + \frac{x^4}{12} + \frac{x^6}{360} + \cdots$$

By replacing x by $2x$ we get

$$2\cosh(2x) - 2 = (2x)^2 + \frac{(2x)^4}{12} + \frac{(2x)^6}{360} + \cdots$$

By combining the last two expansions we have

$$32(\cosh(x) - 1) - 2(\cosh(2x) - 1) = 12x^2 - \frac{48}{360}x^6 + \cdots = 12x^2 - \frac{2}{15}x^6 + \cdots$$

In the case of $\sinh(x)$ we have

$$\sinh(x) = \frac{he^x - \mathrm{e}^{-x}}{2} = x + \frac{x^3}{6} + \frac{x^5}{120} + \cdots$$

By replacing x by $2x$ we get

$$\sinh(2x) = (2x) + \frac{(2x)^3}{6} + \frac{(2x)^5}{120} + \cdots$$

By combining the last two expansions we have

$$8\sinh(x) - \sinh(2x) = 6x - \frac{24}{120}x^5 + \cdots = 6x - \frac{x^5}{15} + \cdots$$

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13. When Fourier series is covered later in the module one of the things that will be done is to determine Fourier coefficients and in many examples this will involve integration by parts. As a practice question now show that when n is a non-zero integer we have the following.

$$\int_0^\pi x\cos(nx)\,\mathrm{d}x = \begin{cases} \frac{-2}{n^2}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

$$\int_0^\pi x\sin(nx)\,\mathrm{d}x = \frac{(-1)^{n+1}\pi}{n}.$$

Solution

In the integration by parts the term in the product that we choose to differentiate is the one which becomes simpler, i.e. we choose x which differentiates tot 1. Thus for the integrand involving $\cos(nx)$ we have

$$\begin{aligned}\int_0^\pi x\cos(nx)\,\mathrm{d}x &= \left[x\frac{\sin(nx)}{n}\right]_0^\pi - \int_0^\pi \frac{\sin(nx)}{n}\,\mathrm{d}x \\ &= \left[\frac{\cos(nx)}{n^2}\right]_0^\pi = \frac{1}{n^2}((-1)^n - 1).\end{aligned}$$

When n is even $(-1)^n = 1$ and the integral is 0 and when n is odd $(-1)^n = -1$ and the integral is $-2/n^2$.

For the integrand involving $\sin(nx)$ we have

$$\begin{aligned}\int_0^\pi x\sin(nx)\,\mathrm{d}x &= \left[x\frac{-\cos(nx)}{n}\right]_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}\,\mathrm{d}x \\ &= -\pi\frac{(-1)^n}{n} + \frac{1}{n^2}\left[\sin(nx)\right]_0^\pi = \frac{(-1)^{n+1}\pi}{n}.\end{aligned}$$