Natural p-BEM for the electric field integral equation on screens *

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Abstract

In this paper we analyse the p-version of the boundary element method for the electric field integral equation on a plane open surface with polygonal boundary. We prove convergence of the p-version with Raviart-Thomas parallelogram elements and derive an a priori error estimate which takes into account the strong singular behaviour of the solution at edges and corners of the surface. Key ingredient of our analysis is the orthogonality of discrete Helmholtz decompositions in a Sobolev space of order $-1/2$.

Key words: p-version, electric field integral equation, time-harmonic electro-magnetic scattering, boundary element method, singularities
AMS Subject Classification: 65N12, 65N15, 78M15, 65N38, 41A10

1 Introduction and formulation of the problem

In this paper we analyse the p-version of the boundary element method (BEM) for the electric field integral equation (EFIE) on an open surface Γ. The EFIE models the scattering of time-harmonic electro-magnetic waves at a perfect conductor, and its solution is the induced electric surface current on Γ, see, e.g., [30]. The basis of our BEM is a variational formulation of the EFIE, called Rumsey's formulation. For smooth surfaces, its boundary element discretisation has been studied by Bendali [4, 5]. With the study of traces of spaces that govern Maxwell's equations in Lipschitz domains [14] there has been some recent progress in the numerical analysis of the EFIE on Lipschitz surfaces. For polyhedral surfaces, Buffa *et al.* and Hiptmair and Schwab [16, 26] studied BEM discretisations of the EFIE with Raviart-Thomas elements of fixed order on refined meshes, i.e., in the framework of the h-version. In particular, the solvability and quasi-optimal convergence of these discretisations have been proved. Moreover, considering lowest order Raviart-Thomas elements and assuming standard Sobolev regularity, Hiptmair and Schwab [26] derived an a priori error estimate in terms of the mesh parameter h. The issues of solvability and convergence of the h-BEM for the EFIE on open Lipschitz surfaces were addressed by Buffa and Christiansen [10]. We note that in [26, 16, 10] the authors focused on

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‡Supported by INTAS Young Scientist Fellowship grant no. 06-1000014-5945.

conforming discretisations of Rumsey's formulation, called natural boundary element method for the EFIE (the approach, we follow in this paper). There are, however, other formulations of mixed type to utilise standard (continuous) basis functions, see [13].

In the engineering literature, the BEM (Galerkin and collocation variants) is called the method of moments and is widely used for electro-magnetic scattering problems. High order versions of the method of moments have been also studied recently (see, e.g., [22, 20]). They are shown to be efficient when dealing with non-regular parts of the solution in combination with methods from physical optics for high-frequency scattering at smooth parts of obstacles [21]. In general, there are two main advantages of high order methods, namely their less vulnerability to numerical dispersion errors and better approximation properties even in the presence of singularities. The influence of the order of basis functions on numerical dispersion has been analysed by Ainsworth [1], and the properties of polynomial approximations of singular functions inherent to first kind integral equations have been studied in [7, 8].

In the p-version of the BEM the mesh is fixed and approximations are improved by increasing polynomial degrees. To the best of our knowledge there are no proofs of convergence for the p-version applied to the EFIE. The analysis of high order approximations for the EFIE on open or closed polyhedral surfaces poses two particular challenges.

First, in order to prove convergence of the method, one usually relies on properties of the continuous and discrete Helmholtz decompositions, and on the proximity in some sense of the discrete decompositions to the continuous one, see [16, 10]. Known techniques are inherently designed towards low order approximations, as it turns out when trying to generalise them to high order methods. For instance, the equivalence of norms in finite-dimensional spaces is usually used (see, e.g., the proof of Lemma 6.2 in [26]). This argument is not available for the p-version. Also, related with appearing singularities (which is the second challenge described below), the proofs of proximity of low-order discrete Helmholtz decompositions to the continuous decomposition utilise an error estimate for the standard Raviart-Thomas interpolation operator in $\mathbf{H}(\mathrm{div}, \Gamma)$ (see, e.g., the proof of Theorem 4.2 in [16]). For the p-version, stability of this operator is guaranteed when the interpolated function is in $\mathbf{H}^s(\mathrm{div}, \Gamma)$ with $s > 1/2$ (see Section 3 for a definition of $\mathbf{H}^s(\mathrm{div}, \Gamma)$), whereas on polyhedral surfaces less regularity has to be accounted for.

Second, the solution to the EFIE on polyhedral surfaces suffers from singular behaviour at edges and corners. This can be deduced from the behaviour of solutions to the Maxwell problem on polyhedral domains as studied by Costabel and Dauge in [18]. Open surfaces represent the least regular case, and there have been no high order approximation results for them whatsoever.

In this paper we deal with both issues. In particular, to prove convergence of the p-version of the BEM for the EFIE we follow the framework presented in [16, Section 4.1]. However, rather than considering $\mathbf{L}^2$-orthogonal discrete Helmholtz decompositions, we consistently employ the $\tilde{\mathbf{H}}^{-1/2}$-inner product and orthogonality (precise definitions will be given below). This turns out to be crucial for the p-version. As for the approximation analysis of singularities, we partly rely on our previous results for the Laplacian, see [7, 8], by using continuity properties of the surface curl operator. The exception is a particular kind of vertex singularity, which does not have a vanishing tangential component on the boundary of Γ and which needs to be treated in a vector fashion (i.e., component-wise approximations are not sufficient for it).

We restrict ourselves to plane open surfaces which can be discretised by parallelogram meshes. A generalisation to smooth curved surfaces is possible by mapping techniques. The case of triangular elements, however, is not an easy generalisation as, for instance, standard p-version approximation results for Raviart-Thomas triangular elements are unknown. The approach presented in this paper is, in principle, applicable to polyhedral surfaces and we expect that all the results can be extended to that case. However, this extension is not straightforward as some technical details make use of the smoothness of Γ, except for its boundary. More general hp-methods, which increase polynomial degrees in combination with mesh refinements, are desirable but are not covered in this paper. The corresponding analysis is a non-trivial extension of our results.

Let us introduce Rumsey's formulation of the electric field integral equation. For a given wave number $k > 0$ and a (tangential or scalar) vector field v we define the single layer operator Ψ_k by

$$\Psi_k v(x) = \frac{1}{4\pi} \int_\Gamma v(y) \frac{e^{ik|x-y|}}{|x-y|}\, dS_y, \qquad x \in \Gamma.$$

Also, denoting by div and ∇ the two-dimensional divergence and gradient operators on Γ, respectively (in the general case being the surface divergence and gradient, respectively, both acting on tangential vector fields), we need the space

$$\begin{aligned} \mathbf{X} = \tilde{\mathbf{H}}_0^{-1/2}(\mathrm{div}, \Gamma) \quad := \quad & \{\mathbf{u} \in \tilde{\mathbf{H}}^{-1/2}(\Gamma);\ \mathrm{div}\,\mathbf{u} \in \tilde{H}^{-1/2}(\Gamma) \text{ and} \\ & \langle \mathbf{u}, \nabla v\rangle + \langle \mathrm{div}\,\mathbf{u}, v\rangle = 0 \quad \text{for all} \quad v \in C^\infty(\bar{\Gamma})\}. \end{aligned}$$

The dual space of $\mathbf{X}$ (with $\mathbf{L}^2(\Gamma)$ as pivot space) is denoted by $\mathbf{X}'$ and $\langle \cdot, \cdot \rangle$ denotes the extension of the $\mathbf{L}^2(\Gamma)$-inner product by duality between $\mathbf{X}$ and $\mathbf{X}'$. Moreover, $\tilde{H}^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$. For a definition of $H^{1/2}(\Gamma)$ see Section 3.1. Throughout, we use boldface symbols for vector fields. The spaces (or sets) of vector fields are also denoted in boldface (e.g., $\mathbf{H}^s(\Gamma) = (H^s(\Gamma))^2$), with norms and inner products being defined component-wise.

Now, for a given tangential vector field $\mathbf{f} \in \mathbf{X}'$ ($\mathbf{f}$ represents the excitation by an incident wave), Rumsey's formulation reads as: *find a complex tangential field* $\mathbf{u} \in \mathbf{X}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) := \langle \Psi_k \mathrm{div}\,\mathbf{u}, \mathrm{div}\,\mathbf{v}\rangle - k^2 \langle \Psi_k \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in \mathbf{X}. \tag{1.1}$$

An outline of this paper is as follows. In the next section we recall regularity results for the EFIE, define the p-version of the BEM for its approximate solution, state the unique solvability and quasi-optimal convergence of this approximation method (Theorem 2.1), and prove an a priori error estimate in terms of the polynomial degree p (Theorem 2.2). In Section 3 we define the needed Sobolev spaces and collect some technical lemmas. Section 4 is devoted to Helmholtz decompositions of $\mathbf{X}$: we define the continuous decomposition, prove its $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$-orthogonality, and introduce the framework of discrete decompositions. Interpolation operators and approximation results for Raviart-Thomas elements in different spaces are discussed in Section 5. These results are then applied to prove the existence and uniqueness of discrete solutions to an auxiliary problem. In Section 6 we prove the solvability and convergence of the p-BEM (Theorem 2.1). Section 7 presents p-approximation error estimates for smooth vector functions (Theorem 7.1) and for general singular vector fields (Theorem 7.2). In Appendix A we first recall the structure of Maxwell singularities (by referring to Costabel and Dauge [18]). Then, using a trace argument, we conclude on the behaviour of singularities inherent to the solution of the EFIE. Finally, in Appendix B we study the approximation of a particular type of vertex singularities. The obtained result is needed to prove our general approximation theorem (Theorem 7.2).

Throughout the paper, C denotes a generic constant which is independent of polynomial degrees p and involved functions, unless stated otherwise.

2 The p-version of the BEM and main results

First, let us determine the typical structure of the solution $\mathbf{u}$ to our model problem (1.1), provided that the right-hand side function $\mathbf{f}$ is sufficiently smooth.

Let V and E denote the sets of vertices and edges of Γ, respectively. For $v \in V$, let $E(v)$ denote the set of edges with v as an end point. Then it follows from the results of [18] that the solution $\mathbf{u}$ of (1.1) has the form

$$\mathbf{u} = \mathbf{u}_{\rm reg} + \sum_{e \in E} \mathbf{u}^e + \sum_{v \in V} \mathbf{u}^v + \sum_{v \in V} \sum_{e \in E(v)} \mathbf{u}^{ev}, \tag{2.1}$$

where (see Section 3.1 for definitions of the Sobolev spaces involved)

$$\mathbf{u}_{\rm reg} \in \mathbf{H}^k_0(\mathrm{div}, \Gamma) \quad \text{with} \quad k > 0$$

and $\mathbf{u}^e$, $\mathbf{u}^v$, and $\mathbf{u}^{ev}$ are the edge, vertex, and edge-vertex singularities, respectively. We deduce the precise form of these singularities from Appendix A, where explicit formulas for singularities inherent to the solution of the EFIE are obtained in the more general case of a piecewise plane (open or closed) surface Γ.

We will use local polar and Cartesian coordinate systems (r_v, θ_v) and (x_{e1}, x_{e2}), both with origin v, such that $e = \{(x_{e1}, x_{e2});\ x_{e2} = 0,\ x_{e1} > 0\}$ and for sufficiently small neighbourhood B_τ of v there holds $\Gamma \cap B_\tau \subset \{(r_v, \theta_v);\ 0 < \theta_v < \omega_v\}$. Here, ω_v denotes the interior angle (on Γ) between the edges meeting at v. For simplicity of notation we write out here only the leading singularities in $\mathbf{u}^e$, $\mathbf{u}^v$, and $\mathbf{u}^{ev}$, thus omitting the corresponding terms of higher regularity. Complete expansions can be written analogously to (A.18), (A.24), (A.26), and (A.29) in Appendix A.

From (A.18) we have

$$\mathbf{u}^e = \mathbf{curl}\Big(x_{e2}^{\gamma_1^e}\,|\log x_{e2}|^{s_1^e}\, b_1^e(x_{e1})\,\chi_1^e(x_{e1})\,\chi_2^e(x_{e2})\Big) + x_{e2}^{\gamma_2^e}\,|\log x_{e2}|^{s_2^e}\,\mathbf{b}_2^e(x_{e1})\,\chi_1^e(x_{e1})\,\chi_2^e(x_{e2}), \tag{2.2}$$

where $\mathbf{curl} = (\partial/\partial x_{e2},\, -\partial/\partial x_{e1})$, $\gamma_1^e,\ \gamma_2^e \geq \frac{1}{2}$, and s_1^e, $s_2^e \geq 0$ are integers. Here, χ_1^e, χ_2^e are C^∞ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of e and $\chi_1^e = 0$ in a neighbourhood of these vertices. Moreover, $\chi_2^e = 1$ for $0 \le x_{e2} \le \delta_e$ and $\chi_2^e = 0$ for $x_{e2} \ge 2\delta_e$ with some $\delta_e \in (0, \frac{1}{2})$. The functions $b_1^e \chi_1^e \in H^{m_1}(e)$ and $\mathbf{b}_2^e \chi_1^e \in \mathbf{H}^{m_2}(e)$ for m_1 and m_2 as large as required.

Similarly, we deduce from (A.24) that

$$\mathbf{u}^v = \mathbf{curl}\Big(r_v^{\lambda_1^v} \, |\log r_v|^{q_1^v} \, \chi^v(r_v) \, \chi_1^v(\theta_v)\Big) + r_v^{\lambda_2^v} \, |\log r_v|^{q_2^v} \, \chi^v(r_v) \, \boldsymbol{\chi}_2^v(\theta_v), \tag{2.3}$$

where λ_1^v, $\lambda_2^v > -\frac{1}{2}$, and q_1^v, $q_2^v \ge 0$ are integers, χ^v is a C^∞ cut-off function with $\chi^v = 1$ for $0 \le r_v \le \tau_v$ and $\chi^v = 0$ for $r_v \ge 2\tau_v$ with some $\tau_v \in (0, \frac{1}{2})$. The functions χ_1^v, $\boldsymbol{\chi}_2^v$ are such that $\chi_1^v \in H^{t_1}(0, \omega_v) \cap H_0^1(0, \omega_v)$, $\boldsymbol{\chi}_2^v \in \mathbf{H}^{t_2}(0, \omega_v)$ for t_1, t_2 as large as required, and $\boldsymbol{\chi}_2^v \cdot \mathbf{n}|_{\partial\Gamma} = 0$.

For the combined edge-vertex singularity $\mathbf{u}^{ev}$ we use (A.26) and (A.29). One has

$$\mathbf{u}^{ev} = \mathbf{u}_1^{ev} + \mathbf{u}_2^{ev},$$

where

$$\begin{aligned} \mathbf{u}_1^{ev} \;&=\; \mathbf{curl}\Big(x_{e1}^{\lambda_1^v - \gamma_1^e} x_{e2}^{\gamma_1^e} |\log x_{e1}|^{\beta_1} |\log x_{e2}|^{\beta_2} \, \chi^v(r_v) \chi^{ev}(\theta_v)\Big) \\ &\qquad + x_{e1}^{\lambda_2^v - \gamma_2^e} x_{e2}^{\gamma_2^e} |\log x_{e1}|^{\beta_3} |\log x_{e2}|^{\beta_4} \, \chi^v(r_v) \chi^{ev}(\theta_v) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned} \tag{2.4}$$

and

$$\mathbf{u}_2^{ev} = \mathbf{curl}\Big(x_{e2}^{\gamma_1^e} \, |\log x_{e2}|^{s_1^e} \, b_3^e(x_{e1}, x_{e2}) \, \chi_2^e(x_{e2})\Big) + x_{e2}^{\gamma_2^e} \, |\log x_{e2}|^{s_2^e} \, \mathbf{b}_4^e(x_{e1}, x_{e2}) \, \chi_2^e(x_{e2}). \tag{2.5}$$

Here, λ_i^v, γ_i^e, s_i^e $(i = 1, 2)$, χ^v, and χ_2^e are as above, $\beta_k \ge 0$ $(k = 1 \ldots, 4)$ are integers, $\beta_1 + \beta_2 = s_1^e + q_1^v$, $\beta_3 + \beta_4 = s_2^e + q_2^v$ with q_1^v, q_2^v being as in (2.3), χ^{ev} is a C^∞ cut-off function with $\chi^{ev} = 1$ for $0 \le \theta_v \le \beta_v$ and $\chi^{ev} = 0$ for $\frac{3}{2}\beta_v \le \theta_v \le \omega_v$ for some $\beta_v \in (0, \min\{\omega_v/2, \pi/8\}]$. The functions b_3^e and $\mathbf{b}_4^e$, when extended by zero onto $\mathbf{R}^{2+} := \{(x_{e1}, x_{e2});\; x_{e2} > 0\}$, lie in $H^{m_1}(\mathbf{R}^{2+})$ and $\mathbf{H}^{m_2}(\mathbf{R}^{2+})$, respectively, with m_1, m_2 as large as required. Note that the supports of $\mathbf{u}_1^{ev}$ and $\mathbf{u}_2^{ev}$ are subsets of the sector $\bar{S}_{ev} = \{(r_v, \theta_v);\; 0 \le r_v \le 2\tau_v,\, 0 \le \theta_v \le \frac{3}{2}\beta_v\}$, cf. Remark A.1 below.

Remark 2.1 *The exponents γ_i^e $(i = 1, 2)$ for edge and vertex-edge singularities in* (2.2), (2.4), (2.5) *satisfy $\gamma_i^e \ge \frac{1}{2}$. However, for our approximation analysis below it suffices to require that $\gamma_i^e > 0$ $(i = 1, 2)$. Note that $\gamma_i^e > 0$ and $\lambda_i^v > -\frac{1}{2}$ $(i = 1, 2)$ are the minimum requirements to guarantee $\mathbf{u} \in \mathbf{X}$.*

For the approximate solution of (1.1) we apply the p-version of the BEM based on Galerkin discretisations with Raviart-Thomas (RT) spaces. In what follows, $p \ge 1$ will always specify a polynomial degree.

We discretise Γ by a fixed mesh $\{\Gamma_j;\; j = 1, \ldots, J\}$ consisting of parallelograms. Let $Q = (-1, 1)^2$ be the reference square. Then for any element Γ_j of the mesh one has $\Gamma_j = T_j(Q)$, where T_j denotes an invertible affine mapping

$$x = T_j(\xi) = B_j \xi + \mathbf{b}_j.$$

Here, $B_j \in \mathbf{R}^{2\times 2}$, $\mathbf{b}_j \in \mathbf{R}^2$, $x = (x_1, x_2) \in \Gamma_j$ and $\xi = (\xi_1, \xi_2) \in Q$.

The affine mapping T_j is used to associate the scalar function u defined on the element Γ_j with the function $\hat{u}$ defined on the reference square Q:

$$u = \hat{u} \circ T_j^{-1} \quad \text{and} \quad \hat{u} = u \circ T_j.$$

Any vector-valued function $\hat{\mathbf{v}}$ defined on Q is transformed to the function $\mathbf{v}$ on Γ_j by using the standard Piola transformation:

$$\mathbf{v} = \mathcal{M}_j(\hat{\mathbf{v}}) = \tfrac{1}{J_j} B_j \hat{\mathbf{v}} \circ T_j^{-1}, \quad \hat{\mathbf{v}} = \mathcal{M}_j^{-1}(\mathbf{v}) = J_j B_j^{-1} \mathbf{v} \circ T_j, \tag{2.6}$$

where $J_j = \det(B_j)$.

