A direct coupling of local discontinuous Galerkin and boundary element methods*

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Abstract

The coupling of local discontinuous Galerkin (LDG) and boundary element methods (BEM), which has been developed recently to solve linear and nonlinear exterior transmission problems, employs a mortar-type auxiliary unknown to deal with the weak continuity of the traces at the interface boundary. As a consequence, the main features of LDG and BEM are maintained and hence the coupled approach benefits from the advantages of both methods. In this paper we propose a direct procedure that, instead of a mortar variable, makes use of a finite element subspace whose functions are required to be continuous only on the coupling boundary. In this way, the normal derivative becomes the only boundary unknown, and hence the total number of unknown functions is reduced by two. We prove the stability of the new discrete scheme and derive an a priori error estimate in the energy norm. The analysis is also extended to the case of nonlinear problems.

Key words: boundary elements, local discontinuous Galerkin method, coupling, error estimates

Mathematics subject classifications (1991): 65N30, 65N38, 65N12, 65N15

1 Introduction

The coupling of local discontinuous Galerkin and boundary element methods, as applied to linear exterior boundary value problems in the plane, has been introduced and analyzed for the first time in [15]. The model problem there is the Poisson equation in an annular domain coupled with the Laplace equation in the surrounding unbounded exterior region. The corresponding extension to a class of nonlinear-linear exterior transmission problems, which is also motivated by previous applications of the LDG method to some nonlinear problems in heat conduction and fluid mechanics (see, e.g. [5], [6], and [21]), was developed recently in [7], [8], and [9]. In these works, the authors consider a nonlinear elliptic equation in divergence form in an annular region coupled with discontinuous transmission conditions on the interface boundary and the Poisson equation in the exterior unbounded domain. In both the linear and nonlinear cases the technique employed resembles the usual coupling of finite element and boundary element methods, but the corresponding analysis becomes quite different. In particular, in order to deal with the weak continuity of the traces at the coupling boundary, a mortar-type auxiliary unknown representing an interior approximation of the normal derivative needs to be

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defined. Hence, different mesh sizes on that boundary and special relationships between them are required. In addition, the continuity and ellipticity estimates of the bilinear form involved hold with different mesh-dependent norms, and Strang-type a priori error estimates instead of the usual Céa's ones are obtained.

In the present paper we simplify the approach from [15] and develop a direct procedure for the coupling of LDG and BEM which does not make use of any mortar unknown but, instead, employs a finite element subspace with functions that are required to be continuous only on the coupling boundary Γ. Consequently, the normal derivative becomes the only boundary unknown and then the total number of unknown functions is reduced by two. In order to introduce the model problem let Ω_0 be a simply connected and bounded domain in $\mathbb{R}^2$ with polygonal boundary Γ_0. Then, given $f \in L^2(\mathbb{R}^2 \setminus \bar{\Omega}_0)$ with compact support, we consider the exterior Dirichlet problem:

$$\begin{gathered} -\Delta u = f \quad \text{in} \quad \mathbb{R}^2 \setminus \bar{\Omega}_0, \quad u = 0 \quad \text{on} \quad \Gamma_0\,, \\ u(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,. \end{gathered} \tag{1.1}$$

Next, let Γ be a closed polygonal curve such that the support of f is inside the annular domain Ω enclosed by Γ_0 and Γ. We assume that this support does not intersect Γ. Then (1.1) can be written as the Poisson equation in Ω:

$$-\Delta u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \Gamma_0\,, \tag{1.2}$$

and the Laplace equation in the exterior domain $\Omega_e := \mathbb{R}^2 \setminus (\bar{\Omega}_0 \cup \bar{\Omega})$:

$$-\Delta u_e = 0 \quad \text{in} \quad \Omega_e, \quad u_e(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,, \tag{1.3}$$

coupled by the transmission conditions:

$$u = u_e \quad \text{on}\, \Gamma \quad \text{and} \quad \partial_{\boldsymbol{\nu}} u = \partial_{\boldsymbol{\nu}} u_e \quad \text{on}\, \Gamma\,. \tag{1.4}$$

