Exercises related to chapter 1: eigenvalues, eigenvectors, matrix norms, plus some revision exercises based on previous modules

1. Determine the eigenvalues and eigenvectors of the following matrices.

$$\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

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2. Let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and let } \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

By first computing $A\underline{v}$ determine all the eigenvalues and eigenvectors of A.

By using your results about the eigenvalues and eigenvectors of matrix A determine the eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}.$$

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3. Let

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

Determine all the eigenvalues and eigenvectors of this matrix.

Prove by induction that

$$A^n = \begin{pmatrix} \alpha^n & n\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix}, \quad n = 1, 2, \ldots$$

Hence or otherwise determine $\lim_{n\to\infty} A^n$ when $|\alpha| < 1$.

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4. If A is a diagonalisable matrix and the spectral radius $\rho(A)$ is less than 1 then explain why $\lim_{n\to\infty} A^n =$ zero matrix.

The result that $A^n \to 0$ as $n \to \infty$ if and only if the spectral radius is less than 1 is actually true for all square matrices (i.e. not just diagonalisable matrices) but the proof in the non-diagonalisable case is longer.

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5. Let A be an invertible matrix and let $\underline{x}$ be such that $\|\underline{x}\| = 1$. Show that for the matrix norm induced by the vector norm we have

$$\frac{1}{\|A^{-1}\|} \le \|A\underline{x}\| \le \|A\|.$$

[Hint: For the lower bound consider vectors of the form $A^{-1}\underline{y}$ with $\|\underline{y}\| = 1$.]

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6. *This was a question in the MA2815 paper in May 2019 exam and was worth in total 5 marks of the 100 marks on the 3 hour exam.*

   Let $\underline{x} = (x_i)$ denote a real $n \times 1$ column vector. Define $\|\underline{x}\|_\infty$.

   Let $A = (a_{ij})$ be an $n \times n$ real matrix. The matrix norm induced by the ∞-vector norm is given by

$$\|A\|_\infty = \max\{\|A\underline{x}\|_\infty : \ \|\underline{x}\|_\infty = 1\} = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

   In the case of the 3×3 matrix A given by

$$A = \begin{pmatrix} -4 & 3 & 2 \\ -1 & -1 & 8 \\ 1 & 1 & 10 \end{pmatrix}$$

   determine $\|A\|_\infty$.

   Give any vector $\underline{x}$ with $\|\underline{x}\|_\infty = 1$ such that $\|A\underline{x}\|_\infty = \|A\|_\infty$ and indicate whether or not the vector $\underline{x}$ given is an eigenvector of the matrix A.

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7. *This was a question in the MA2815 paper in May 2018 exam and was worth in total 2 marks of the 70 marks on the 3 hour exam.*

   Let D be the following 3×3 matrix.

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -5 \end{pmatrix}.$$

   If $\underline{x}$ is a 3×1 vector then give the components of $D\underline{x}$ and $D^{-1}\underline{x}$ and give the ∞-matrix norms $\|D\|_\infty$ and $\|D^{-1}\|_\infty$.

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8. *This was a question in the MA2815 paper in May 2017 exam and was worth in total 4 marks of the 70 marks on the 3 hour exam.*

   Let $\underline{x} = (x_i)$ denote a $n \times 1$ real column vector. Define the 1-norm $\|\underline{x}\|_1$.

   Let $A = (a_{ij})$ denote a $n \times n$ real matrix. The matrix 1-norm induced by the vector 1-norm is given by

$$\|A\|_1 = \max\{\|A\underline{x}\|_1 : \ \|\underline{x}\|_1 = 1\} = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

   Determine $\|A\|_1$ in the case of the 3×3 matrix A given by

$$A = \begin{pmatrix} -4 & 1 & 1 \\ 2 & -5 & 0 \\ 1 & 3 & -4 \end{pmatrix}.$$

   For this 3×3 matrix give any vector $\underline{x}$ with $\|\underline{x}\|_1 = 1$ such that $\|A\underline{x}\|_1 = \|A\|_1$.

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9. *This was a question in the MA2815 paper in April/May 2015 exam and was worth in total 5 marks of the 70 marks on the 3 hour exam.*

   Let $\underline{x} = (x_i)$ denote a real $n \times 1$ column vector. Define $\|\underline{x}\|_\infty$.

   Let $A = (a_{ij})$ be an $n \times n$ real matrix. The matrix norm induced by the ∞-vector norm is given by

$$\|A\|_\infty = \max\left\{\|A\underline{x}\|_\infty :\ \|\underline{x}\|_\infty = 1\right\} = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

   In the case of the 3×3 matrix A given by

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -5 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

   determine $\|A\|_\infty$.

   For this matrix give a vector $\underline{x}$ with $\|\underline{x}\|_\infty = 1$ such that $\|A\underline{x}\|_\infty = \|A\|_\infty$.

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10. The ∞ vector norm of $\underline{x} \in \mathbb{R}^n$ is defined by

$$\|\underline{x}\|_\infty = \max\left\{|x_i| :\ 1 \le i \le n\right\}.$$

   Let $A = (a_{ij})$ denote a $n \times n$ matrix and let $\underline{x}$ denote a $n \times 1$ real column vector. Show that if $\|\underline{x}\|_\infty = 1$ then

$$|(A\underline{x})_i| \le \sum_{j=1}^{n} |a_{ij}|.$$

   Further determine any vector $\underline{x}$ such that

$$|(A\underline{x})_i| = \sum_{j=1}^{n} |a_{ij}|.$$

   Hence prove the result

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = \text{maximum } \textbf{row} \text{ sum of absolute values.}$$

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11. Let λ be an eigenvalue of an $n \times n$ matrix A and let $\|A\|$ denote any matrix norm induced by a vector norm. Show that

$$|\lambda| \le \|A\|.$$

   By using the results $\|A\|_1 = \|A^T\|_\infty$ and $\|A\|_2^2 = \rho(A^T A)$ show that

$$\|A\|_2^2 \le \|A\|_1 \|A\|_\infty.$$

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12. When the finite difference method is considered later in the module the explanation of the method will involve Taylor expansions about various points. Based on what you have done already about Maclaurin expansions determine the Maclaurin expansions of the following giving all non-zero terms up to the one involving x^6 in your answer.

    (a) $2(\cosh(x) - 1)$.

    (b) $\sinh(x)$.

    (c) $32(\cosh(x) - 1) - 2(\cosh(2x) - 1)$.

    (d) $8\sinh(x) - \sinh(2x)$.

    Please note that

$$\cosh(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}.$$

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13. When Fourier series is covered later in the module one of the things that will be done is to determine Fourier coefficients and in many examples this will involve integration by parts. As a practice question now show that when n is a non-zero integer we have the following.

$$\begin{aligned} \int_0^\pi x\cos(nx)\,\mathrm{d}x &= \begin{cases} \frac{-2}{n^2}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \\ \int_0^\pi x\sin(nx)\,\mathrm{d}x &= \frac{(-1)^{n+1}\pi}{n}. \end{aligned}$$