Further, $\mathcal{P}_p(I)$ denotes the set of polynomials of degree $\le p$ on an interval $I \subset \mathbf{R}$. By $\mathcal{P}_{p_1,p_2}(Q)$ we denote the set of polynomials on Q of degree $\le p_1$ in ξ_1 and of degree $\le p_2$ in ξ_2. For $p_1 = p_2 = p$ we will use the notation $\mathcal{P}_p(Q) = \mathcal{P}_{p,p}(Q)$. If K is an arbitrary parallelogram in $\mathbf{R}^2$, then we will denote by $\mathcal{P}_p(K)$ the set of polynomials v on K such that $v \circ M \in \mathcal{P}_p(Q)$, where $M : Q \to K$ is an invertible affine mapping.

Now we can define the RT-spaces on the reference element (see, e.g., [9, 31]):

$$\mathbf{V}_p^{\mathrm{RT}}(Q) = \mathcal{P}_{p,p-1}(Q) \times \mathcal{P}_{p-1,p}(Q). \tag{2.7}$$

Then using transformations (2.6), we set

$$\mathbf{X}_p(\Gamma) := \{\mathbf{v} \in \mathbf{X}^0;\ \mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}) \in \mathbf{V}_p^{\mathrm{RT}}(Q),\ j = 1, \ldots, J\}, \tag{2.8}$$

where the space $\mathbf{X}^0 = \mathbf{H}_0(\mathrm{div}, \Gamma) \subset \mathbf{X}$ is defined in §3.1. Then the p-version of the Galerkin BEM for the EFIE is: *Find* $\mathbf{u}_p \in \mathbf{X}_p(\Gamma)$ *such that*

$$a(\mathbf{u}_p, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}_p(\Gamma). \tag{2.9}$$

Let us formulate the result which states the unique solvability of (2.9) and quasi-optimal convergence of the p-version of the BEM for the EFIE.

Theorem 2.1 *There exists* $p_0 \ge 1$ *such that for any* $\mathbf{f} \in \mathbf{X}'$ *and for arbitrary* $p \ge p_0$ *the discrete problem* (2.9) *is uniquely solvable and the* p*-version of the Galerkin BEM generated by RT-elements converges quasi-optimally, i.e.,*

$$\|\mathbf{u} - \mathbf{u}_p\| \le C \inf\{\|\mathbf{u} - \mathbf{v}\|;\ \mathbf{v} \in \mathbf{X}_p(\Gamma)\}. \tag{2.10}$$

Here, $\mathbf{u} \in \mathbf{X}$ *is the solution of* (1.1), $\mathbf{u}_p \in \mathbf{X}_p(\Gamma)$ *is the solution of* (2.9), $\|\cdot\|$ *denotes the norm in* $\mathbf{X}$, *and* $C > 0$ *is a constant independent of* p.

The proof of Theorem 2.1 is given in Section 6 below.

The next statement specifies the convergence rate for the p-version of the BEM applied to the EFIE on the plane screen Γ.

Theorem 2.2 *Let* $\mathbf{u} \in \mathbf{X}$ *be the solution of* (1.1) *with sufficiently smooth given function* $\mathbf{f} \in \mathbf{X}'$ *such that representation* (2.1)–(2.5) *holds. Let* $v_0 \in V$, $e_0 \in E(v_0)$ *be such that*

$$\min\{\lambda_1^{v_0} + 1/2, \lambda_2^{v_0} + 1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\} = \min_{v \in V, e \in E(v)} \min\{\lambda_1^v + 1/2, \lambda_2^v + 1/2, \gamma_1^e, \gamma_2^e\}$$

with λ_i^v, γ_i^e $(i = 1, 2)$ *being as in* (2.2)–(2.5). *Then denote*

$$\beta = \begin{cases} \max\{q_1^{v_0} + s_1^{e_0} + \frac{1}{2},\, q_2^{v_0} + s_2^{e_0} + \frac{1}{2}\} & \text{if } \lambda_i^{v_0} = \gamma_i^{e_0} - \frac{1}{2} \text{ for } i = 1, 2, \\ \max\{q_1^{v_0} + s_1^{e_0} + \frac{1}{2},\, q_2^{v_0} + s_2^{e_0}\} & \text{if } \lambda_1^{v_0} = \gamma_1^{e_0} - \frac{1}{2},\ \lambda_2^{v_0} \neq \gamma_2^{e_0} - \frac{1}{2}, \\ \max\{q_1^{v_0} + s_1^{e_0},\, q_2^{v_0} + s_2^{e_0} + \frac{1}{2}\} & \text{if } \lambda_1^{v_0} \neq \gamma_1^{e_0} - \frac{1}{2},\ \lambda_2^{v_0} = \gamma_2^{e_0} - \frac{1}{2}, \\ \max\{q_1^{v_0} + s_1^{e_0},\, q_2^{v_0} + s_2^{e_0}\} & \text{otherwise} \end{cases} \tag{2.11}$$

with the numbers $s_i^{e_0}$ *and* $q_i^{v_0}$ $(i = 1, 2)$ *given in* (2.2) *and* (2.3), *respectively. Then for every* $p \geq p_0$ (p_0 *is given by Theorem* 2.1), *the BE approximation* $\mathbf{u}_p \in \mathbf{X}_p(\Gamma)$ *defined by* (2.9) *satisfies*

$$\|\mathbf{u} - \mathbf{u}_p\| \leq C p^{-2\min\{\lambda_1^{v_0}+1/2, \lambda_2^{v_0}+1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\}} (1 + \log p)^\beta,$$

where $C > 0$ *is a constant independent of* p.

Proof. Considering enough singularity terms in representation (2.1) we obtain a sufficiently high regularity for the function $\mathbf{u}_{\text{reg}} \in \mathbf{H}_0^k(\text{div}, \Gamma)$. Then, due to the quasi-optimal convergence (2.10) of the p-BEM, the assertion follows immediately from the general approximation result given in Theorem 7.2 below. □

3 Preliminaries

3.1 Functional spaces, norms, and inner products

First of all, let us recall the Sobolev spaces and norms that will be used, see [27].

For a Lipschitz domain $\Omega \subset \mathbf{R}^n$ and an integer s, let $H^s(\Omega)$ be the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H^s(\Omega)}^2 = \|u\|_{H^{s-1}(\Omega)}^2 + |u|_{H^s(\Omega)}^2 \quad (s \geq 1),$$

where

$$|u|_{H^s(\Omega)}^2 = \int_\Omega |D^s u(x)|^2\, dx, \quad \text{and} \quad H^0(\Omega) = L^2(\Omega).$$

Here, $|D^s u(x)|^2 = \sum_{|\alpha|=s} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \ldots, x_n)$. For a positive non-integer $s = m + \sigma$ with integer $m \geq 0$ and $0 < \sigma < 1$, the norm in $H^s(\Omega)$ is

$$\|u\|_{H^s(\Omega)}^2 = \|u\|_{H^m(\Omega)}^2 + |u|_{H^s(\Omega)}^2$$

with semi-norm

$$|u|^2_{H^s(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} \, dx \, dy.$$

The Sobolev spaces $\tilde{H}^s(\Omega)$ for $s \in (0,1)$ and for a bounded Lipschitz domain Ω are defined by interpolation. We use the real K-method of interpolation (see [27]) to define

$$\tilde{H}^s(\Omega) = \Big(L^2(\Omega), H^t_0(\Omega)\Big)_{\frac{s}{t},2} \quad (1/2 < t \le 1,\ 0 < s < t).$$

Here, $H^t_0(\Omega)$ $(0 < t \le 1)$ is the completion of $C^\infty_0(\Omega)$ in $H^t(\Omega)$ and we identify $H^1_0(\Omega)$ and $\tilde{H}^1(\Omega)$. Note that the Sobolev spaces $H^s(\Omega)$ also satisfy the interpolation property

$$H^s(\Omega) = \Big(L^2(\Omega), H^1(\Omega)\Big)_{s,2} \quad (0 < s < 1)$$

with equivalent norms. Furthermore, the semi-norm $|\cdot|_{H^1(\Omega)}$ is a norm in $\tilde{H}^1(\Omega)$ due to the Poincaré inequality.

For $s \in [-1,0)$ the Sobolev spaces and their norms are defined by duality with $L^2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space:

$$H^s(\Omega) = (\tilde{H}^{-s}(\Omega))', \quad \tilde{H}^s(\Omega) = (H^{-s}(\Omega))',$$

$$\|u\|_{H^s(\Omega)} = \sup_{0 \ne v \in \tilde{H}^{-s}(\Omega)} \frac{|\langle u, v\rangle|}{\|v\|_{\tilde{H}^{-s}(\Omega)}}, \quad \|u\|_{\tilde{H}^s(\Omega)} = \sup_{0 \ne v \in H^{-s}(\Omega)} \frac{|\langle u, v\rangle|}{\|v\|_{H^{-s}(\Omega)}},$$

where

$$\langle u, v\rangle = \langle u, v\rangle_{0,\Omega} := \int_\Omega u(x)\bar{v}(x) dx$$

denotes the extension of the $L^2(\Omega)$-inner product by duality (and $\bar{v}$ is the complex conjugate of v).

Now, let Γ be a plane open surface with polygonal boundary and let $x = (x_1, x_2) \in \Gamma$. For the space $\tilde{H}^{-1/2}(\Gamma)$, besides the norm introduced above, we will also need an inner product. We define it by

$$\langle u, v\rangle_{-\frac{1}{2},\Gamma} := \langle \Psi u, v\rangle_{0,\Gamma} = \langle u, \Psi v\rangle_{0,\Gamma} \quad \forall\, u, v \in \tilde{H}^{-1/2}(\Gamma),$$

where

$$\Psi u(x) := \Psi_0 u(x) = \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|} \, dS_y, \qquad \Psi:\ \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is the single layer potential operator of the Laplacian, cf. [17]. When acting on tangential vector fields, we will denote the single layer potential operator by $\boldsymbol{\Psi}$.

The spaces $\mathbf{H}^s(\Gamma)$ and $\tilde{\mathbf{H}}^s(\Gamma)$ of vector fields defined on Γ can be introduced similarly as above for any $s \ge -1$. The norms and inner products in these spaces are defined componentwise and the usual convention $\mathbf{H}^0(\Gamma) = \tilde{\mathbf{H}}^0(\Gamma) = \mathbf{L}^2(\Gamma)$ holds. Besides that, we will use two other families of spaces of vector fields on Γ. These are $\mathbf{H}^s_{\perp,0}(\Gamma)$ and $\tilde{\mathbf{H}}^s_\perp(\Gamma)$ with $0 \le s \le 1$.

The space $\mathbf{H}^s_{\perp,0}(\Gamma)$ is defined as the completion of the space $\mathbf{C}^\infty_{\perp,0}(\Gamma) := \{\mathbf{v} = (v_1, v_2);\ v_i \in C^\infty(\Gamma),\ i = 1,2,\ \mathbf{v}\cdot\mathbf{n}|_{\partial\Gamma} = 0\}$ in $\mathbf{H}^s(\Gamma)$. Here, $\mathbf{n}$ denotes the unit outer normal vector to $\partial\Gamma$. Then identifying $\mathbf{H}^1_{\perp,0}(\Gamma)$ and $\tilde{\mathbf{H}}^1_\perp(\Gamma)$ we define the space $\tilde{\mathbf{H}}^s_\perp(\Gamma)$ by interpolation

$$\tilde{\mathbf{H}}^s_\perp(\Gamma) = \Big(\mathbf{L}^2(\Gamma), \mathbf{H}^t_{\perp,0}(\Gamma)\Big)_{\frac{s}{t},2} \quad (1/2 < t \le 1,\ 0 < s < t).$$

Analogously to the scalar case there holds $\mathbf{H}^s(\Gamma) = \tilde{\mathbf{H}}^s(\Gamma) = \tilde{\mathbf{H}}^s_\perp(\Gamma) = \mathbf{H}^s_{\perp,0}(\Gamma)$ if $0 < s < \frac{1}{2}$, and $\tilde{\mathbf{H}}^s_\perp(\Gamma) = \mathbf{H}^s_{\perp,0}(\Gamma)$ if $\frac{1}{2} < s < 1$, with equivalent respective norms.

We will use standard differential operators acting on scalar functions

$$\nabla u = (\partial u/\partial x_1,\ \partial u/\partial x_2), \quad \mathbf{curl}\, u = (\partial u/\partial x_2,\ -\partial u/\partial x_1)$$

and on 2D vector fields (here, $\mathbf{v} = (v_1, v_2)$)

$$\operatorname{div}\mathbf{v} = \partial v_1/\partial x_1 + \partial v_2/\partial x_2, \quad \operatorname{curl}\mathbf{v} = \partial v_2/\partial x_1 - \partial v_1/\partial x_2.$$

Then we set:

$$\mathbf{H}(\mathrm{curl}, \Gamma) := \{\mathbf{v} \in \mathbf{L}^2(\Gamma);\ \operatorname{curl}\mathbf{v} \in L^2(\Gamma)\},$$

$$\mathbf{H}^k(\mathrm{div}, \Gamma) := \{\mathbf{v} \in \mathbf{H}^k(\Gamma);\ \operatorname{div}\mathbf{v} \in H^k(\Gamma)\}, \quad k \ge 0,$$

$$\tilde{\mathbf{H}}^s(\mathrm{div}, \Gamma) := \{\mathbf{v} \in \tilde{\mathbf{H}}^s(\Gamma);\ \operatorname{div}\mathbf{v} \in \tilde{H}^s(\Gamma)\}, \quad s \in [-1/2, 0],$$

$$H^s_*(\Gamma) := \{v \in H^s(\Gamma);\ \langle v, 1\rangle = 0\}, \quad s > -1/2, \qquad H^0_*(\Gamma) = L^2_*(\Gamma),$$

$$\mathcal{H}(\Gamma) := \{v \in H^1_*(\Gamma);\ \Delta v \in \tilde{H}^{-1/2}(\Gamma)\}.$$

Here, Δ denotes the standard Laplace operator, $\Delta = \operatorname{div}\nabla$.

The spaces $\mathbf{H}^k(\mathrm{div}, \Gamma)$ and $\tilde{\mathbf{H}}^s(\mathrm{div}, \Gamma)$ are equipped with their graph norms denoted by $\|\cdot\|_{\mathbf{H}^k(\mathrm{div},\Gamma)}$ and $\|\cdot\|_{\tilde{\mathbf{H}}^s(\mathrm{div},\Gamma)}$, respectively. For $k = 0$ we drop the superscript in the notation of the space, $\mathbf{H}^0(\mathrm{div}, \Gamma) = \mathbf{H}(\mathrm{div}, \Gamma)$.

By $\mathbf{H}^k_0(\mathrm{div}, \Gamma)$ with $k \ge 0$ (respectively, by $\mathbf{X}^s = \tilde{\mathbf{H}}^s_0(\mathrm{div}, \Gamma)$ with $s \in [-1/2, 0]$) we denote the subspace of elements $\mathbf{u} \in \mathbf{H}^k(\mathrm{div}, \Gamma)$ (respectively, $\mathbf{u} \in \tilde{\mathbf{H}}^s(\mathrm{div}, \Gamma)$) such that for all $v \in C^\infty(\bar{\Gamma})$ there holds

$$\langle \mathbf{u}, \nabla v\rangle + \langle \operatorname{div}\mathbf{u}, v\rangle = 0. \tag{3.1}$$

We note that if $\mathbf{u} \in \mathbf{X}^s$ with $s \in [-1/2, 0]$ then identity (3.1) holds for any $v \in H^{1-s}(\Gamma)$ by density. In particular, $\mathbf{X}^s$ is a closed subspace of $\tilde{\mathbf{H}}^s(\mathrm{div}, \Gamma)$. For $s = -\frac{1}{2}$ we drop the superscript in the notation of this space, $\mathbf{X}^{-1/2} = \mathbf{X}$, and also drop the subscript in the notation of the corresponding norm, $\|\cdot\|_{\tilde{\mathbf{H}}^{-1/2}(\mathrm{div},\Gamma)} = \|\cdot\|$.

3.2 Auxiliary lemmas

The following variational problem on Γ will be a useful tool in our analysis: *Given* $\psi \in H^s(\Gamma)$, $s > -\frac{1}{2}$ *(or* $\psi \in \tilde{H}^s(\Gamma)$, $-1 \le s \le -\frac{1}{2}$*), find* $\phi \in H^1_*(\Gamma)$ *such that*

$$\langle \nabla \phi, \nabla \tilde{\phi} \rangle = -\langle \psi, \tilde{\phi} \rangle \quad \forall \tilde{\phi} \in H^1_*(\Gamma). \tag{3.2}$$

We will need the following standard regularity result for this problem (see, e.g., [23]).

Lemma 3.1 *If* $\psi \in H^s(\Gamma)$, $s > -\frac{1}{2}$ *(respectively,* $\psi \in \tilde{H}^s(\Gamma)$, $-1 \le s \le -\frac{1}{2}$*), then there exists a unique solution* ϕ *to problem* (3.2)*. Moreover, there holds* $\phi \in H^{1+r}(\Gamma)$ *and*

$$\|\phi\|_{H^{1+r}(\Gamma)} \le C \|\psi\|_{H^s(\Gamma)} \quad \textit{(respectively, } \|\phi\|_{H^{1+r}(\Gamma)} \le C \|\psi\|_{\tilde{H}^s(\Gamma)})$$

for any $r < \min\{s^*, s+1\}$*, where* $s^* = \frac{\pi}{\omega}$ *and* ω *denotes the maximal internal angle at the vertices of* Γ*.*

To state the second auxiliary result, we denote by $R_p(\Gamma)$ the set of piecewise polynomials of degree p defined on the partition of Γ, i.e.,

$$R_p(\Gamma) := \{v \in L^2(\Gamma);\ v|_{\Gamma_j} \circ T_j \in \mathcal{P}_p(Q),\ j = 1, \dots, J\}. \tag{3.3}$$

The following lemma states the inverse inequality for such piecewise polynomials. We refer to [24] for a proof.

Lemma 3.2 *Let* $v \in R_p(\Gamma)$*. If* $v \in H^r(\Gamma)$ *(respectively,* $v \in \tilde{H}^r(\Gamma)$ *) for a real number* $r \le 1$*, then for* $s \le r$ *there holds*

$$\|v\|_{H^r(\Gamma)} \le C\, p^{2(r-s)} \|v\|_{H^s(\Gamma)} \quad \textit{(respectively, } \|v\|_{\tilde{H}^r(\Gamma)} \le C\, p^{2(r-s)} \|v\|_{\tilde{H}^s(\Gamma)}).$$

Here, C *is a positive constant independent of* p*.*

4 Decompositions

In this section we introduce direct orthogonal decompositions of the energy space $\mathbf{X}$ and of the discrete space $\mathbf{X}_p(\Gamma)$.