Here, $\partial_{\boldsymbol{\nu}} u$ denotes the normal derivative of u with normal vector pointing outside Ω. The purpose of this work is to solve numerically (1.1) by means of a new LDG-BEM coupling which, similarly to [15], consists of applying the LDG to (1.2) and the BEM to (1.3). As already mentioned the main advantage of the method to be presented here is the reduction of the total number of unknowns, whereas the advantage of the approach from [15] is the explicit splitting, through a suitable mortar variable, of the LDG and BEM modules. The remainder of this work is organized as follows. In Section 2 we introduce the boundary integral equation formulation in Ω_e, define the LDG method in Ω, and establish the resulting coupled LDG-BEM approach. Next, in Section 3 we prove the unique solvability and stability of our discrete scheme. The associated a priori error analysis is provided in Section 4. Then, in Section 5 we describe a Lagrange multiplier based implementation of the coupled scheme which maintains the discontinuous character of the LDG method. Finally, in Section 6 we extend our analysis to the class of nonlinear problems studied in [7], [8], and [9].

Throughout this paper, c and C denote positive constants, independent of the parameters and functions involved, and may take different values at different occurrences. Given any linear space V, the corresponding vector valued space $V \times V$ endowed with the product norm will be denoted by $\mathbf{V}$. If $\mathcal{O}$ is an open set, its closure, or a polygonal curve, and $s \in \mathbb{R}$, then $|\cdot|_{s,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$ denote the seminorm and norm in the Sobolev space $H^s(\mathcal{O})$. In particular, the norms of $H^s(\Gamma)$ are denoted by $\|\cdot\|_{s,\Gamma}$. Also, $\langle\cdot,\cdot\rangle$ denotes both the $L^2(\Gamma)$ inner product and its extension to the duality pairing of $H^{-s}(\Gamma) \times H^s(\Gamma)$.

2 The coupled LDG-BEM approach

2.1 The boundary integral formulation in the exterior domain

We use Green's representation formula for u_e in Ω_e,

$$u_e(\mathbf{x}) \,=\, \int_\Gamma \partial_{\boldsymbol{\nu}(\boldsymbol{y})}\, E(\mathbf{x},\boldsymbol{y})\, u(\boldsymbol{y})\, ds_{\boldsymbol{y}} \,-\, \int_\Gamma E(\mathbf{x},\boldsymbol{y})\, \lambda(\boldsymbol{y})\, ds_{\boldsymbol{y}} \,+\, c \qquad \forall\, \mathbf{x} \,\in\, \Omega_e\,, \tag{2.1}$$

where $E(\boldsymbol{x},\boldsymbol{y}) := -\dfrac{1}{2\pi}\log|\boldsymbol{x}-\boldsymbol{y}|$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$, $\lambda = \partial_{\boldsymbol{\nu}} u$, and c is a constant. Note that we made use of the transmission conditions (1.4). It is well-known that (2.1) gives rise to the following system of boundary integral equations:

$$\begin{array}{rcrcl} \mathcal{W}u & - & (\frac{1}{2}\mathcal{I}-\mathcal{K}')\lambda & = & -\lambda \quad \text{on} \quad \Gamma\,, \\ (\frac{1}{2}\mathcal{I}-\mathcal{K})u & + & \mathcal{V}\lambda \,+\, c & = & 0 \quad \text{on} \quad \Gamma\,, \end{array} \tag{2.2}$$

where $\mathcal{V}$, $\mathcal{K}$, $\mathcal{K}'$, and $\mathcal{W}$ are the boundary integral operators associated with the single, double, adjoint of the double, and hypersingular layer potentials, respectively. We recall from [12] that their main mapping properties are given by $\mathcal{V}: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $\mathcal{K}: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $\mathcal{K}': H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and $\mathcal{W}: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and that they are defined as follows:

$$\begin{array}{rcll} \mathcal{V}\mu(\mathbf{x}) & := & \displaystyle\int_\Gamma E(\mathbf{x},\boldsymbol{y})\,\mu(\boldsymbol{y})\,ds_{\boldsymbol{y}} & \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,, \quad \forall\,\mu \in H^{-1/2}(\Gamma)\,, \\ \mathcal{K}\psi(\mathbf{x}) & := & \displaystyle\int_\Gamma \partial_{\boldsymbol{\nu}(\boldsymbol{y})}\,E(\mathbf{x},\boldsymbol{y})\,\psi(\boldsymbol{y})\,ds_{\boldsymbol{y}} & \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,, \quad \forall\,\psi \in H^{1/2}(\Gamma)\,, \\ \mathcal{K}'\mu(\mathbf{x}) & := & \displaystyle\int_\Gamma \partial_{\boldsymbol{\nu}(\mathbf{x})}E(\mathbf{x},\boldsymbol{y})\,\mu(\boldsymbol{y})\,ds_{\boldsymbol{y}} & \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,, \quad \forall\,\mu \in H^{-1/2}(\Gamma)\,, \\ \mathcal{W}\psi(\mathbf{x}) & := & -\partial_{\boldsymbol{\nu}(\mathbf{x})}\displaystyle\int_\Gamma \partial_{\boldsymbol{\nu}(\boldsymbol{y})}E(\mathbf{x},\boldsymbol{y})\,\psi(\boldsymbol{y})\,ds_{\boldsymbol{y}} & \forall\,(a.e.)\,\mathbf{x} \in \Gamma\,, \quad \forall\,\psi \in H^{1/2}(\Gamma)\,. \end{array}$$

Here, $\partial_{\boldsymbol{\nu}(\mathbf{x})}$ stands for the normal derivative operator at $\mathbf{x} \in \Gamma$.

Next, according to the behaviour of u at infinity (cf. (1.1)), we observe that λ belongs to $H_0^{-1/2}(\Gamma)$ where

$$H_0^{-1/2}(\Gamma) := \{\mu \in H^{-1/2}(\Gamma) : \quad \langle \mu, 1\rangle = 0\}\,.$$

We also remark in advance that the analysis of (2.2) and its discrete counterpart below will depend on the symmetry of $\mathcal{W}$ and the ellipticity of $\mathcal{V}$ and $\mathcal{W}$:

$$\begin{array}{rcll} \langle \mathcal{W}\varphi, \psi\rangle & = & \langle \mathcal{W}\psi, \varphi\rangle & \forall\,\varphi,\,\psi \in H^{1/2}(\Gamma)\,, \\ \langle \mu, \mathcal{V}\mu\rangle & \geq & C\,\|\mu\|^2_{-1/2,\Gamma} & \forall\,\mu \in H_0^{-1/2}(\Gamma)\,, \\ \langle \mathcal{W}\psi, \psi\rangle & \geq & C\,\|\psi\|^2_{1/2,\Gamma,0} & \forall\,\psi \in H^{1/2}(\Gamma)\,, \end{array} \tag{2.3}$$

where $\|\cdot\|_{1/2,\Gamma,0}$ stands for a seminorm in $H^{1/2}(\Gamma)$. More precisely, according to the decomposition $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbb{R}$, with

$$H_0^{1/2}(\Gamma) := \{\psi \in H^{1/2}(\Gamma) : \quad \langle 1, \psi\rangle = 0\}\,,$$

we define

$$\|\psi\|_{1/2,\Gamma,0} := \|\tilde{\psi}\|_{1/2,\Gamma} \quad \forall\,\psi = \tilde{\psi} + c \,\in\, H^{1/2}(\Gamma)\,, \quad \tilde{\psi} \in H_0^{1/2}(\Gamma),\; c \in \mathbb{R}\,. \tag{2.4}$$

Equivalently, $\|\cdot\|_{1/2,\Gamma,0}$ denotes the quotient space norm

$$\|\psi\|_{1/2,\Gamma,0} := \inf_{c\in\mathbb{R}} \|\psi + c\|_{1/2,\Gamma} \qquad \forall\,\psi \in H^{1/2}(\Gamma)\,.$$