4.1 Helmholtz decomposition

Following [12, 10] we decompose $\mathbf{X}$ using the mapping

$$\Lambda : \begin{cases} \mathbf{X} & \to \quad \mathbf{L}^2(\Gamma), \\ \mathbf{u} & \mapsto \quad \nabla f, \end{cases} \tag{4.1}$$

where f solves the Neumann problem: *Find* $f \in H^1_*(\Gamma)$ *such that*

$$\langle \nabla f, \nabla g \rangle = -\langle \operatorname{div} \mathbf{u}, g \rangle \quad \forall g \in H^1_*(\Gamma). \tag{4.2}$$

One has $\operatorname{div}\Lambda\mathbf{u} = \operatorname{div}\mathbf{u}$. Moreover,

$$Ker\Lambda = \{\mathbf{u} \in \mathbf{X};\ \operatorname{div}\mathbf{u} = 0\} \quad \text{and} \quad \Lambda(\Lambda\mathbf{u}) = \Lambda\mathbf{u}.$$

Thus, Λ is a continuous projector, $\Lambda : \mathbf{X} \to \mathbf{X}$. Denoting

$$\mathbf{V} := Im\,\Lambda, \qquad \mathbf{W} := Ker\,\Lambda = \{\mathbf{u} \in \mathbf{X};\ \operatorname{div}\mathbf{u} = 0\}, \tag{4.3}$$

which are closed subspaces of $\mathbf{X}$, one has the Helmholtz decomposition

$$\mathbf{X} = \mathbf{V} \oplus \mathbf{W}. \tag{4.4}$$

Note that

$$\mathbf{V} = \nabla\mathcal{H}(\Gamma), \qquad \mathbf{W} = \mathbf{curl}\,\tilde{H}^{1/2}(\Gamma) \tag{4.5}$$

so that (4.4) can be written as (cf., [12, Theorem 5.4])

$$\mathbf{X} = \nabla\mathcal{H}(\Gamma) \oplus \mathbf{curl}\,\tilde{H}^{1/2}(\Gamma).$$

Theorem 4.1 *Decomposition* (4.4) *is* $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$*-orthogonal.*

Proof. For any $\mathbf{v} \in \mathbf{V}$, $\mathbf{w} \in \mathbf{W}$ one has by (4.5)

$$\mathbf{v} = \nabla f, \quad \mathbf{w} = \mathbf{curl}\,g \quad \text{for some } f \in \mathcal{H}(\Gamma),\ g \in \tilde{H}^{1/2}(\Gamma).$$

We consider $(\mathbf{F}_n)_n \subset \mathbf{C}_0^\infty(\Gamma)$ with $\mathbf{F}_n \to \nabla f$ in $\mathbf{L}^2(\Gamma)$ $(n \to \infty)$. Then, making use of the continuities $\mathbf{\Psi} : \mathbf{L}^2(\Gamma) \to \mathbf{H}^1(\Gamma)$, $\operatorname{curl} : \mathbf{H}^1(\Gamma) \to L^2(\Gamma)$, $\operatorname{curl} : \mathbf{L}^2(\Gamma) \to \tilde{H}^{-1}(\Gamma)$, $\Psi : \tilde{H}^{-1}(\Gamma) \to L^2(\Gamma)$ (see [29, 14]), and noting the commutativity property $\operatorname{curl}\mathbf{\Psi}\mathbf{F}_n = \Psi(\operatorname{curl}\mathbf{F}_n)$ (see [28, Lemma 2.3], cf. also [25, Lemma 4.2]), one proves the relation

$$\operatorname{curl}\mathbf{\Psi}\nabla f = \Psi(\operatorname{curl}\nabla f).$$

By a similar density argument it follows that $\operatorname{curl}\nabla f = 0$ (see also [14, Theorem 5.1]). Hence, integrating by parts and using the density $C_0^\infty(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$, we obtain

$$\langle \mathbf{v}, \mathbf{curl}\,g\rangle_{-\frac{1}{2},\Gamma} = \langle \mathbf{\Psi}\nabla f, \mathbf{curl}\,g\rangle = \langle \operatorname{curl}\mathbf{\Psi}\nabla f, g\rangle = \langle \Psi(\operatorname{curl}\nabla f), g\rangle = 0.$$

Therefore, $\langle \mathbf{v}, \mathbf{w}\rangle_{-\frac{1}{2},\Gamma} = 0$ and the proof is finished. □

4.2 Discrete decomposition

The Helmholtz decomposition (4.4) was used in [10] to prove an inf-sup condition for the electric field integral operator and to establish the unique solvability of the EFIE on Γ. The discretisation of the EFIE by the Galerkin BEM is based on a sequence $\{\mathbf{X}_n\}_n$ of finite dimensional subspaces $\mathbf{X}_n \subset \mathbf{X}$. However, Helmholtz decompositions of functions in $\mathbf{X}_n$ may give functions which are not discrete. That is why the discrete inf-sup condition (and thus, the unique solvability of (2.9) and quasi-optimal convergence of the BEM) cannot be deduced by standard arguments, which are usually applied to conforming Galerkin discretisations of coercive variational problems.

In [10], sufficient conditions were found to prove the well-posedness of the Galerkin BEM applied to problem (1.1). The main idea there was to consider discrete decompositions $\mathbf{X}_n = \mathbf{V}_n \oplus \mathbf{W}_n$, which are in some sense close to the Helmholtz decomposition of $\mathbf{X}$ when $n \to \infty$. The abstract formulation of this approach is given in [16]. In particular, the following theorem holds (see Proposition 4.1, Corollary 4.2, and Theorem 4.5 in [10] and also [16, Theorem 4.1]).

Theorem 4.2 *Let $\{\mathbf{X}_n\}_n$ be a sequence of closed subspaces $\mathbf{X}_n \subset \mathbf{X}$ with decompositions $\mathbf{X}_n = \mathbf{V}_n \oplus \mathbf{W}_n$ which are stable with respect to complex conjugation and which satisfy the following assumptions:*

(A1) *the family $\{\mathbf{X}_n\}_n$ is dense in the space $\mathbf{X}$, namely*

$$\overline{\bigcup_n \mathbf{X}_n} = \mathbf{X};$$

(A2) *the spaces $\mathbf{V}_n$ and $\mathbf{W}_n$ are such that $\mathbf{W}_n \subset \mathbf{W}$ and*

$$\sup_{\mathbf{v}_n \in \mathbf{V}_n \setminus \{\mathbf{0}\}} \inf_{\mathbf{v} \in \mathbf{V}} \frac{\|\mathbf{v}_n - \mathbf{v}\|}{\|\mathbf{v}_n\|} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.6}$$

Then there exists n_0 such that for all $\mathbf{f} \in \mathbf{X}'$ and $n \geq n_0$ the Galerkin system

$$a(\mathbf{u}_n, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in \mathbf{X}_n$$

has a unique solution $\mathbf{u}_n \in \mathbf{X}_n$ which converges quasi-optimally, i.e.,

$$\|\mathbf{u} - \mathbf{u}_n\| \leq C \inf\{\|\mathbf{u} - \mathbf{v}\|;\ \mathbf{v} \in \mathbf{X}_n\},$$

where $\mathbf{u} \in \mathbf{X}$ is the solution of (1.1).

In this paper we discretise the EFIE by the p-version of the Galerkin BEM based on the sequence of the RT-subspaces $\mathbf{X}_p(\Gamma) \subset \mathbf{X}$ (see (2.7)–(2.9)). To prove the well-posedness of (2.9) (see Theorem 2.1) we will use the abstract convergence result of Theorem 4.2 above. To that end one needs to consider discrete decompositions of $\mathbf{X}_p(\Gamma)$. We set

$$\mathbf{X}_p(\Gamma) = \mathbf{V}_p \oplus \mathbf{W}_p, \tag{4.7}$$

where

$$\mathbf{W}_p := \{\mathbf{w}_p \in \mathbf{X}_p(\Gamma);\ \operatorname{div}\mathbf{w}_p = 0\}, \tag{4.8}$$

$$\mathbf{V}_p := \{\mathbf{v}_p \in \mathbf{X}_p(\Gamma);\ \langle \mathbf{v}_p, \mathbf{w}_p\rangle_{-\frac{1}{2},\Gamma} = 0 \quad \forall \mathbf{w}_p \in \mathbf{W}_p\}. \tag{4.9}$$

Thus, $\mathbf{V}_p$ and $\mathbf{W}_p$ are orthogonal with respect to the $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$-inner product by definition. Decomposition (4.7) is stable with respect to complex conjugation.

Remark 4.1 *In* [10, 16], $\mathbf{L}^2(\Gamma)$*-orthogonal discrete decompositions were introduced for finite dimensional subspaces based on RT and Brezzi-Douglas-Marini (BDM) boundary elements. It has been proved that these decompositions satisfy assumptions* (A1), (A2) *of Theorem* 4.2 *with respect to the mesh parameter* h*, i.e., in the framework of the* h*-version of the BEM for the EFIE. It turns out that, for the* p*-version, the* $\mathbf{L}^2(\Gamma)$*-orthogonality of decomposition* (4.7) *is not sufficient to prove* (A2) *with standard techniques. That is why we need* $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$*-orthogonality instead.*

5 Interpolation operators

In this section we will introduce the interpolation operators acting on vector fields and recall some of their properties.

First let us consider the standard (element-wise) L^2-projection onto the set of piecewise polynomials. We denote this L^2-projection by $\Pi_p^0 : L^2(\Gamma) \to R_p(\Gamma)$, where $R_p(\Gamma)$ is defined in (3.3). There holds the following approximation result.

Lemma 5.1 *For any* $r \geq 0$ *there holds*

$$\|v - \Pi_p^0 v\|_{L^2(\Gamma)} \leq C\, p^{-r}\, \|v\|_{H^r(\Gamma)} \qquad \forall v \in H^r(\Gamma)$$

where $C > 0$ *is independent of* p *and* v.

This assertion follows from the local approximation result of [2, Lemma 4.5] (see [34] for the proof).

Now let us introduce the interpolation operators acting on vector fields $\mathbf{v} \in \mathbf{H}(\operatorname{div}, \Gamma)$. By Π_p^{RT} we denote the standard RT-interpolation operator $\Pi_p^{\mathrm{RT}} :\ \mathbf{H}(\operatorname{div}, \Gamma) \to \mathbf{X}_p(\Gamma)$ (see, e.g., [31, Chapter 2, Section 7] for the definition). Using the same notation as in (2.6), this operator satisfies

$$\langle \widehat{\Pi_p^{\mathrm{RT}}\mathbf{v}} - \hat{\mathbf{v}},\ \hat{\mathbf{w}}\rangle_{0,Q} = 0 \quad \forall \hat{\mathbf{w}} \in \mathcal{P}_{p-2,p-1}(Q) \times \mathcal{P}_{p-1,p-2}(Q). \tag{5.1}$$

Hence

$$\langle \Pi_p^{\mathrm{RT}}\mathbf{v} - \mathbf{v},\ \mathbf{w}\rangle_{0,\Gamma} = 0 \quad \forall \mathbf{w} \in \mathbf{X}_{p-1}^{\mathrm{curl}}(\Gamma),$$

where $\mathbf{X}_p^{\mathrm{curl}}(\Gamma)$ denotes a discrete subspace of $\mathbf{H}(\operatorname{curl}, \Gamma)$ based on the Nédélec elements of the first type, namely,

$$\mathbf{X}_p^{\mathrm{curl}}(\Gamma) := \{\mathbf{v} \in \mathbf{H}(\operatorname{curl}, \Gamma);\ \mathcal{M}_j^{-1}(\mathbf{v}|_{\Gamma_j}) \in \mathcal{P}_{p-1,p}(Q) \times \mathcal{P}_{p,p-1}(Q),\ j = 1, \ldots, J\}.$$

For the $\mathbf{L}^2$-estimate of the error of the RT-interpolation (in terms of polynomial degrees p), we cite the following result from [33] (see Lemma 4.1 therein).

Lemma 5.2 *If $\mathbf{u} \in \mathbf{H}^r(\Gamma)$ with $r > \frac{1}{2}$, then for any $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon)$ such that*

$$\|\mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u}\|_{\mathbf{L}^2(\Gamma)} \le C\, p^{-(r-1/2-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\Gamma)}.$$

On the other hand, in [19] the $\mathbf{H}(\mathrm{curl})$-conforming projection-based interpolation operator $\Pi_p^{\mathrm{curl}} : \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\mathrm{curl}, \Gamma) \to \mathbf{X}_p^{\mathrm{curl}}(\Gamma)$, $r > 0$ has been introduced and analysed. Using the isomorphism of the curl and the div operator in 2D (and, as a consequence, the isomorphism of the Nédélec and RT elements), we reformulate the main results of [19] in the $\mathbf{H}(\mathrm{div})$-settings. We will denote by Π_p^{div} the corresponding $\mathbf{H}(\mathrm{div})$-conforming projection-based interpolation operator.

Lemma 5.3 [19, Proposition 2] *For $r > 0$ the operator $\Pi_p^{\mathrm{div}} : \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\mathrm{div}, \Gamma) \to \mathbf{H}(\mathrm{div}, \Gamma)$ is bounded, with norm independent of the polynomial degree p.*

This result implies the $\mathbf{L}^2$-stability of Π_p^{div}: there exists a positive constant $C > 0$ independent of p such that for any $\mathbf{u} \in \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\mathrm{div}, \Gamma)$, $r > 0$, there holds

$$\|\Pi_p^{\mathrm{div}}\mathbf{u}\|_{\mathbf{L}^2(\Gamma)} \le C\left(\|\mathbf{u}\|_{\mathbf{H}^r(\Gamma)} + \|\mathrm{div}\,\mathbf{u}\|_{L^2(\Gamma)}\right). \tag{5.2}$$

The following approximation result is Theorem 3 in [19].

Lemma 5.4 *Let $\mathbf{u} \in \mathbf{H}^r(\mathrm{div}, \Gamma)$ with $0 < r < 1$, and let $0 < \varepsilon < r$. Then there exists $C > 0$, depending on ε but independent of p such that*

$$\|\mathbf{u} - \Pi_p^{\mathrm{div}}\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\Gamma)} \le C\, p^{-(r-\varepsilon)}\, \|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\Gamma)}.$$

It is essential that both interpolation operators, Π_p^{RT} and Π_p^{div}, satisfy the commuting diagram property:

$$\mathrm{div}\,(\Pi_p^{\mathrm{RT}}\mathbf{u}) = \Pi_{p-1}^0(\mathrm{div}\,\mathbf{u}) \qquad \forall \mathbf{u} \in \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\mathrm{div}, \Gamma), \quad r > 1/2, \tag{5.3}$$

$$\mathrm{div}\,(\Pi_p^{\mathrm{div}}\mathbf{u}) = \Pi_{p-1}^0(\mathrm{div}\,\mathbf{u}) \qquad \forall \mathbf{u} \in \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\mathrm{div}, \Gamma), \quad 0 < r < 1; \tag{5.4}$$

here we refer to [9, Proposition 3.7] and [19, Proposition 3], respectively.

The interpolation operators introduced above become a useful tool in the analysis of Galerkin discretisations of mixed variational formulations for elliptic boundary value problems. Let us demonstrate this for a certain auxiliary problem, which will be used further in Section 7.1. Given $\mathbf{u} \in \mathbf{H}_0^r(\mathrm{div}, \Gamma)$, $r > 0$, we consider the following mixed variational problem: *Find $(\mathbf{z}, f) \in (\mathbf{H}_0(\mathrm{div}, \Gamma), L_*^2(\Gamma))$ such that*

$$\begin{aligned} \langle \mathbf{z}, \mathbf{v}\rangle + \langle \mathrm{div}\,\mathbf{v}, f\rangle &= \langle \mathbf{u}, \mathbf{v}\rangle && \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{div}, \Gamma), \\ \langle \mathrm{div}\,\mathbf{z}, g\rangle &= \langle \mathrm{div}\,\mathbf{u}, g\rangle && \forall g \in L_*^2(\Gamma). \end{aligned} \tag{5.5}$$

The unique solvability of (5.5) is proved by usual techniques (see [9]). In our case it is clear that the pair $(\mathbf{u}, 0)$ solves (5.5).

A conforming Galerkin approximation of problem (5.5) based on RT-elements reads as: *Find* $(\mathbf{z}_p, f_p) \in (\mathbf{X}_p(\Gamma), R^*_{p-1}(\Gamma))$ *for* $p \geq 1$ *such that*

$$\begin{aligned} \langle \mathbf{z}_p, \mathbf{v}\rangle + \langle \operatorname{div} \mathbf{v}, f_p\rangle &= \langle \mathbf{u}, \mathbf{v}\rangle \qquad \forall \mathbf{v} \in \mathbf{X}_p(\Gamma), \\ \langle \operatorname{div} \mathbf{z}_p, g\rangle &= \langle \operatorname{div} \mathbf{u}, g\rangle \qquad \forall g \in R^*_{p-1}(\Gamma). \end{aligned} \tag{5.6}$$

Here, $R^*_p(\Gamma) := \{g \in R_p(\Gamma); \langle g, 1\rangle = 0\}$ and $R_p(\Gamma)$ is defined by (3.3).

We now prove the unique solvability of (5.6). Observe that for any given $g_p \in R^*_{p-1}(\Gamma)$ we can solve the Neumann problem analogous to (3.2) to find a function $\phi \in H^1_*(\Gamma)$ such that $\Delta\,\phi = g_p$ on Γ. Then applying the regularity result of Lemma 3.1 we have $\phi \in H^{1+r}(\Gamma)$, $0 < r \leq r_0$, $r_0 > \frac{1}{2}$ and

$$\|\nabla\,\phi\|_{\mathbf{H}^r(\Gamma)} \leq C\|\phi\|_{H^{1+r}(\Gamma)} \leq C\|g_p\|_{L^2(\Gamma)}. \tag{5.7}$$

Therefore, $\nabla\,\phi \in \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\operatorname{div}, \Gamma)$ and the interpolant $\Pi^{\operatorname{div}}_p \nabla\,\phi \in \mathbf{X}_p(\Gamma)$ is well defined, due to Lemma 5.3. Moreover, (5.4) yields

$$\operatorname{div}\,(\Pi^{\operatorname{div}}_p \nabla\,\phi) = g_p.$$

Hence, using (5.2) and (5.7) we prove the discrete Ladyzhenskaya-Babuška-Brezzi condition:

$$\begin{aligned} \sup_{\mathbf{v}_p \in \mathbf{X}_p(\Gamma)\setminus\{\mathbf{0}\}} \frac{\langle \operatorname{div} \mathbf{v}_p, g_p\rangle}{\|\mathbf{v}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)}} &\geq \frac{\langle \operatorname{div}\,(\Pi^{\operatorname{div}}_p \nabla\,\phi), g_p\rangle}{\|\Pi^{\operatorname{div}}_p \nabla\,\phi\|_{\mathbf{H}(\operatorname{div},\Gamma)}} \\ &\geq \frac{\|g_p\|^2_{L^2(\Gamma)}}{C\left(\|\nabla\,\phi\|_{\mathbf{H}^r(\Gamma)} + \|\operatorname{div} \nabla\,\phi\|_{L^2(\Gamma)}\right) + \|\operatorname{div}\,(\Pi^{\operatorname{div}}_p \nabla\,\phi)\|_{L^2(\Gamma)}} \\ &\geq \tilde{C}\,\|g_p\|_{L^2(\Gamma)} \qquad \forall g_p \in R^*_{p-1}(\Gamma). \end{aligned}$$

This condition along with the property $\operatorname{div} \mathbf{X}_p(\Gamma) = R^*_{p-1}(\Gamma)$ ensures existence, uniqueness, and asymptotic quasi-optimality of the solution $(\mathbf{z}_p, f_p)$ to (5.6) (see [9]). Then we rewrite (5.6) as

$$\langle \mathbf{u} - \mathbf{z}_p, \mathbf{v}\rangle = \langle \operatorname{div} \mathbf{v}, f_p\rangle \qquad \forall \mathbf{v} \in \mathbf{X}_p(\Gamma), \tag{5.8}$$

$$\langle \operatorname{div}\,(\mathbf{u} - \mathbf{z}_p), g\rangle = 0 \qquad \forall g \in R_{p-1}(\Gamma) \tag{5.9}$$

(note that (5.9) holds for any $g \in R_{p-1}(\Gamma)$, because $\langle \operatorname{div}\,(\mathbf{u} - \mathbf{z}_p), c\rangle = 0$ for any constant c). Furthermore, recalling that $\mathbf{z} = \mathbf{u}$ and $f = 0$, we have

$$\|\mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)} \leq \|\mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)} + \|f_p\|_{L^2(\Gamma)} \leq C\left(\|\mathbf{z}\|_{\mathbf{H}(\operatorname{div},\Gamma)} + \|f\|_{L^2(\Gamma)}\right) = C\|\mathbf{u}\|_{\mathbf{H}(\operatorname{div},\Gamma)} \tag{5.10}$$

and

$$\begin{aligned} \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)} &\leq \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)} + \|f - f_p\|_{L^2(\Gamma)} \\ &\leq C\left(\inf_{\mathbf{v}_p \in \mathbf{X}_p(\Gamma)} \|\mathbf{u} - \mathbf{v}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)} + \inf_{g_p \in R^*_{p-1}(\Gamma)} \|f - g_p\|_{L^2(\Gamma)}\right) \\ &= C \inf_{\mathbf{v}_p \in \mathbf{X}_p(\Gamma)} \|\mathbf{u} - \mathbf{v}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)}. \end{aligned} \tag{5.11}$$

We find by (5.10)

$$\|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{H}(\mathrm{div},\Gamma)} \le C\|\mathbf{u}\|_{\mathbf{H}(\mathrm{div},\Gamma)}. \tag{5.12}$$

From (5.11) using the commuting diagram property (5.3), Lemma 5.1, and Lemma 5.2 we have for any $r > \frac{1}{2}$

$$\begin{aligned}
\|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{H}(\mathrm{div},\Gamma)} &\le C\left(\|\mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u}\|_{\mathbf{L}^2(\Gamma)} + \|\mathrm{div}\,(\mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u})\|_{L^2(\Gamma)}\right) \\
&= C\left(\|\mathbf{u} - \Pi_p^{\mathrm{RT}}\mathbf{u}\|_{\mathbf{L}^2(\Gamma)} + \|\mathrm{div}\,\mathbf{u} - \Pi_{p-1}^{0}\mathrm{div}\,\mathbf{u})\|_{L^2(\Gamma)}\right) \\
&\le Cp^{-(r-1/2-\tilde\varepsilon)}\|\mathbf{u}\|_{\mathbf{H}^r(\mathrm{div},\Gamma)}, \qquad \tilde\varepsilon > 0.
\end{aligned} \tag{5.13}$$

The estimate in (5.13) can be improved to give a sub-optimal p-approximation result (see estimate (5.14) below). The argument is based on interpolation between (5.12) and (5.13). It was first used in [3] for the scalar case, and we refer to Lemma 4.1 and Theorem 4.2 in [34] for the case of vector fields. Thus we have proved the following auxiliary result.

Lemma 5.5 *Given any $k > 0$, $\varepsilon > 0$ and $\mathbf{u} \in \mathbf{H}_0^k(\mathrm{div},\Gamma)$, there exists a pair $(\mathbf{z}_p, f_p) \in (\mathbf{X}_p(\Gamma), R_{p-1}^*(\Gamma))$ solving* (5.6) *and satisfying* (5.8), (5.9). *Moreover, there exists a constant $C > 0$ independent of p and $\mathbf{u}$ but depending on ε and k such that*

$$\|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{H}(\mathrm{div},\Gamma)} \le Cp^{-(k-\varepsilon)}\|\mathbf{u}\|_{\mathbf{H}^k(\mathrm{div},\Gamma)}. \tag{5.14}$$

6 Proof of Theorem 2.1

In this section we prove Theorem 2.1 relying on the abstract convergence result of Theorem 4.2. One needs to check that assumptions (A1) and (A2) are satisfied. First, we note that the family $\{\mathbf{X}_p(\Gamma)\}_p$ of RT-spaces is dense in $\mathbf{X}^0$. Since the injection $\mathbf{X}^0 \subset \mathbf{X}$ is dense as well (see, e.g., [15, Lemma 2.4]), we conclude that the family $\{\mathbf{X}_p(\Gamma)\}_p$ satisfies assumption (A1) of Theorem 4.2.

Further, from the definitions of $\mathbf{W}$ and $\mathbf{W}_p$ (compare (4.3) and (4.8)) it is clear that $\mathbf{W}_p \subset \mathbf{W}$. Thus, it remains to prove that the subspace $\mathbf{V}_p$ defined by (4.9) satisfies assumption (4.6). In particular, we will show below that there exists a sequence $\{\delta_p\}_p$, $\delta_p \to 0$ as $p \to \infty$, such that for any given $\mathbf{v}_p \in \mathbf{V}_p$ there exists $\mathbf{v} \in \mathbf{V}$ satisfying

$$\|\mathbf{v}_p - \mathbf{v}\| \le \delta_p\|\mathbf{v}_p\|. \tag{6.1}$$

The proof of this statement consists of four steps.

Step 1: Construction of the function $\mathbf{v}$ for given $\mathbf{v}_p$. Given $\mathbf{v}_p \in \mathbf{V}_p$, we solve the Neumann problem to find $f \in H_*^1(\Gamma)$ such that

$$\langle \nabla f, \nabla g\rangle = -\langle \mathrm{div}\,\mathbf{v}_p, g\rangle \quad \forall g \in H_*^1(\Gamma). \tag{6.2}$$

We set $\mathbf{v} := \nabla f$. By definition of $\mathbf{V}$ there holds $\mathbf{v} \in \mathbf{V}$, see (4.1)–(4.3). Moreover,

$$\mathrm{div}\,\mathbf{v} = \mathrm{div}\,\mathbf{v}_p. \tag{6.3}$$

Note that $\operatorname{div}\mathbf{v}_p \in H^{-1/2+\varepsilon}(\Gamma)$ for any $\varepsilon \in (0,1)$. Therefore, the regularity result for problem (6.2) reads as (see Lemma 3.1): there exists $r > \frac{1}{2}$ such that $f \in H^{1+r}(\Gamma)$. Moreover, using the continuity of the gradient as a mapping $H^{1+r}(\Gamma) \to \mathbf{H}^r(\Gamma)$, we have

$$\|\mathbf{v}\|_{\mathbf{H}^r(\Gamma)} \le C\,\|f\|_{H^{1+r}(\Gamma)} \le C\,\|\operatorname{div}\mathbf{v}_p\|_{H^{-1/2+\varepsilon}(\Gamma)}, \qquad r > 1/2,\quad \varepsilon \in (0,1). \tag{6.4}$$

In view of (6.3) the desired estimate in (6.1) reduces to the inequality

$$\|\mathbf{v}_p - \mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le \delta_p \|\mathbf{v}_p\|. \tag{6.5}$$

Step 2: Reducing $\|\mathbf{v}_p - \mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)}$ to the $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$-error of the RT-interpolation. Since $\mathbf{v} \in \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\operatorname{div},\Gamma)$, $r > \frac{1}{2}$, we can apply the RT-interpolation operator Π_p^{RT} (see Section 5) to define $\mathbf{v}_p^{\mathrm{RT}} := \Pi_p^{\mathrm{RT}}\,\mathbf{v} \in \mathbf{X}_p(\Gamma)$. Now recalling (5.3) and using (6.3) we find

$$\operatorname{div}\mathbf{v}_p^{\mathrm{RT}} = \operatorname{div}\mathbf{v}_p = \operatorname{div}\mathbf{v}.$$

Hence, $(\mathbf{v}_p - \mathbf{v}_p^{\mathrm{RT}}) \in \mathbf{W}_p \subset \mathbf{W}$. This fact together with the orthogonalities $\mathbf{V} \perp \mathbf{W}$ and $\mathbf{V}_p \perp \mathbf{W}_p$ with respect to the $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$-inner product implies the equalities:

$$\langle \mathbf{v}, \mathbf{v}_p - \mathbf{v}_p^{\mathrm{RT}} \rangle_{-\frac{1}{2},\Gamma} = \langle \mathbf{v}_p, \mathbf{v}_p - \mathbf{v}_p^{\mathrm{RT}} \rangle_{-\frac{1}{2},\Gamma} = 0.$$

Therefore,

$$\|\mathbf{v} - \mathbf{v}_p\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le C\,\langle \mathbf{v} - \mathbf{v}_p, \mathbf{v} - \mathbf{v}_p \rangle_{-\frac{1}{2},\Gamma} = C\,\langle \mathbf{v} - \mathbf{v}_p, \mathbf{v} - \mathbf{v}_p^{\mathrm{RT}} \rangle_{-\frac{1}{2},\Gamma},$$

which gives

$$\|\mathbf{v} - \mathbf{v}_p\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le C\,\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)}. \tag{6.6}$$

Step 3: Reducing $\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)}$ to $\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma)}$. We start with the definition of the norm in $\tilde{\mathbf{H}}^{-1/2}$ on an arbitrary element Γ_j:

$$\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma_j)} = \sup_{\mathbf{w} \in \mathbf{H}^{1/2}(\Gamma_j)\setminus\{\mathbf{0}\}} \frac{\langle \mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}, \mathbf{w} \rangle_{0,\Gamma_j}}{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\Gamma_j)}}.$$

For any $\mathbf{w} \in \mathbf{H}^{1/2}(\Gamma_j)$, let us denote by $\mathbf{w}_p$ the local (component-wise) L^2-projection of $\mathbf{w}$ onto the set of polynomials of degree $(p-2)$ on Γ_j. Then, recalling (5.1), we have for $p > 2$

$$\begin{aligned}
\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma_j)} &= \sup_{\mathbf{w} \in \mathbf{H}^{1/2}(\Gamma_j)\setminus\{\mathbf{0}\}} \frac{\langle \mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}, \mathbf{w} - \mathbf{w}_p \rangle_{0,\Gamma_j}}{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\Gamma_j)}} \\
&\le \|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma_j)} \sup_{\mathbf{w} \in \mathbf{H}^{1/2}(\Gamma_j)\setminus\{\mathbf{0}\}} \frac{\|\mathbf{w} - \mathbf{w}_p\|_{\mathbf{L}^2(\Gamma_j)}}{\|\mathbf{w}\|_{\mathbf{H}^{1/2}(\Gamma_j)}} \\
&\le C\,p^{-1/2}\,\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma_j)}.
\end{aligned} \tag{6.7}$$

Here we also applied Lemma 5.1 restricted to the element Γ_j. Since

$$\|\mathbf{u}\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le \sum_j \|\mathbf{u}|_{\Gamma_j}\|^2_{\tilde{\mathbf{H}}^{-1/2}(\Gamma_j)},$$

we square (6.7) and sum up the results over all elements to obtain

$$\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le C\, p^{-1/2}\, \|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma)}. \tag{6.8}$$

Step 4: Estimating $\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma)}$ and conclusion. Since $\mathbf{v} \in \mathbf{H}^r(\Gamma)$ for some $r > \frac{1}{2}$, we apply Lemma 5.2 and then inequality (6.4) to obtain

$$\|\mathbf{v} - \Pi_p^{\mathrm{RT}}\mathbf{v}\|_{\mathbf{L}^2(\Gamma)} \le Cp^{-(r-1/2-\tilde{\varepsilon})}\|\mathbf{v}\|_{\mathbf{H}^r(\Gamma)} \le Cp^{-(r-1/2-\tilde{\varepsilon})}\|\operatorname{div} \mathbf{v}_p\|_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma)} \tag{6.9}$$

for any $\varepsilon \in (0,1)$ and arbitrary $\tilde{\varepsilon} \in (0, r - \frac{1}{2})$. Then making use of the inverse inequality (see Lemma 3.2) we estimate

$$\|\operatorname{div} \mathbf{v}_p\|_{\mathbf{H}^{-1/2+\varepsilon}(\Gamma)} \le Cp^{2\varepsilon}\|\operatorname{div} \mathbf{v}_p\|_{\mathbf{H}^{-1/2}(\Gamma)} \le Cp^{2\varepsilon}\|\operatorname{div} \mathbf{v}_p\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le Cp^{2\varepsilon}\|\mathbf{v}_p\|. \tag{6.10}$$

Now we put together (6.6) and (6.8)–(6.10). Then selecting ε small enough we have

$$\|\mathbf{v} - \mathbf{v}_p\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} \le Cp^{-(r-\bar{\varepsilon})}\|\mathbf{v}_p\|, \qquad r > 1/2, \quad 0 < \bar{\varepsilon} < r.$$

This gives (6.5), which implies (6.1). Therefore, the subspace $\mathbf{V}_p$ satisfies (4.6). Thus we have shown that the discrete $\tilde{\mathbf{H}}^{-1/2}(\Gamma)$-orthogonal decomposition (4.7) verifies assumption (A2) of Theorem 4.2, and the proof is finished.

7 Approximation results

7.1 Approximation of smooth vector functions

In this sub-section we prove the following p-approximation result for vector fields $\mathbf{u} \in \mathbf{H}_0^k(\operatorname{div}, \Gamma)$.

Theorem 7.1 *Given any $k > 0$, $\varepsilon > 0$ and $\mathbf{u} \in \mathbf{H}_0^k(\operatorname{div}, \Gamma)$, there exists $\mathbf{z}_p \in \mathbf{X}_p(\Gamma)$ such that for $0 \le s \le \frac{1}{2}$*

$$\|\mathbf{u} - \mathbf{z}_p\|_{\tilde{\mathbf{H}}^{-s}(\operatorname{div},\Gamma)} \le Cp^{-(k+s-\varepsilon)}\|\mathbf{u}\|_{\mathbf{H}^k(\operatorname{div},\Gamma)}. \tag{7.1}$$

Here, $C > 0$ is a constant independent of p, s and $\mathbf{u}$ but depending on ε and k.

Proof. Let $\mathbf{u} \in \mathbf{H}_0^k(\operatorname{div}, \Gamma)$ and $k > 0$. Then applying Lemma 5.5 we find a function $\mathbf{z}_p \in \mathbf{X}_p(\Gamma)$ satisfying equalities (5.8), (5.9) and such that estimate (7.1) holds with $s = 0$. We now prove (7.1) for $s \in (0, \frac{1}{2})$. First, we write the negative-order norm of $(\mathbf{u} - \mathbf{z}_p)$:

$$\|\mathbf{u} - \mathbf{z}_p\|_{\tilde{\mathbf{H}}^{-s}(\Gamma)} = \sup_{\mathbf{w} \in \mathbf{H}^s(\Gamma)\setminus\{\mathbf{0}\}} \frac{\langle \mathbf{u} - \mathbf{z}_p, \mathbf{w}\rangle}{\|\mathbf{w}\|_{\mathbf{H}^s(\Gamma)}}. \tag{7.2}$$

Given $\mathbf{w} \in \mathbf{H}^s(\Gamma) = \tilde{\mathbf{H}}^s(\Gamma)$ ($s \in (0, \frac{1}{2})$), we consider the Neumann problem (cf. (3.2)): *Find* $\varphi \in H^1_*(\Gamma)$ *such that*

$$\langle \nabla \varphi, \nabla \phi \rangle = \langle \operatorname{div} \mathbf{w}, \phi \rangle \quad \forall \phi \in H^1_*(\Gamma). \tag{7.3}$$

Since $\operatorname{div} \mathbf{w} \in \tilde{H}^{s-1}(\Gamma)$ and $s \in (0, \frac{1}{2})$, the regularity result for φ reads as (see Lemma 3.1):

$$\varphi \in H^{1+r}(\Gamma) \ \text{ for any } 0 < r \le s < \tfrac{1}{2}. \tag{7.4}$$

Moreover, there holds

$$\|\varphi\|_{H^{1+r}(\Gamma)} \le C \, \|\operatorname{div} \mathbf{w}\|_{\tilde{H}^{s-1}(\Gamma)} \le C \, \|\mathbf{w}\|_{\mathbf{H}^s(\Gamma)}. \tag{7.5}$$

Then we set

$$\mathbf{q} := \mathbf{w} + \nabla \varphi \in \mathbf{H}^r(\Gamma). \tag{7.6}$$

It follows from (7.3) that $\operatorname{div} \nabla \varphi = -\operatorname{div} \mathbf{w}$. Hence

$$\operatorname{div} \mathbf{q} = \operatorname{div} \mathbf{w} + \operatorname{div} \nabla \varphi = 0.$$

Furthermore, we have by (7.4)–(7.6)

$$\|\mathbf{q}\|_{\mathbf{H}^r(\Gamma)} \le \|\mathbf{w}\|_{\mathbf{H}^r(\Gamma)} + C\|\varphi\|_{H^{1+r}(\Gamma)} \le \|\mathbf{w}\|_{\mathbf{H}^r(\Gamma)} + C\|\mathbf{w}\|_{\mathbf{H}^s(\Gamma)} \le C\|\mathbf{w}\|_{\mathbf{H}^s(\Gamma)}. \tag{7.7}$$

Now, we use (7.6) to represent the numerator in (7.2) as

$$\langle \mathbf{u} - \mathbf{z}_p, \mathbf{w} \rangle = \langle \mathbf{u} - \mathbf{z}_p, \mathbf{q} - \nabla \varphi \rangle = \langle \mathbf{u} - \mathbf{z}_p, \mathbf{q} \rangle + \langle \operatorname{div} (\mathbf{u} - \mathbf{z}_p), \varphi \rangle \tag{7.8}$$

(here we also used equality (3.1) and the fact that $(\mathbf{u} - \mathbf{z}_p) \in \mathbf{X}^0$).

Since $\mathbf{q} \in \mathbf{H}^r(\Gamma) \cap \mathbf{H}(\operatorname{div}, \Gamma)$ for an $r \in (0, \frac{1}{2})$ we can apply the interpolation operator $\Pi_p^{\operatorname{div}}$ to $\mathbf{q}$. Recalling that $\mathbf{z}_p$ satisfies (5.8) and $\mathbf{q}$ is divergence-free, we use commutativity property (5.4) and the approximation result of Lemma 5.4 to obtain for the first term on the right-hand side of (7.8):

$$\begin{aligned}
\langle \mathbf{u} - \mathbf{z}_p, \mathbf{q} \rangle &= \langle \mathbf{u} - \mathbf{z}_p, \Pi_p^{\operatorname{div}} \mathbf{q} \rangle + \langle \mathbf{u} - \mathbf{z}_p, \mathbf{q} - \Pi_p^{\operatorname{div}} \mathbf{q} \rangle \\
&= \langle \operatorname{div} \Pi_p^{\operatorname{div}} \mathbf{q}, f_p \rangle + \langle \mathbf{u} - \mathbf{z}_p, \mathbf{q} - \Pi_p^{\operatorname{div}} \mathbf{q} \rangle = \langle \mathbf{u} - \mathbf{z}_p, \mathbf{q} - \Pi_p^{\operatorname{div}} \mathbf{q} \rangle \\
&\le \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{L}^2(\Gamma)} \, \|\mathbf{q} - \Pi_p^{\operatorname{div}} \mathbf{q}\|_{\mathbf{L}^2(\Gamma)} = \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{L}^2(\Gamma)} \, \|\mathbf{q} - \Pi_p^{\operatorname{div}} \mathbf{q}\|_{\mathbf{H}(\operatorname{div},\Gamma)} \\
&\le C p^{-(r-\varepsilon_1)} \Big(\|\mathbf{q}\|_{\mathbf{H}^r(\Gamma)} + \|\operatorname{div} \mathbf{q}\|_{H^r(\Gamma)} \Big) \, \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{L}^2(\Gamma)} \\
&= C p^{-(r-\varepsilon_1)} \|\mathbf{q}\|_{\mathbf{H}^r(\Gamma)} \, \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{L}^2(\Gamma)} \\
&\le C p^{-(r-\varepsilon_1)} \|\mathbf{w}\|_{\mathbf{H}^s(\Gamma)} \, \|\mathbf{u} - \mathbf{z}_p\|_{\mathbf{L}^2(\Gamma)}, \qquad 0 < \varepsilon_1 < r;
\end{aligned} \tag{7.9}$$

for the last step we used (7.7).

To estimate the second term on the right-hand side of (7.8) we use (5.9), (7.5) and apply Lemma 5.1:

$$\begin{aligned}\langle \operatorname{div}(\mathbf{u}-\mathbf{z}_p), \varphi\rangle &= \langle \operatorname{div}(\mathbf{u}-\mathbf{z}_p), \varphi - \Pi^0_{p-1}\varphi\rangle \le \|\operatorname{div}(\mathbf{u}-\mathbf{z}_p)\|_{L^2(\Gamma)}\, \|\varphi - \Pi^0_{p-1}\varphi\|_{L^2(\Gamma)} \\ &\le Cp^{-(r+1)}\|\operatorname{div}(\mathbf{u}-\mathbf{z}_p)\|_{L^2(\Gamma)}\, \|\varphi\|_{H^{r+1}(\Gamma)} \\ &\le Cp^{-(r+1)}\|\operatorname{div}(\mathbf{u}-\mathbf{z}_p)\|_{L^2(\Gamma)}\, \|\mathbf{w}\|_{\mathbf{H}^s(\Gamma)}. \end{aligned} \tag{7.10}$$

Combining (7.9), (7.10) and making use of representation (7.8), we obtain by (7.2)

$$\|\mathbf{u}-\mathbf{z}_p\|_{\tilde{\mathbf{H}}^{-s}(\Gamma)} \le Cp^{-(r-\varepsilon_1)}\|\mathbf{u}-\mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)}. \tag{7.11}$$

By the same argument as in (7.10) we also prove

$$\|\operatorname{div}(\mathbf{u}-\mathbf{z}_p)\|_{\tilde{H}^{-s}(\Gamma)} = \sup_{v\in H^s(\Gamma)\setminus\{0\}} \frac{\langle \operatorname{div}(\mathbf{u}-\mathbf{z}_p), v\rangle}{\|v\|_{H^s(\Gamma)}} \le Cp^{-s}\|\operatorname{div}(\mathbf{u}-\mathbf{z}_p)\|_{L^2(\Gamma)}. \tag{7.12}$$

Setting $r := s$ and combining (7.11), (7.12) we derive

$$\|\mathbf{u}-\mathbf{z}_p\|_{\tilde{\mathbf{H}}^{-s}(\operatorname{div},\Gamma)} \le Cp^{-(s-\varepsilon_1)}\|\mathbf{u}-\mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)}.$$

Hence, using (7.1) with $s = 0$ to estimate the norm $\|\mathbf{u}-\mathbf{z}_p\|_{\mathbf{H}(\operatorname{div},\Gamma)}$, we prove the assertion of the theorem for any $s \in (0, \frac{1}{2})$. For $s = \frac{1}{2}$ the assertion then immediately follows, because $\|\cdot\|_{\tilde{\mathbf{H}}^{-1/2}(\operatorname{div},\Gamma)} \le \|\cdot\|_{\tilde{\mathbf{H}}^{-1/2+\varepsilon}(\operatorname{div},\Gamma)}$ for any small $\varepsilon > 0$. □

7.2 General approximation result

By (2.2)–(2.5) we conclude that any singular function $\mathbf{u}^s$ in (2.1) ($s = e$, v, or ev) can be written as

$$\mathbf{u}^s = \mathbf{curl}\, w^s + \mathbf{v}^s = \mathbf{curl}\, w^s + (v_1^s,\, v_2^s), \tag{7.13}$$

where $w^s \in \tilde{H}^{1/2}(\Gamma)$ for $s = e,\, v,\, ev$, $\mathbf{v}^s \in \tilde{\mathbf{H}}^{1/2}(\Gamma)$ for $s = e,\, ev$, and $\mathbf{v}^s \in \tilde{\mathbf{H}}^{1/2}_{\perp}(\Gamma)$ for $s = v$. It is important to note that the functions w^s, v_1^s, v_2^s ($s = e,\, v,\, ev$) are scalar singularities inherent to the solution of the boundary integral equation with hypersingular integral operator for the Laplacian on Γ (or on a closed piecewise plane surface $\tilde{\Gamma} \supset \Gamma$) and with possibly singular right-hand side. Polynomial approximations of these scalar singularities in fractional order Sobolev spaces were analysed in [7, 6].

In the following theorem we prove a general approximation result for the vector function $\mathbf{u}$ given by (2.1)–(2.5).

Theorem 7.2 *Let the function* $\mathbf{u}$ *be given by* (2.1)–(2.5) *on* Γ *with* $\gamma_1^e,\, \gamma_2^e > 0$ *and* $\lambda_1^v,\, \lambda_2^v > -\frac{1}{2}$. *Also, let* $v_0 \in V$, $e_0 \in E(v_0)$ *be such that*

$$\min\{\lambda_1^{v_0} + 1/2, \lambda_2^{v_0} + 1/2, \gamma_1^{e_0}, \gamma_2^{e_0}\} = \min_{v\in V, e\in E(v)} \min\{\lambda_1^v + 1/2, \lambda_2^v + 1/2, \gamma_1^e, \gamma_2^e\}.$$

Then for any small $\varepsilon > 0$ *and for every* $p \geq 1$ *there exists* $\mathbf{u}_p \in \mathbf{X}_p(\Gamma)$ *such that*

$$\|\mathbf{u} - \mathbf{u}_p\| \leq C \max\Big\{p^{-(k+1/2-\varepsilon)}, \\ p^{-2\min\{\lambda_1^{v0}+1/2,\lambda_2^{v0}+1/2,\gamma_1^{e0},\gamma_2^{e0}\}}(1+\log p)^{\beta}\Big\}, \tag{7.14}$$

where β *is defined by* (2.11) *and the constant* $C > 0$ *is independent of* p.

Proof. If $p = 1$, then we set $\mathbf{u}_p \equiv \mathbf{0}$ on Γ, and (7.14) is valid. Let $p \geq 2$. For the partition $\{\Gamma_j;\ j = 1,\ldots,J\}$ of Γ we define

$$S_p(\Gamma) := \{v \in C^0(\Gamma);\ v|_{\Gamma_j} \circ T_j \in \mathcal{P}_p(Q),\ j = 1,\ldots,J\} \quad \text{and} \quad S_p^0(\Gamma) := S_p(\Gamma) \cap H_0^1(\Gamma).$$

Let us also define the following functions of p:

$$f_j^e(p) := p^{-2\gamma_j^e}(1+\log p)^{s_j^e}, \quad f_j^v(p) := p^{-2(\lambda_j^v+1/2)}(1+\log p)^{q_j^v},$$

$$f_j^{ev}(p) := p^{-2\min\{\lambda_j^v+1/2,\gamma_j^e\}}(1+\log p)^{\tilde\beta_j}, \quad \tilde\beta_j := \begin{cases} q_j^v + s_j^e + \frac{1}{2} & \text{if } \lambda_j^v = \gamma_j^e - \frac{1}{2}, \\ q_j^v + s_j^e & \text{otherwise.} \end{cases} \tag{7.15}$$

Here, $j = 1, 2$ and γ_j^e, λ_j^v, s_j^e, q_j^v are the same numbers as in (2.2)–(2.5).

Any singular vector field $\mathbf{u}^s$ ($s = e$, v, or ev) in (2.1) can be decomposed as in (7.13). We first use the results of [7, 6] to find piecewise polynomial approximations to the scalar functions w^s ($s = e, v, ev$) and v_i^s ($i = 1, 2$, $s = e, ev$) (see Theorems 3.3, 3.5 in [7] and Theorem 4.1, Remark 4.1 in [6]): there exist $w_p^s \in S_p^0(\Gamma)$ ($s = e, v, ev$) and $v_{i,p}^s \in S_{p-1}^0(\Gamma)$ ($i = 1, 2$, $s = e, ev$) such that

$$\|w^s - w_p^s\|_{\tilde H^{1/2}(\Gamma)} \leq C\, f_1^s(p), \qquad s = e,\ v,\ ev, \tag{7.16}$$

$$\|v_i^s - v_{i,p}^s\|_{\tilde H^{1/2}(\Gamma)} \leq C\, f_2^s(p-1) \leq C\, f_2^s(p), \qquad s = e,\ ev, \quad i = 1, 2, \tag{7.17}$$

where $C > 0$ is a positive constant independent of p.

Let $\mathbf{v}_p^s = (v_{1,p}^s,\ v_{2,p}^s)$ for $s = e, ev$. We observe that $\mathbf{v}_p^s \in \mathbf{H}(\mathrm{div}, \Gamma)$, $\mathbf{v}_p^s \cdot \mathbf{n}|_{\partial\Gamma} = 0$, and for any element Γ_j there holds

$$\mathcal{M}_j^{-1}(\mathbf{v}_p^s|_{\Gamma_j}) = \det(B_j) B_j^{-1}(\mathbf{v}_p^s|_{\Gamma_j}) \circ T_j \in \mathcal{P}_{p-1}(Q) \times \mathcal{P}_{p-1}(Q) \subset \mathbf{V}_p^{\mathrm{RT}}(Q).$$

Therefore $\mathbf{v}_p^s \in \mathbf{X}_p(\Gamma)$ for $s = e, ev$. Moreover, since $\mathbf{v}^s \in \tilde{\mathbf{H}}^{1/2}(\Gamma)$ and $v_{i,p}^s \in S_{p-1}^0(\Gamma)$ for $s = e, ev$ and $i = 1, 2$, we estimate by (7.17)

$$\|\mathbf{v}^s - \mathbf{v}_p^s\|_{\tilde{\mathbf{H}}_\perp^{1/2}(\Gamma)} \leq C\|\mathbf{v}^s - \mathbf{v}_p^s\|_{\tilde{\mathbf{H}}^{1/2}(\Gamma)} \leq C \sum_{i=1}^{2} \|v_i^s - v_{i,p}^s\|_{\tilde H^{1/2}(\Gamma)} \leq C\, f_2^s(p), \quad s = e,\ ev. \tag{7.18}$$

The vector field $\mathbf{v}^v \in \tilde{\mathbf{H}}_\perp^{1/2}(\Gamma)$ is approximated directly by applying Theorem B.1 (see Appendix B): there exists $\mathbf{v}_p^v \in \mathbf{X}_p(\Gamma)$ such that

$$\|\mathbf{v}^v - \mathbf{v}_p^v\|_{\tilde{\mathbf{H}}_\perp^{1/2}(\Gamma)} \leq C\, f_2^v(p). \tag{7.19}$$

On the other hand, it is easy to check that for $s = e,\, v,\, ev$

$$\begin{aligned}
\mathcal{M}_j^{-1}(\mathbf{curl}\, w_p^s|_{\Gamma_j}) &= \det(B_j)B_j^{-1}(\mathbf{curl}\, w_p^s|_{\Gamma_j}) \circ T_j \\
&= \det(B_j)B_j^{-1}\Big(\tfrac{\partial w_p^s}{\partial x_{e2}}|_{\Gamma_j},\, -\tfrac{\partial w_p^s}{\partial x_{e1}}|_{\Gamma_j}\Big)^T \circ T_j \\
&= \det(B_j)B_j^{-1}\Big(\nabla \hat{w}_{p,j}^s \cdot \mathbf{b}_{j,2},\, -\nabla \hat{w}_{p,j}^s \cdot \mathbf{b}_{j,1}\Big)^T,
\end{aligned}$$

where $\hat{w}_{p,j}^s = w_p^s|_{\Gamma_j} \circ T_j$ and $\mathbf{b}_{j,k}$ is the k-th column of the matrix B_j^{-1}. Hence

$$\begin{aligned}
\mathcal{M}_j^{-1}(\mathbf{curl}\, w_p^s|_{\Gamma_j}) &= \det(B_j)\Big(\nabla \hat{w}_{p,j}^s \cdot (0,\, \det(B_j^{-1})),\, -\nabla \hat{w}_{p,j}^s \cdot (\det(B_j^{-1}),\, 0)\Big) \\
&= \Big(\tfrac{\partial \hat{w}_{p,j}^s}{\partial \xi_2},\, -\tfrac{\partial \hat{w}_{p,j}^s}{\partial \xi_1}\Big).
\end{aligned}$$

Since $\hat{w}_{p,j}^s \in \mathcal{P}_{p,p}(Q)$, we have $\mathcal{M}_j^{-1}(\mathbf{curl}\, w_p^s|_{\Gamma_j}) \in \mathcal{P}_{p,p-1}(Q) \times \mathcal{P}_{p-1,p}(Q) = \mathbf{V}_p^{\mathrm{RT}}(Q)$. Moreover, $\mathbf{curl}\, w_p^s \cdot \mathbf{n}|_{\partial\Gamma} = 0$, because w_p^s vanishes on $\partial\Gamma$ and $\mathbf{curl}\, w_p^s \in \mathbf{H}(\mathrm{div}, \Gamma)$, because $\mathrm{div}(\mathbf{curl}\, w_p^s) \equiv 0$ on Γ. Therefore $\mathbf{curl}\, w_p^s \in \mathbf{X}_p(\Gamma)$ for $s = e,\, v,\, ev$.

Thus, we have proved that $\mathbf{u}_p^s := \mathbf{curl}\, w_p^s + \mathbf{v}_p^s \in \mathbf{X}_p(\Gamma)$ for $s = e,\, v,\, ev$. To derive the error estimate for this approximation we recall that the operators $\mathbf{curl} : \tilde{H}^{1/2}(\Gamma) \to \tilde{\mathbf{H}}^{-1/2}(\Gamma)$ and $\mathrm{div} : \tilde{\mathbf{H}}_\perp^{1/2}(\Gamma) \to \tilde{H}^{-1/2}(\Gamma)$ are continuous (see [12]). Therefore, one has

$$\begin{aligned}
\|\mathbf{u}^s - \mathbf{u}_p^s\| &\le \|\mathbf{curl}\,(w^s - w_p^s)\| + \|\mathbf{v}^s - \mathbf{v}_p^s\| \\
&= \|\mathbf{curl}\,(w^s - w_p^s)\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} + \|\mathbf{v}^s - \mathbf{v}_p^s\|_{\tilde{\mathbf{H}}^{-1/2}(\Gamma)} + \|\mathrm{div}(\mathbf{v}^s - \mathbf{v}_p^s)\|_{\tilde{H}^{-1/2}(\Gamma)} \\
&\le C\left(\|w^s - w_p^s\|_{\tilde{H}^{1/2}(\Gamma)} + \|\mathbf{v}^s - \mathbf{v}_p^s\|_{\mathbf{L}^2(\Gamma)} + \|\mathbf{v}^s - \mathbf{v}_p^s\|_{\tilde{\mathbf{H}}_\perp^{1/2}(\Gamma)}\right).
\end{aligned}$$

Hence we obtain by (7.16), (7.18), and (7.19)

$$\|\mathbf{u}^s - \mathbf{u}_p^s\| \le C \max\Big\{f_1^s(p),\; f_2^s(p)\Big\}, \quad s = e,\, v,\, ev. \tag{7.20}$$

For the regular part $\mathbf{u}_{\mathrm{reg}}$ of $\mathbf{u}$ in (2.1), we use the approximation result of Theorem 7.1 giving a discrete vector function $\mathbf{u}_{\mathrm{reg},p} \in \mathbf{X}_p(\Gamma)$ which satisfies

$$\|\mathbf{u}_{\mathrm{reg}} - \mathbf{u}_{\mathrm{reg},p}\| \le C\, p^{-(k+1/2-\varepsilon)}\, \|\mathbf{u}_{\mathrm{reg}}\|_{\mathbf{H}^k(\mathrm{div},\Gamma)}, \qquad \varepsilon > 0. \tag{7.21}$$

Setting

$$\mathbf{u}_p := \mathbf{u}_{\mathrm{reg},p} + \sum_{e\in E} \mathbf{u}_p^e + \sum_{v\in V} \mathbf{u}_p^v + \sum_{v\in V}\sum_{e\in E(v)} \mathbf{u}_p^{ev} \in \mathbf{X}_p(\Gamma),$$

combining estimates (7.20), (7.21), using expressions (7.15) for the functions $f_j^s(p)$ in (7.20), and applying the triangle inequality, we prove (7.14). □

A Singularities of electromagnetic fields on surfaces

Throughout this section we denote by Γ a piecewise smooth (open or closed) Lipschitz surface in $\mathbf{R}^3$. Assuming that Γ has plane faces $\Gamma^{(i)}$ and straight edges e_j, we derive expressions for typical edge and vertex singularities inherent to the solution of the electric field integral equation on Γ.

If Γ is a closed surface, we will denote by Ω the Lipschitz polyhedron bounded by Γ, i.e., $\Gamma = \partial\Omega$. In the case of an open surface Γ, we first introduce a piecewise plane closed Lipschitz surface $\tilde{\Gamma}$ which contains Γ, and then denote by Ω the Lipschitz polyhedron bounded by $\tilde{\Gamma}$, i.e., $\tilde{\Gamma} = \partial\Omega$. For each face $\Gamma^{(i)} \subset \Gamma$ there exists a constant unit normal vector $\boldsymbol{\nu}_i$, which is an outer normal vector to Ω. These vectors are then blended into a unit normal vector $\boldsymbol{\nu}$ defined almost everywhere on Γ.

In addition to Sobolev spaces introduced in Section 3.1 we define

$$\mathbf{L}^2_t(\Gamma) := \{\mathbf{u} \in \mathbf{L}^2(\Gamma);\ \mathbf{u}\cdot\boldsymbol{\nu} = 0 \text{ on } \Gamma\}.$$

To apply trace arguments we will need the "tangential components trace" mapping $\pi_\tau : \mathbf{C}^\infty(\bar{\Omega}) \to \mathbf{L}^2_t(\Gamma)$ and the "tangential trace" mapping $\gamma_\tau : \mathbf{C}^\infty(\bar{\Omega}) \to \mathbf{L}^2_t(\Gamma)$, which are defined as $\mathbf{u} \mapsto \boldsymbol{\nu} \times (\mathbf{u} \times \boldsymbol{\nu})|_\Gamma$ and $\mathbf{u}|_\Gamma \times \boldsymbol{\nu}$, respectively (here, $(\cdot)|_\Gamma$ denotes the standard trace operator acting on vector fields, $(\cdot)|_\Gamma : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s-1/2}(\Gamma)$ for $s \in (\frac{1}{2}, \frac{3}{2})$). The adjoint operator for the mapping π_τ is denoted by i_π; this operator identifies two-dimensional tangential vector fields (sections of the tangent bundle of Γ) with three-dimensional vector fields on Γ (having zero normal component).

In this section, among the tangential differential operators defined on Γ we will need the vector surface curl,

$$\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2_t(\Gamma),$$

which is defined by localisation to each face $\Gamma^{(i)}$ (see [11, 12] for the definition and properties of this operator for both closed and open surfaces).

Now let us consider the vector field $\mathbf{U} = (U_1, U_2, U_3)$ and let $\mathbf{U}$ be the magnetic part of the electromagnetic field solving the boundary value problem for the time-harmonic Maxwell equations in the interior and/or exterior of Ω (if Γ is closed) or in $\mathbf{R}^3\backslash\Gamma$ (if Γ is open). It is known that the jump of the magnetic field $\mathbf{U}$ across Γ solves the EFIE on Γ. We will denote the solution of this boundary integral equation (in its variational formulation) by $\mathbf{u}$.

The function $\mathbf{U}$ has a singular behaviour near corners and edges of Γ. Let us recall the explicit formulas for these singularities which are given in [18]. To that end we fix a vertex v and an edge $\bar{e} \ni v$ of Γ. In a neighbourhood of v, the polyhedron Ω coincides locally with a polyhedral cone Γ_v, and in a neighbourhood of e, Ω coincides locally with a wedge $W_e = \Gamma_e \times \mathbf{R}$, where Γ_e is a plane sector of opening $\omega_e \neq \pi$. We will use three local coordinate systems with origin v: Cartesian coordinates (x, y, z) such that $Oz \supset e$ with $O = (0, 0, 0)$, spherical coordinates $(\rho_v, \theta_v, \varphi_v)$ corresponding to Γ_v, and cylindrical coordinates (r_e, θ_e, z_e) corresponding to W_e.

According to [18, Definition 4.5], the edge singularities of the magnetic field $\mathbf{U}$ can be written as

$$\chi^e(r_e, z_e)\mathbf{U}^{\gamma,k}_e, \quad k = 1,\, 2,\, 3, \tag{A.1}$$

where $\chi^e(r_e, z_e)$ is a C^∞ cut-off function with support away from vertices and other edges of $\partial\Omega$, $\chi^e(r_e, z_e) = 1$ in a neighbourhood of a point on e, and $\mathbf{U}_e^{\gamma,k}$ are generating functions of the following types (cf. [18, Lemma 4.4]):

$$\text{Type 1:} \quad \mathbf{U}_e^{\gamma,1} = (\mathbf{U}_T^{\gamma,+}, 0) = (\nabla\Psi_{\text{Neu}}^{\gamma+1}, 0), \qquad \gamma + 1 \in \Lambda_{\text{Neu}}(\Gamma_e),\ \gamma > -1; \tag{A.2}$$

$$\text{Type 2:} \quad \mathbf{U}_e^{\gamma,2} = (\mathbf{0}, \Psi_{\text{Neu}}^{\gamma}), \qquad \gamma \in \Lambda_{\text{Neu}}(\Gamma_e),\ \gamma > 0; \tag{A.3}$$

$$\text{Type 3:} \quad \mathbf{U}_e^{\gamma,3} = (\mathbf{U}_T^{\gamma,-}, 0), \qquad \gamma - 1 \in \Lambda_{\text{Neu}}(\Gamma_e),\ \gamma > 1; \tag{A.4}$$

here $\Psi_{\text{Neu}}^{\gamma}$ are the Neumann Laplace plane singularities in Γ_e, $\Lambda_{\text{Neu}}(\Gamma_e)$ is the corresponding set of singular exponents, and $\mathbf{U}_T^{\gamma,\pm}$ are the magnetic Maxwell plane singularities in Γ_e. Below we will also need the Dirichlet Laplace plane singularities in Γ_e denoted by $\Psi_{\text{Dir}}^{\gamma}$ and the corresponding set $\Lambda_{\text{Dir}}(\Gamma_e)$ of singular exponents. One has (see [18, Lemmas 2.1 and 2.2])

$$\Lambda_{\text{Dir}}(\Gamma_e) = \Lambda_{\text{Neu}}(\Gamma_e) = \begin{cases} \{\frac{k\pi}{\omega_e},\ k \in Z,\ k \neq 0\} & \text{if } \omega_e \neq 2\pi, \\ \{\frac{k}{2},\ k < 0 \text{ or } k \text{ odd}\} & \text{if } \omega_e = 2\pi. \end{cases} \tag{A.5}$$

Then for any $\bar\gamma \in \Lambda_{\text{Dir}}(\Gamma_e)$ and $\gamma \in \Lambda_{\text{Neu}}(\Gamma_e)$ there holds

$$\Psi_{\text{Dir}}^{\bar\gamma}(r_e, \theta_e) = \begin{cases} r_e^{\bar\gamma} \sin\bar\gamma\theta_e & \text{if } \bar\gamma \notin N, \\ r_e^{\bar\gamma}(\log r_e \sin\bar\gamma\theta_e + \theta_e\cos\bar\gamma\theta_e) - \frac{1}{\omega_e}\Big(-\frac{y}{\sin\omega_e}\Big)^{\bar\gamma} & \text{if } \bar\gamma \in N \end{cases} \tag{A.6}$$

and

$$\Psi_{\text{Neu}}^{\gamma}(r_e, \theta_e) = \begin{cases} r_e^{\gamma} \cos\gamma\theta_e & \text{if } \gamma \notin N, \\ r_e^{\gamma}(\log r_e \cos\gamma\theta_e - \theta_e\sin\gamma\theta_e) + \frac{1}{\omega_e}\Big(-\frac{y}{\sin\omega_e}\Big)^{\gamma} & \text{if } \gamma \in N. \end{cases} \tag{A.7}$$

If $\gamma + 1 \in \Lambda_{\text{Neu}}(\Gamma_e)$ then (cf. [18, Lemma 3.1])

$$\mathbf{U}_T^{\gamma,+} = \begin{cases} (r_e^{\gamma}\cos\gamma\theta_e,\ -r_e^{\gamma}\sin\gamma\theta_e) & \text{if } \gamma \notin N, \\ \Big(r_e^{\gamma}(\log r_e\cos\gamma\theta_e - \theta_e\sin\gamma\theta_e),\ -r_e^{\gamma}(\log r_e\sin\gamma\theta_e + \theta_e\cos\gamma\theta_e)\Big) & \text{if } \gamma \in N; \end{cases} \tag{A.8}$$

if $\gamma - 1 \in \Lambda_{\text{Neu}}(\Gamma_e)$ then

$$\mathbf{U}_T^{\gamma,-} = \begin{cases} (r_e^{\gamma}\cos\gamma\theta_e,\ r_e^{\gamma}\sin\gamma\theta_e) & \text{if } \gamma \notin N, \\ \Big(r_e^{\gamma}(\log r_e\cos\gamma\theta_e - \theta_e\sin\gamma\theta_e),\ r_e^{\gamma}(\log r_e\sin\gamma\theta_e + \theta_e\cos\gamma\theta_e)\Big) & \text{if } \gamma \in N. \end{cases} \tag{A.9}$$

The vertex singularities corresponding to non-integer magnetic Maxwell singular exponents have the form (cf. [18, Lemma 4.1])

$$\chi^v(\rho_v)\mathbf{U}_v^{\lambda,k}, \quad k = 1,\ 2,\ 3, \tag{A.10}$$

where $\chi^v(\rho_v)$ is a C^∞ cut-off function such that $\chi^v(\rho_v) = 1$ in a neighbourhood of the vertex v, and $\mathbf{U}_v^{\lambda,k}$ are generating functions of the following three types:

$$\text{Type 1:} \quad \mathbf{U}_v^{\lambda,1} = \nabla\Phi_{\text{Neu}}^{\lambda+1}, \qquad \lambda + 1 \in \Lambda_{\text{Neu}}(\Gamma_v),\ \lambda > -3/2; \tag{A.11}$$

$$\text{Type 2:}\quad \mathbf{U}_v^{\lambda,2} = \nabla\Phi_{\rm Dir}^{\lambda}\times\mathbf{x}, \qquad \lambda\in\Lambda_{\rm Dir}(\Gamma_v),\ \lambda>-1/2; \tag{A.12}$$

$$\text{Type 3:}\quad \mathbf{U}_v^{\lambda,3} = (2\lambda-1)\Phi_{\rm Neu}^{\lambda-1}\,\mathbf{x}-\rho_v^2\,\nabla\Phi_{\rm Neu}^{\lambda-1}, \qquad \lambda-1\in\Lambda_{\rm Neu}(\Gamma_v),\ \lambda>1/2; \tag{A.13}$$

here $\mathbf{x}=(x,y,z)$, $\Phi_{\rm Dir}^{\lambda}=\rho_v^{\lambda}\phi_{\rm Dir}(\theta_v,\varphi_v)$ are the Dirichlet Laplace vertex singularities in Γ_v with $\lambda\in\Lambda_{\rm Dir}(\Gamma_v)=\Big\{-\frac12\pm\sqrt{\mu+\frac14};\ \mu\in\sigma(\Delta_{G_v}^{\rm Dir})\Big\}$, $\sigma(\Delta_{G_v}^{\rm Dir})$ is the spectrum of the Laplace-Beltrami operator with Dirichlet conditions on a spherical polygonal domain $G_v:=\Gamma_v\cap\mathbf{S}^2$, $\mathbf{S}^2$ is the unit sphere centred in v, $\phi_{\rm Dir}(\theta_v,\varphi_v)$ spans the eigenspace of $\Delta_{G_v}^{\rm Dir}$ corresponding to the eigenvalue $\mu=\lambda(\lambda+1)$, and $\Phi_{\rm Neu}^{\lambda\pm1}$, $\Lambda_{\rm Neu}(\Gamma_v)$, $\phi_{\rm Neu}(\theta_v,\varphi_v)$ are the corresponding Neumann analogues.

Note that the eigenfunctions of the Laplace-Beltrami operator on G_v (subject to Dirichlet or Neumann boundary conditions on ∂G_v) can be decomposed into singular functions (corner singularities on G_v) and a smooth remainder (see (3.24) in [36]). We will specify this decomposition locally in the neighbourhood of our fixed edge-vertex pair (e,v). This can be done by using a C^∞ cut-off function $\chi^{ev}(\theta_v)$ such that $\chi^{ev}(\theta_v)=1$ in a neighbourhood of $\theta_v=0$. One has

$$\phi_{\rm b.c.}(\theta_v,\varphi_v)=w(\theta_v,\varphi_v)+\chi^{ev}(\theta_v)\sum_{\gamma+2q<s_0}c_{\gamma,q}\,\theta_v^{2q}\,\Psi_{\rm b.c.}^{\gamma}(\theta_v,\varphi_v). \tag{A.14}$$

Here, $w\in H^{1+d}(\Gamma_v)$ with $d=\min\{\gamma+2q;\ \gamma\in\Lambda_{\rm b.c.}(\Gamma_e)\cap(0,+\infty),\ q\ge0 \text{ integer},\ \gamma+2q\ge s_0\}$ for some given $s_0>0$, $c_{\gamma,q}\in\mathbf{R}$, the subscript "b.c." refers to the type of the boundary condition applied (b.c. = Dir or b.c. = Neu), so that the sets $\Lambda_{\rm b.c.}(\Gamma_e)$ are defined by (A.5), and the functions $\Psi_{\rm b.c.}^{\gamma}$ are defined by (A.6) or (A.7).

Since $\sin\theta_v\simeq\theta_v$ for small values of θ_v we have the following decomposition of $\phi_{\rm b.c.}(\theta_v,\varphi_v)$ which is equivalent to (A.14):

$$\phi_{\rm b.c.}(\theta_v,\varphi_v)=w(\theta_v,\varphi_v)+\chi^{ev}(\theta_v)\sum_{\gamma+2q<s_0}c_{\gamma,q}\,\sin^{2q}\theta_v\,\Psi_{\rm b.c.}^{\gamma}(\sin\theta_v,\varphi_v). \tag{A.15}$$

For positive integer exponents λ, the generating functions $\mathbf{U}_v^{\lambda,k}$ in (A.11)–(A.13) will include additional singular terms of the type $\rho_v^{\lambda}\log\rho_v$. These terms appear due to corresponding logarithmic singularities for the Laplacian (cf. (3.5) in [36]).

Now we use the above formulas to find the expressions for corresponding singularities in the solution $\mathbf{u}$ of the EFIE on Γ. Let us fix a face $\Gamma^{(1)}\subset\Gamma$ such that $e\subset\partial\Gamma^{(1)}$. Thus, $\Gamma^{(1)}$ is a plane open surface with polygonal boundary. We assume that $\Gamma^{(1)}\subset Oyz$. Then $\boldsymbol{\nu}_1=(1,0,0)$ and using trace arguments (see [11, 12]) one has on $\Gamma^{(1)}$:

$$\mathbf{u}=i_\pi^{-1}(\gamma_\tau(\mathbf{U}))=i_\pi^{-1}(\mathbf{U}|_{\Gamma^{(1)}}\times\boldsymbol{\nu}_1)=\Big(U_3|_{\Gamma^{(1)}},\ -U_2|_{\Gamma^{(1)}}\Big). \tag{A.16}$$

Using (A.16) with (A.2)–(A.4), (A.7)–(A.9) and recalling that on the face $\Gamma^{(1)}\subset Oyz$ there holds $r_e=y$, $\theta_e=\frac{\pi}{2}$, $z_e=z$, we obtain the generating functions for the corresponding edge singularities on $\Gamma^{(1)}$:

$$\mathbf{u}_e^{\gamma,1}=(0,y^{\gamma}\sin\tfrac{\pi\gamma}{2}), \qquad \gamma+1\in\Lambda_{\rm Neu}(\Gamma_e)\backslash N,\ \gamma>-1;$$

$$\mathbf{u}_e^{\gamma,2} = (y^\gamma \cos\tfrac{\pi\gamma}{2}, 0), \qquad \gamma \in \Lambda_{\text{Neu}}(\Gamma_e)\backslash N,\ \gamma > 0;$$

$$\mathbf{u}_e^{\gamma,3} = (0, -y^\gamma \sin\tfrac{\pi\gamma}{2}), \qquad \gamma - 1 \in \Lambda_{\text{Neu}}(\Gamma_e)\backslash N,\ \gamma > 1.$$

For integral singular exponents, the expressions for $\mathbf{u}_e^{\gamma,k}$ ($k = 1,2,3$) will also include $\log y$-factors, cf. (A.7)–(A.9). Summarising the above and using (A.1), we now write the expression for the edge singularity $\mathbf{u}^e$ on $\Gamma^{(1)}$ in a more general form:

$$\begin{aligned} \mathbf{u}^e \ &= \ \sum_{j=1}^{m_{e,1}} \sum_{s=0}^{s_j^{e,1}} \Big(\tilde{b}^e_{j,s,1}(z) y^{\gamma_j^{e,1}},\ \tilde{b}^e_{j,s,2}(z) y^{\gamma_j^{e,1}-1}\Big)\, |\log y|^s\, \chi_1^e(z)\, \chi_2^e(y) \\ &\quad + \sum_{j=1}^{m_{e,2}} \sum_{s=0}^{s_j^{e,2}} \Big(0,\ \tilde{b}^e_{j,s}(z) y^{\gamma_j^{e,2}}\, |\log y|^s\, \chi_1^e(z)\, \chi_2^e(y)\Big), \end{aligned} \tag{A.17}$$

where the singularity exponents satisfy $\gamma_{j+1}^{e,1} \geq \gamma_j^{e,1} \geq \frac{1}{2}$, $\gamma_{j+1}^{e,2} \geq \gamma_j^{e,2} \geq \frac{3}{2}$, and $m_{e,1}$, $m_{e,2}$, $s_j^{e,1} \geq 0$, $s_j^{e,2} \geq 0$ are integers. Here, χ_1^e, χ_2^e are C^∞ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of e and $\chi_1^e = 0$ in a neighbourhood of these vertices. Moreover, $\chi_2^e = 1$ for $0 \leq y \leq \delta_e$ and $\chi_2^e = 0$ for $y \geq 2\delta_e$ with some $\delta_e \in (0, \frac{1}{2})$. The functions $\tilde{b}^e_{j,s}\chi_1^e$, $\tilde{b}^e_{j,s,1}\chi_1^e$, $\tilde{b}^e_{j,s,2}\chi_1^e \in H^{\tilde{m}}(e)$ for $\tilde{m}$ as large as required.

Note that for any given smooth functions $\tilde{b}_1(z)$ and $\tilde{b}_2(z)$ there exist sufficiently smooth scalar functions $b_1(z)$, $b_2(z)$ and a smooth vector function $\mathbf{f}_{\text{reg}}(y,z)$ such that for any $\gamma \neq 0$ there holds

$$\Big(\tilde{b}_1(z) y^\gamma,\ \tilde{b}_2(z) y^{\gamma-1}\Big)\chi_2^e(y) = \mathbf{curl}_{\Gamma^{(1)}}\Big(y^\gamma\, b_1(z)\, \chi_2^e(y)\Big) + \Big(y^\gamma\, b_2(z)\, \chi_2^e(y),\ 0\Big) + \mathbf{f}_{\text{reg}}(y,z),$$

where $\mathbf{curl}_{\Gamma^{(1)}} = (\partial/\partial z,\ -\partial/\partial y)$ and $\mathbf{f}_{\text{reg}}(y,z)$ is a smooth function with both components vanishing in a δ_e-neighbourhood of the edge e. Using the analogous formula with incorporated logarithmic terms we can write the pure edge singularity $\mathbf{u}^e$ in (A.17) as

$$\begin{aligned} \mathbf{u}^e \ &= \ \sum_{j=1}^{m_{e,1}} \sum_{s=0}^{s_j^{e,1}} \mathbf{curl}_{\Gamma^{(1)}} \Big(y^{\gamma_j^{e,1}}\, |\log y|^s\, b^e_{j,s,1}(z)\, \chi_1^e(z)\, \chi_2^e(y)\Big) \\ &\quad + \sum_{j=1}^{m_{e,2}} \sum_{s=0}^{s_j^{e,2}} y^{\gamma_j^{e,2}}\, |\log y|^s\, \mathbf{b}^e_{j,s,2}(z)\, \chi_1^e(z)\, \chi_2^e(y), \end{aligned} \tag{A.18}$$

where $m_{e,1}$, $m_{e,2}$, $s_j^{e,1}$, $s_j^{e,2}$, χ_1^e, χ_2^e are as in (A.17), $\gamma_{j+1}^{e,1} \geq \gamma_j^{e,1} \geq \frac{1}{2}$, $\gamma_{j+1}^{e,2} \geq \gamma_j^{e,2} \geq \frac{1}{2}$, $b^e_{j,s,1}\chi_1^e \in H^{m_1}(e)$ and $\mathbf{b}^e_{j,s,2}\chi_1^e \in \mathbf{H}^{m_2}(e)$ for m_1 and m_2 as large as required.

Now we proceed to the vertex singularities of $\mathbf{u}$. Let us focus on the case where λ is not a positive integer (according to [18, Lemma 4.1], $\lambda = 0$ and $\lambda = -1$ do not belong to the set of singular exponents).

Let $\lambda > -3/2$ and $\lambda + 1 \in \Lambda_{\text{Neu}}(\Gamma_v)$. Then, using (A.16) with (A.10), (A.11), one has for vertex singularities of the first type:

$$\mathbf{u}_v^{\lambda,1} = \chi^v(\rho_v)\left(\frac{\partial \Phi_{\text{Neu}}^{\lambda+1}}{\partial z}\bigg|_{\varphi_v=\frac{\pi}{2}}, -\frac{\partial \Phi_{\text{Neu}}^{\lambda+1}}{\partial y}\bigg|_{\varphi_v=\frac{\pi}{2}}\right) = \chi^v(\rho_v)\,\mathbf{curl}_{\Gamma^{(1)}}\left(\rho_v^{\lambda+1}\phi_{\text{Neu}}(\theta_v, \tfrac{\pi}{2})\right). \tag{A.19}$$

Observe that

$$\chi^v(\rho_v)\,\mathbf{curl}_{\Gamma^{(1)}} f(\rho_v,\theta_v) = \frac{\partial \chi^v}{\partial \rho_v} f(\rho_v,\theta_v)\begin{pmatrix} -\cos\theta_v \\ \sin\theta_v \end{pmatrix} + \mathbf{curl}_{\Gamma^{(1)}}\left(\chi^v(\rho_v) f(\rho_v,\theta_v)\right).$$

Hence, using decomposition (A.15) of $\phi_{\text{Neu}}(\theta_v,\phi_v)$ with non-integers $\gamma \in \Lambda_{\text{Neu}}(\Gamma_e)\cap(0,+\infty)$ we write (A.19) as

$$\begin{aligned}
\mathbf{u}_v^{\lambda,1} &= \rho_v^{\lambda+1}\frac{\partial \chi^v}{\partial \rho_v}\left(w(\theta_v,\tfrac{\pi}{2}) + \sum_{\gamma+2q<s_0} c_{\gamma,q}\sin^{2q}\theta_v\,\Psi^{\gamma}_{\text{Neu}}(\sin\theta_v,\tfrac{\pi}{2})\chi^{ev}(\theta_v)\right)\begin{pmatrix} -\cos\theta_v \\ \sin\theta_v \end{pmatrix} \\
&\quad + \sum_{\gamma+2q<s_0} \mathbf{curl}_{\Gamma^{(1)}}\left(c_{\gamma,q}\rho_v^{\lambda+1}\sin^{2q}\theta_v\,\Psi^{\gamma}_{\text{Neu}}(\sin\theta_v,\tfrac{\pi}{2})\chi^v(\rho_v)\chi^{ev}(\theta_v)\right) \\
&\quad + \mathbf{curl}_{\Gamma^{(1)}}\left(\rho_v^{\lambda+1}\chi^v(\rho_v) w(\theta_v,\tfrac{\pi}{2})\right) \\
&= \mathbf{u}_0 + \sum_{\gamma+2q<s_0}\mathbf{a}_{\gamma,q}(y,z) + \sum_{\gamma+2q<s_0}\mathbf{curl}_{\Gamma^{(1)}}\left(\tilde{c}^{(1)}_{\gamma,q}\rho_v^{\lambda+1}\sin^{\gamma+2q}\theta_v\chi^v(\rho_v)\chi^{ev}(\theta_v)\right) \\
&\quad + \mathbf{curl}_{\Gamma^{(1)}}\left(\rho_v^{\lambda+1}\chi^v(\rho_v)\tilde{w}_1(\theta_v)\right).
\end{aligned} \tag{A.20}$$

Let us discuss the terms on the right-hand side of (A.20). We assume that s_0 is large enough, so that all smooth functions described below are as regular as required.

1. The function $\mathbf{u}_0$ is smooth, because $\partial\chi^v/\partial\rho_v = 0$ near the vertex.
2. For the same reason as in the previous step and due to the fact that $\rho_v \sin\theta_v|_{\Gamma^{(1)}} = y$, one can write $\mathbf{a}_{\gamma,q}$ as edge singularities

   $$\mathbf{a}_{\gamma,q}(y,z) = \mathbf{k}_{\gamma,q}(y,z)y^{\gamma+2q}\chi_1^e(z)\chi_2^e(y),$$

   where $\mathbf{k}_{\gamma,q}$ is smooth and χ_1^e, χ_2^e are the same as in (A.17).
3. One has $\tilde{c}^{(1)}_{\gamma,q} \in \mathbf{R}$. Then, using an idea from von Petersdorff, cf. [35, (2.22)–(2.24)], we rewrite the terms $\rho_v^{\lambda+1}\sin^{\gamma+2q}\theta_v$ as

   $$\rho_v^{\lambda+1}\sin^{\gamma+2q}\theta_v = \sum_{l=0}^{L} b_l z^{\lambda+1-\gamma-2q-2l}y^{\gamma+2q+2l} + \rho_v^{\lambda+1}w_1(\theta_v), \tag{A.21}$$

   where w_1 is a sufficiently smooth function, b_l are real numbers, and $L \geq 0$ is an integer that depends on the needed regularity of w_1; moreover, w_1 vanishes along with its derivatives up to a certain order (depending on L) at $\theta_v = 0$.

4. The function $\tilde{w}_1$ (see the last term in (A.20)) is smooth.

For vertex singularities of the second type, we use (A.16) with (A.10), (A.12) and then apply (A.15) to decompose $\phi_{\rm Dir}(\theta_v,\varphi_v)$. As a result, we have for $\lambda \in \Lambda_{\rm Dir}(\Gamma_v)$, $\lambda > -1/2$ and for non-integers $\gamma \in \Lambda_{\rm Dir}(\Gamma_e) \cap (0,+\infty)$:

$$
\begin{aligned}
\mathbf{u}_v^{\lambda,2} &= \rho_v^\lambda\,\chi^v(\rho_v) \begin{pmatrix} -\frac{\partial\phi_{\rm Dir}}{\partial\varphi_v}\big|_{\varphi_v=\frac{\pi}{2}} \\ -\left(\frac{\partial\phi_{\rm Dir}}{\partial\varphi_v}\frac{\sin\varphi_v\,\cos\theta_v}{\sin\theta_v} - \frac{\partial\phi_{\rm Dir}}{\partial\theta_v}\,\cos\varphi_v\right)\Big|_{\varphi_v=\frac{\pi}{2}} \end{pmatrix} \\
&= \rho_v^\lambda\,\chi^v(\rho_v)\,\mathbf{w}_2(\theta_v) + \chi^v(\rho_v)\,\chi^{ev}(\theta_v) \sum_{\gamma+2q<s_0} \tilde{c}^{(2)}_{\gamma,q}\,\rho_v^\lambda \begin{pmatrix} \sin^{\gamma+2q}\theta_v \\ \sin^{\gamma+2q-1}\theta_v\,\cos\theta_v \end{pmatrix} \qquad \text{(A.22)}
\end{aligned}
$$

where $\mathbf{w}_2(\theta_v) = (w_{2,1}(\theta_v), w_{2,2}(\theta_v))$ with $w_{2,1}$ and $w_{2,2}$ sufficiently smooth due to Dirichlet boundary conditions for $\phi_{\rm Dir}(\theta_v,\varphi_v)$ at $\theta_v = 0$ and at $\varphi_v = \frac{\pi}{2}$, $\tilde{c}^{(2)}_{\gamma,q} \in \mathbf{R}$, and the terms $\rho_v^\lambda \sin^{\gamma+2q}\theta_v$, $\rho_v^\lambda \sin^{\gamma+2q-1}\theta_v$ can be treated similarly to (A.21).

Analogously, for vertex singularities of the third type, we obtain by (A.16), (A.10), (A.13), and (A.15) for $\lambda > 1/2$ such that $(\lambda-1) \in \Lambda_{\rm Neu}(\Gamma_v)$ and for non-integers $\gamma \in \Lambda_{\rm Neu}(\Gamma_e)\cap(0,+\infty)$:

$$
\begin{aligned}
\mathbf{u}_v^{\lambda,3} &= \rho_v^\lambda\,\chi^v(\rho_v) \begin{pmatrix} \left(\lambda\phi_{\rm Neu}\,\cos\theta_v + \frac{\partial\phi_{\rm Neu}}{\partial\theta_v}\,\sin\theta_v\right)\Big|_{\varphi_v=\frac{\pi}{2}} \\ -\left(\lambda\phi_{\rm Neu}\,\sin\theta_v\,\sin\varphi_v - \frac{\partial\phi_{\rm Neu}}{\partial\varphi_v}\frac{\cos\varphi_v}{\sin\theta_v} - \frac{\partial\phi_{\rm Neu}}{\partial\theta_v}\,\cos\theta_v\,\sin\varphi_v\right)\Big|_{\varphi_v=\frac{\pi}{2}} \end{pmatrix} \\
&= \rho_v^\lambda\,\chi^v(\rho_v)\,\mathbf{w}_3(\theta_v) + \chi^v(\rho_v)\,\chi^{ev}(\theta_v) \sum_{\gamma+2q<s_0} \tilde{c}^{(3)}_{\gamma,q}\,\rho_v^\lambda \begin{pmatrix} \sin^{\gamma+2q}\theta_v\,f_1(\theta_v) \\ \sin^{\gamma+2q-1}\theta_v\,f_2(\theta_v) \end{pmatrix} \qquad \text{(A.23)}
\end{aligned}
$$

where $\mathbf{w}_3(\theta_v)$ is as smooth as necessary, $\tilde{c}^{(3)}_{\gamma,q} \in \mathbf{R}$, f_1 and f_2 are smooth (trigonometric) functions of θ_v. As before, the terms $\rho_v^\lambda \sin^{\gamma+2q}\theta_v$ and $\rho_v^\lambda \sin^{\gamma+2q-1}\theta_v$ can be dealt with as in (A.21).

For integer values of $\gamma \in \Lambda_{\rm Dir}(\Gamma_e) = \Lambda_{\rm Neu}(\Gamma_e)$ the expressions for $\phi_{\rm Dir}(\theta_v,\varphi_v)$ and $\phi_{\rm Neu}(\theta_v,\varphi_v)$ include additional terms with $\log(\sin\theta_v)$-factors, appearing due to corresponding terms in (A.6) and (A.7). This results in additional terms with $\log y$-factors in the first expression in (A.21) and an additional term with $\log(\sin\theta_v)$-factor in the last term in (A.21). The latter produces the function $w_1(\theta_v)\,\log(\sin\theta_v)$, which is as smooth as necessary due to the high order root of w_1 at $\theta_v = 0$.

If λ is a positive integer, then similarly as above the expression for $\mathbf{u}_v^{\lambda,k}$ will include additional $\rho_v^\lambda \log\rho_v$-singularities.

Thus, summarising the above, we conclude that the singular fields $\mathbf{u}_v^{\lambda,k}$ $(k = 1,2,3)$ on $\Gamma^{(1)}$ comprise two main contributions (we omit the smooth remainder $\mathbf{u}_0$ and the edge singularities

$\mathbf{a}_{\gamma,q}$ appearing in (A.20)): these are purely radial singularities of the type $\rho_v^\lambda \log^s \rho_v$ (for each component of the singular field) in a neighbourhood of the vertex and the combined edge-vertex singularities of the type $z^{\lambda-\gamma} y^\gamma \log y$ in a neighbourhood of the edge-vertex. The purely radial singularity $\mathbf{u}^v$ on $\Gamma^{(1)}$ can be written in the following general form

$$\begin{aligned}\mathbf{u}^v \;&=\; \sum_{i=1}^{n_{v,1}} \sum_{t=0}^{q_i^{v,1}} B_{it}^{v,1} \, \mathbf{curl}_{\Gamma^{(1)}} \Big(\rho_v^{\lambda_i^{v,1}+1} \, |\log \rho_v|^t \, \chi^v(\rho_v) \, \chi_{1,i,t}^v(\theta_v) \Big) \\ &\quad + \sum_{i=1}^{n_{v,2}} \sum_{t=0}^{q_i^{v,2}} B_{it}^{v,2} \, \rho_v^{\lambda_i^{v,2}} \, |\log \rho_v|^t \, \chi^v(\rho_v) \, \boldsymbol{\chi}_{2,i,t}^v(\theta_v), \end{aligned} \tag{A.24}$$

where $\lambda_{i+1}^{v,1} \geq \lambda_i^{v,1} > -\frac{3}{2}$, $\lambda_{i+1}^{v,2} \geq \lambda_i^{v,2} > -\frac{1}{2}$, and $n_{v,1}$, $n_{v,2}$, $q_i^{v,1} \geq 0$, $q_i^{v,2} \geq 0$ are integers, $B_{it}^{v,1}$, $B_{it}^{v,2}$ are real numbers, and $\chi^v(\rho_v)$ is the same cut-off function as in (A.10). The functions $\chi_{1,i,t}^v \in H^{t_1}(0,\omega_v)$ and $\boldsymbol{\chi}_{2,i,t}^v \in \mathbf{H}^{t_2}(0,\omega_v)$ for t_1 and t_2 as large as required. Here, ω_v denotes the interior angle (on $\Gamma^{(1)}$) between the edges meeting at v.

To specify the combined edge-vertex singularities of the form $z^{\lambda-\gamma} y^\gamma \log y$ mentioned above let us define the cut-off functions χ^v and χ^{ev} such that $\chi^v = 1$ for $0 \leq \rho_v \leq \tau_v$ and $\chi^v = 0$ for $\rho_v \geq 2\tau_v$ with some $\tau_v \in (0, \frac{1}{2})$; $\chi^{ev} = 1$ for $0 \leq \theta_v \leq \beta_v$ and $\chi^{ev} = 0$ for $\frac{3}{2}\beta_v \leq \theta_v \leq \omega_v$ with some $\beta_v \in (0, \min\{\omega_v/2, \pi/8\}]$. First, without loss of generality we omit logarithmic factors and consider the edge-vertex singularities in the sector $S_{ev}^0 = \{(\rho_v, \theta_v, \frac{\pi}{2});\; 0 < \rho_v < \tau_v,\, 0 < \theta_v < \beta_v\}$ (note that $\chi^v = \chi^{ev} = 1$ in S_{ev}^0). It follows from the above presentation that these singularities are either vector curls of $z^{\lambda_1+1-\gamma_1} y^{\gamma_1}$ with $\gamma_1 \geq \frac{1}{2}$, $\lambda_1 > -\frac{3}{2}$ (cf. (A.20), (A.21)) or vector functions $\mathbf{u} = (C_1 z^{\lambda_2-\gamma_2} y^{\gamma_2},\, C_2 z^{\lambda_2-\gamma_2+1} y^{\gamma_2-1})$ with $\gamma_2 \geq \frac{1}{2}$, $\lambda_2 > -\frac{1}{2}$, and $C_1, C_2 \in \mathbf{R}$ (cf. (A.22), (A.23)). In the latter case there holds

$$\begin{aligned}\mathbf{u} \;&=\; C_1\, (z^{\lambda_2-\gamma_2} y^{\gamma_2},\, 0) + C_2\, (0,\, z^{\lambda_2-\gamma_2+1} y^{\gamma_2-1}) \\ &=\; C_3\, \mathbf{curl}_{\Gamma^{(1)}} (z^{\lambda_2+1-\gamma_2} y^{\gamma_2}) + C_4\, (z^{\lambda_2-\gamma_2} y^{\gamma_2},\, 0) \end{aligned} \tag{A.25}$$

for some real numbers C_3, C_4. Note that cut-off functions can be easily incorporated in (A.25): we then obtain additional terms on the right-hand side. These terms correspond to pure edge singularities $(y^{\gamma_2},\, y^{\gamma_2})$ and purely radial singularities $(\rho_v^{\lambda_2},\, \rho_v^{\lambda_2})$.

Thus, we conclude that any edge-vertex singularity in tensor product form can be represented as a linear combination of the vector curl of $z^{\lambda+1-\gamma} y^\gamma$ with $\gamma \geq \frac{1}{2}$, $\lambda > -\frac{3}{2}$ and vector functions $(z^{\lambda'-\gamma'} y^{\gamma'},\, 0)$ with $\gamma' \geq \frac{1}{2}$, $\lambda' > -\frac{1}{2}$. We write these singularities in the following general form:

$$\begin{aligned}\mathbf{u}_1^{ev} &= \sum_{j=1}^{m_{e,1}} \sum_{i=1}^{n_{v,1}} \sum_{s=0}^{s_j^{e,1}} \sum_{t=0}^{q_i^{v,1}} \sum_{l=0}^{s} B_{ijlts}^{ev,1} \, \mathbf{curl}_{\Gamma^{(1)}} \Big(z^{\lambda_i^{v,1}+1-\gamma_j^{e,1}} \, y^{\gamma_j^{e,1}} \, |\log z|^{s+t-l} \, |\log y|^l \, \chi^v(\rho_v) \, \chi^{ev}(\theta_v) \Big) \\ &\quad + \sum_{j=1}^{m_{e,2}} \sum_{i=1}^{n_{v,2}} \sum_{s=0}^{s_j^{e,2}} \sum_{t=0}^{q_i^{v,2}} \sum_{l=0}^{s} B_{ijlts}^{ev,2} \, z^{\lambda_i^{v,2}-\gamma_j^{e,2}} \, y^{\gamma_j^{e,2}} \, |\log z|^{s+t-l} \, |\log y|^l \, \chi^v(\rho_v) \, \chi^{ev}(\theta_v) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned} \tag{A.26}$$

where s is an integer, $B^{ev,1}_{ijlts}$, $B^{ev,2}_{ijlts}$ are real numbers, and all remaining parameters as well as the cut-off functions χ^v, χ^{ev} are as before.

It follows from the main regularity result of [18] (see Theorem 4.7 therein) that, besides edge singularities (A.1) and vertex singularities (A.10), the decomposition of the vector field $\mathbf{U}$ also includes edge-vertex singularities. These singularities can be written using a convolution operator analogously to the Laplace problem (cf. Theorem 2.6 and Theorem 4.7 in [18]). In [36, Theorem 6] it has been shown that these combined singularities (for the Laplace problem) can be written as a combination of singularities of the following two types (for simplicity, we omit logarithmic factors):

$$C\,\rho_v^{\lambda^v-\gamma^e}\,y^{\gamma^e}\,\chi^v(\rho_v)\,\chi^{ev}(\theta_v) \tag{A.27}$$

and

$$\alpha(\rho_v)\,y^{\gamma^e}\,\chi^v(\rho_v)\,\chi^{ev}(\theta_v), \tag{A.28}$$

where γ^e and λ^v are the exponents for corresponding pure edge and pure radial singularities, respectively, χ^v and χ^{ev} are the same cut-off functions as above, $C \in \mathbf{R}$, and $\alpha \in H^{s-\gamma^e}_{-\gamma^e}(0,+\infty)$. Here, $H^{s-\gamma^e}_{-\gamma^e}(0,+\infty) \subset H^{s-\gamma^e}_{\text{loc}}(0,+\infty)$ is a weighted Sobolev space and s is as large as needed.

Analogously, we conclude that both components of the combined edge-vertex singularities of the vector field $\mathbf{U}$ can be written as combinations of (A.27) and (A.28). Taking traces (see (A.16)) and treating the terms $\rho_v^{\lambda^v-\gamma^e}\,y^{\gamma^e} = \rho_v^{\lambda^v}\,\sin^{\gamma^e}\theta_v$ as in (A.21), it is easy to see that the singularities of $\mathbf{u}$ generated by (A.27) are covered by (A.26). To deal with the singularities generated by (A.28) we note that $\alpha \in H^{s-\gamma^e}_{-\gamma^e}(0,+\infty)$ has a sufficiently high order root at 0 for s large enough, see [32, page 731]. Then, for a properly selected cut-off function $\chi^e_2(y)$ in (A.18), one has (cf. [32, page 734])

$$\alpha(\rho_v)\,\chi^v(\rho_v)\,\chi^{ev}(\theta_v) = \chi(y,z)\,\chi^e_2(y).$$

Here, χ is a function that can be smoothly extended by zero onto $\tilde{\Gamma}^{(1)} := \{(0,y,z);\ y>0\}$ to lie in $H^s(\tilde{\Gamma}^{(1)})$. Thus, the edge-vertex singularities generated by (A.28) extend the corresponding edge singularities (A.18) till the vertex v. They can be written in the general form

$$\begin{aligned}\mathbf{u}^{ev}_2 \;&=\; \sum_{j=1}^{m_{e,1}}\sum_{s=0}^{s^{e,1}_j}\mathbf{curl}_{\Gamma^{(1)}}\Big(y^{\gamma^{e,1}_j}\,|\log y|^s\,\chi^e_{j,s,1}(y,z)\,\chi^e_2(y)\Big)\\ &\quad+\sum_{j=1}^{m_{e,2}}\sum_{s=0}^{s^{e,2}_j} y^{\gamma^{e,2}_j}\,|\log y|^s\,\boldsymbol{\chi}^e_{j,s,2}(y,z)\,\chi^e_2(y).\end{aligned} \tag{A.29}$$

Here, $m_{e,1}$, $m_{e,2}$, $\gamma^{e,1}_j$, $\gamma^{e,2}_j$, $s^{e,1}_j$, $s^{e,2}_j$, and χ^e_2 are as in (A.18). The functions $\chi^e_{j,s,1}$ and $\boldsymbol{\chi}^e_{j,s,2}$, when extended by zero onto $\tilde{\Gamma}^{(1)}$, lie in $H^{m_1}(\tilde{\Gamma}^{(1)})$ and $\mathbf{H}^{m_2}(\tilde{\Gamma}^{(1)})$, respectively, with m_1, m_2 as large as required.

Remark A.1 *We note that the supports of* $\mathbf{u}^{ev}_1$ *and* $\mathbf{u}^{ev}_2$ *are subsets of the plane sector* $\bar{S}_{ev} = \{(\rho_v,\theta_v,\frac{\pi}{2});\ 0 \le \rho_v \le 2\tau_v,\ 0 \le \theta_v \le \frac{3}{2}\beta_v\}$.

Remark A.2 *If* Γ *is an open surface and the face* $\Gamma^{(1)}$ *is such that* $\partial\Gamma \cap \partial\Gamma^{(1)}$ *contains at least one edge then an additional assumption must be imposed on the smooth functions* $\chi^v_{1,i,t}$ *and* $\boldsymbol{\chi}^v_{2,i,t}$ *in* (A.24) *to guarantee that the normal component of* $\mathbf{u}^v$ *vanishes on* $\partial\Gamma \cap \partial\Gamma^{(1)}$. *Note that one can always obtain the normal components of the edge singularity and of the combined edge-vertex singularities to be vanishing on* $\partial\Gamma \cap \partial\Gamma^{(1)}$ *by a proper choice of the corresponding cut-off functions.*

B Approximation of vertex singularities

In this appendix we analyse the polynomial approximation of the radially singular vector field $\mathbf{v} \in \tilde{\mathbf{H}}^{1/2}_{\perp}(\Gamma)$. The results below are needed for the approximation analysis of the vertex singularities given by (2.3).

Let us fix a vertex v of Γ. Denoting by (r, θ) local polar coordinates (with origin at v), we consider the singular vector field

$$\mathbf{u} = (u_1, u_2) = r^{\lambda} |\log r|^{\beta} \chi(r) \mathbf{w}(\theta), \tag{B.1}$$

where $\lambda > -1/2$, $\beta \geq 0$ is an integer, χ is a C^{∞} cut-off function such that $\chi = 1$ for $0 \leq r \leq \tau$ and $\chi = 0$ for $r \geq 2\tau$ with some $\tau \in (0, \frac{1}{2})$, $\mathbf{w} = (w_1, w_2) \in \mathbf{H}^m(0, \omega_v)$ for m as large as required, and $\mathbf{w} \cdot \mathbf{n}|_{\partial\Gamma} = 0$. If $\lambda = 0$, we assume that β is a positive integer, so that $\mathbf{u}$ has only a logarithmic singularity in this case. Note that the function $\mathbf{u}$ in (B.1) corresponds to the second term in (2.3).

Let $\bar{A}_v := \cup\{\bar{\Gamma}_j;\ v \in \bar{\Gamma}_j\}$. Assuming that the cut-off function χ in (B.1) is such that $\operatorname{supp} \mathbf{u} \subset \bar{A}_v$, we study approximations of $\mathbf{u}$ by $\mathbf{H}_0(\mathrm{div}, \Gamma)$-conforming vector fields with piecewise polynomial components. Our analysis relies on the p-approximation result for scalar vertex singularities

$$u = r^{\lambda} |\log r|^{\beta} \chi(r) w(\theta), \tag{B.2}$$

where λ, β, χ are as in (B.1) and $w \in H^m(0, \omega_v)$ with m as large as required. This result is formulated in the following lemma (see [6, Theorem 3.2]).

Lemma B.1 *Let* $\Gamma_j \subset A_v$ *and let* u *be given by* (B.2) *on* Γ_j. *Then there exists a sequence* $u_p \in \mathcal{P}_p(\Gamma_j)$, $p = 1, 2, \ldots$, *such that for* $0 \leq s < \min\{1, \lambda + 1\}$

$$\|u - u_p\|_{H^s(\Gamma_j)} \leq C\, p^{-2(\lambda+1-s)} (1 + \log p)^{\beta}.$$

Moreover, $u_p(0,0) = 0$, $u_p = 0$ *on the sides* $l_i \subset \partial\Gamma_j$, $\bar{l}_i \not\ni v$, *and*

$$\|u - u_p\|_{L^2(l_k)} \leq C\, p^{-2(\lambda+1/2)} (1 + \log p)^{\beta} \quad \text{for each side } l_k \subset \partial\Gamma_j,\ v \in \bar{l}_k.$$

We will also need the following lemma, which is proved in [32, Lemma 9.2].

Lemma B.2 *Let K be a parallelogram, and let l be a side of K with vertices A, B. Let $w_p \in \mathcal{P}_p(l)$ be such that $w_p(A) = w_p(B) = 0$ and $\|w_p\|_{L^2(l)} \le f(p)$. Then there exists $u_p \in \mathcal{P}_p(K)$ such that $u_p = w_p$ on l, $u_p = 0$ on $\partial K \backslash l$, and for $0 \le s \le 1$*

$$\|u_p\|_{H^s(K)} \le C\, p^{-1+2s}\, f(p).$$

Now we are ready to state and prove the needed approximation result for the singular vector field $\mathbf{u}$ in (B.1).

Theorem B.1 *Let $\mathbf{u}$ be given by (B.1) with $\lambda > -\frac{1}{2}$ and an integer $\beta \ge 0$. Then there exists a sequence $\boldsymbol{\phi} \in \mathbf{X}_p(\Gamma)$, $p = 1, 2, \ldots$, such that*

$$\|\mathbf{u} - \boldsymbol{\phi}\|_{\tilde{\mathbf{H}}_{\perp}^{1/2}(\Gamma)} \le C\, p^{-2(\lambda+1/2)}\, (1 + \log p)^{\beta}. \tag{B.3}$$

Proof. If $p = 1$, then we set $\boldsymbol{\phi} \equiv \mathbf{0}$ on Γ, and (B.3) holds. Let $p \ge 2$. First, we approximate $\mathbf{u}$ component-wise on a separate element $K \subset A_v$. Let $\mathcal{A}(K) = \{l_i\}$ contain those sides $l_i \subset \partial K$ for which $v \in \bar{l}_i$, and let $\mathcal{B}(K)$ be the union of the other sides of K. Then, applying Lemma B.1 to each component u_k ($k = 1, 2$) of the vector field $\mathbf{u}$ on K, we find polynomials $\varphi_k \in \mathcal{P}_{p-1}(K)$ such that $\varphi_k = 0$ at the vertex v and on the sides $l_i \in \mathcal{B}(K)$. Moreover, for $k = 1, 2$

$$\|u_k - \varphi_k\|_{H^s(K)} \le C\, p^{-2(\lambda+1-s)}\, (1 + \log p)^{\beta}, \quad 0 \le s < \min\{1, \lambda + 1\}, \tag{B.4}$$

$$\|u_k - \varphi_k\|_{L^2(l)} \le C\, p^{-2(\lambda+1/2)}\, (1 + \log p)^{\beta} \quad \text{for every } l \in \mathcal{A}(K). \tag{B.5}$$

Suppose now that $\Gamma_1, \Gamma_2 \subset A_v$ are two elements having the common edge $\tilde{l} = \bar{\Gamma}_1 \cap \bar{\Gamma}_2 \in \mathcal{A}(\Gamma_1) \cap \mathcal{A}(\Gamma_2)$ and assume that $\mathcal{A}(\Gamma_1) = \{l, \tilde{l}\}$ with $l \subset e$. Here, e is an edge of Γ such that $v \in \bar{e}$. (The case of a convex corner with only one element can be dealt with analogously.) Let us denote by $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ (respectively, by $\boldsymbol{\psi} = (\psi_1, \psi_2)$) the above component-wise approximation of $\mathbf{u}$ on Γ_1 (respectively, on Γ_2). We will adjust the function $\boldsymbol{\varphi}$ on Γ_1 to find an $\mathbf{H}(\mathrm{div}, \Gamma_1 \cup \Gamma_2)$-conforming vector field having zero normal component on $l \subset e$.

We denote by $\mathbf{n} = (n_1, n_2)$ and $\tilde{\mathbf{n}} = (\tilde{n}_1, \tilde{n}_2)$ the unit outer normal vectors to Γ_1 for edges l and $\tilde{l}$, respectively. It is clear that $|n_1 \tilde{n}_2| + |\tilde{n}_1 n_2| > 0$. Therefore, without loss of generality we can assume that $\tilde{n}_1 \ne 0$ and $n_2 \ne 0$.

Recalling that $\mathbf{u} \cdot \mathbf{n}|_l = (u_1 n_1 + u_2 n_2)|_l = 0$, we consider the normal trace

$$g_1 = \boldsymbol{\varphi} \cdot \mathbf{n}|_l = (\varphi_1 n_1 + \varphi_2 n_2)|_l$$

vanishing at the end-points of l. One has by (B.5)

$$\|g_1\|_{L^2(l)} = \|(u_1 - \varphi_1) n_1 + (u_2 - \varphi_2) n_2\|_{L^2(l)} \le C\, p^{-2(\lambda+1/2)}\, (1 + \log p)^{\beta}.$$

Then we use Lemma B.2 to find a polynomial $z_1 \in \mathcal{P}_{p-1}(\Gamma_1)$ such that

$$z_1 = g_1 \;\text{ on } l, \qquad z_1 = 0 \;\text{ on } \partial\Gamma_1 \backslash l,$$

and for $0 \le s \le 1$

$$\|z_1\|_{H^s(\Gamma_1)} \le C\, p^{-2(\lambda+1-s)}\, (1+\log p)^{\beta}. \tag{B.6}$$

We set

$$\tilde{\varphi}_2 = \varphi_2 - \tfrac{1}{n_2}\, z_1 \quad \text{on} \ \ \Gamma_1. \tag{B.7}$$

Then $\tilde{\varphi}_2 \in \mathcal{P}_{p-1}(\Gamma_1)$,

$$(\varphi_1 n_1 + \tilde{\varphi}_2 n_2)|_l = (\varphi_1 n_1 + \varphi_2 n_2)|_l - z_1|_l = 0,$$

$\tilde{\varphi}_2 = \varphi_2$ on $\partial\Gamma_1 \backslash l$, and the norms $\|u_2 - \tilde{\varphi}_2\|_{H^s(\Gamma_1)}$, $\|u_2 - \tilde{\varphi}_2\|_{L^2(\tilde{l})}$ are bounded as in (B.4), (B.5), respectively.

Now we consider the jump of the normal component along the common side $\tilde{l}$ of Γ_1 and Γ_2

$$g_2 = [\boldsymbol{\psi} - (\varphi_1, \tilde{\varphi}_2)] \cdot \tilde{\mathbf{n}}|_{\tilde{l}} = [(\psi_1 - \varphi_1)\tilde{n}_1 + (\psi_2 - \varphi_2)\tilde{n}_2]|_{\tilde{l}},$$

which vanishes at the end points of $\tilde{l}$. Using again Lemma B.2 we find a polynomial $z_2 \in \mathcal{P}_{p-1}(\Gamma_1)$ such that

$$z_2 = g_2 \ \ \text{on} \ \tilde{l}, \qquad z_2 = 0 \ \ \text{on} \ \partial\Gamma_1 \backslash \tilde{l},$$

and the norm $\|z_2\|_{H^s(\Gamma_1)}$ for $0 \le s \le 1$ is bounded as in (B.6).

Then we define the vector function $\boldsymbol{\phi} = (\phi_1, \phi_2)$ on $\Gamma_1 \cup \Gamma_2$ as follows

$$\boldsymbol{\phi} = \boldsymbol{\psi} \ \ \text{on} \ \Gamma_2, \qquad \boldsymbol{\phi} = \Big(\varphi_1 + \tfrac{1}{\tilde{n}_1}\, z_2,\ \tilde{\varphi}_2\Big) \ \ \text{on} \ \Gamma_1$$

with $\tilde{\varphi}_2$ defined by (B.7). It is easy to see that $\phi_1 = \phi_2 = 0$ on $\mathcal{B}(\Gamma_1) \cup \mathcal{B}(\Gamma_2)$, $\boldsymbol{\phi} \cdot \tilde{\mathbf{n}}$ is continuous along $\tilde{l}$ (thus, $\boldsymbol{\phi} \in \mathbf{H}(\mathrm{div}, \Gamma_1 \cup \Gamma_2)$), $\boldsymbol{\phi} \cdot \mathbf{n}|_l = 0$, and for $j = 1, 2$ there holds

$$\mathcal{M}_j^{-1}(\boldsymbol{\phi}|_{\Gamma_j}) = \det(B_j) B_j^{-1}(\boldsymbol{\phi}|_{\Gamma_j}) \circ T_j \in \mathcal{P}_{p-1}(Q) \times \mathcal{P}_{p-1}(Q) \subset \mathbf{V}_p^{\mathrm{RT}}(Q).$$

Moreover, the norms $\|u_k - \phi_k\|_{H^s(\Gamma_1)}$ and $\|u_k - \phi_k\|_{H^s(\Gamma_2)}$ are bounded as in (B.4) for $k = 1, 2$ and for any $0 \le s < \min\{1, \lambda+1\}$. Repeating the above procedure for all elements $\Gamma_j \subset A^v$ we construct a vector function $\boldsymbol{\phi} = (\phi_1, \phi_2) \in \mathbf{X}_p(A_v)$ such that $\phi_1 = \phi_2 = 0$ on $\partial A_v \backslash \partial\Gamma$ and for $k = 1, 2$

$$\|u_k - \phi_k\|_{H^s(\Gamma_j)} \le C\, p^{-2(\lambda+1-s)}\, (1+\log p)^{\beta}, \quad 0 \le s < \min\{1, \lambda+1\}. \tag{B.8}$$

Now we extend both components of $\boldsymbol{\phi}$ by zero onto $\Gamma \backslash A_v$ (keeping the notation $\boldsymbol{\phi}$ for the extension). Then $\boldsymbol{\phi} \in \mathbf{X}_p(\Gamma)$ and for $0 \le s < \min\{1, \lambda+1\}$ there holds

$$\|\mathbf{u} - \boldsymbol{\phi}\|_{\tilde{\mathbf{H}}^s_\perp(\Gamma)} \le C\, p^{-2(\lambda+1-s)}\, (1+\log p)^{\beta}. \tag{B.9}$$

In fact, for $s = 0$ estimate (B.9) on Γ immediately follows from inequalities (B.8) on individual elements. If $1/2 < s < \min\{1, \lambda+1\}$, we use the fact that $\mathbf{H}^s_{\perp,0}(\Gamma) = \tilde{\mathbf{H}}^s_\perp(\Gamma)$ for these values of s and then apply Lemma 3.1 of [7] to each component of $(\mathbf{u} - \boldsymbol{\phi})$:

$$\|\mathbf{u} - \boldsymbol{\phi}\|^2_{\tilde{\mathbf{H}}^s_\perp(\Gamma)} \le C \|\mathbf{u} - \boldsymbol{\phi}\|^2_{\mathbf{H}^s(\Gamma)} \le C \sum_{k=1}^{2} \sum_{j:\Gamma_j \subset A_v} \|u_k - \phi_k\|^2_{H^s(\Gamma_j)}.$$

Then (B.9) follows again from (B.8).

Finally, for $0 < s \le 1/2$ estimate (B.9) is obtained via interpolation between $\mathbf{H}^0(\Gamma) = \tilde{\mathbf{H}}^0_\perp(\Gamma)$ and $\tilde{\mathbf{H}}^{s'}_\perp(\Gamma)$ with some $s' \in (\frac{1}{2}, \min\{1, \lambda + 1\})$. Taking $s = \frac{1}{2}$ in (B.9) we prove (B.3). $\square$

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