2.2 The LDG formulation in the interior domain

The setting and analysis of the LDG formulation in Ω require several notations, definitions, and assumptions that we recall from [15]. Let $\mathcal{T}_h$ be a shape regular triangulation of $\bar{\Omega}$ (with possible hanging nodes) made up of straight triangles K with diameter h_K and unit outward normal to ∂K given by $\boldsymbol{\nu}_K$. As usual, the index h denotes $h := \max_{K\in\mathcal{T}_h} h_K$. Then, the edges of $\mathcal{T}_h$ are defined as follows. An *interior edge of* $\mathcal{T}_h$ *is the* (non-empty) interior of $\partial K \cap \partial K'$ where K and K' are two adjacent elements of $\mathcal{T}_h$. Similarly, a *boundary edge* of $\mathcal{T}_h$ is the (non-empty) interior of $\partial K \cap \Gamma_0$ or $\partial K \cap \Gamma$ where K is an element of $\mathcal{T}_h$ which has an edge on Γ_0 or Γ. For each edge e, h_e represents its length. In addition, we define $\mathcal{E}(K)$:={edges of K}, $\mathcal{E}_h^{\text{int}}$: set of interior edges (counted only once), $\mathcal{E}_h^{\Gamma}$: set of edges on Γ, $\mathcal{E}_h^{\Gamma_0}$: set of edges on Γ_0, and I_h: interior grid generated by the triangulation, that is $I_h := \cup\{e: \ e \in \mathcal{E}_h^{\text{int}}\}$. Also, we let Γ_h and Γ_h^0 be the induced meshes on the boundaries Γ and Γ_0, whose lists of edges are $\mathcal{E}_h^{\Gamma}$ and $\mathcal{E}_h^{\Gamma_0}$, respectively.

In what follows we assume that $\mathcal{T}_h$ is a *locally quasi-uniform* mesh, i.e. there exists $l > 1$, independent of the meshsize h, such that $l^{-1} \leq \frac{h_K}{h_{K'}} \leq l$ for each pair K, $K' \in \mathcal{T}_h$ sharing an interior edge. We notice that the hypotheses on the triangulation imply that the cardinality of $\mathcal{E}(K)$ is uniformly bounded, and that for each $e \in \mathcal{E}(K)$ there holds $h_K \leq C\, l\, h_e$.

Now we consider integers $m \geq 1$ and $r \geq 0$ with $r \geq m-1$, and define the finite element spaces

$$V_h := \prod_{K\in\mathcal{T}_h} P_m(K) \quad \text{and} \quad \mathbf{\Sigma}_h := \prod_{K\in\mathcal{T}_h} \mathbf{P}_r(K)\,. \tag{2.5}$$

Hereafter, given an integer $k \geq 0$ and a domain $S \subseteq \mathbb{R}^2$, $P_k(S)$ denotes the space of polynomials of degree at most k on S. For each $v := \{v_K\}_{K\in\mathcal{T}_h} \in V_h$ and $\boldsymbol{\tau} := \{\boldsymbol{\tau}_K\}_{K\in\mathcal{T}_h} \in \mathbf{\Sigma}_h$, the components v_K and $\boldsymbol{\tau}_K$ coincide with the restrictions $v|_K$ and $\boldsymbol{\tau}|_K$, when v and $\boldsymbol{\tau}$ are identified as elements in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively. Further, when no confusion arises, we omit the subscript K and just write v and $\boldsymbol{\tau}$.

Next, given $s > 1/2$, let

$$H^s(\mathcal{T}_h) := \prod_{K\in\mathcal{T}_h} H^s(K)\,, \qquad L^2(I_h) := \prod_{e\in\mathcal{E}_h^{\text{int}}} L^2(e)\,,$$

$$P_0(I_h) := \prod_{e\in\mathcal{E}_h^{\text{int}}} P_0(e) \quad \text{and} \quad P_0(I_h \cup \Gamma_h^0) := \prod_{e\in\mathcal{E}_h^{\text{int}}\cup\mathcal{E}_h^{\Gamma_0}} P_0(e)\,.$$

An analogue remark to the one given before, concerning components and restrictions of the elements in V_h and $\mathbf{\Sigma}_h$, is valid here for each of the product spaces above. Also, we will not use any symbol for the trace on edges, provided it is clear from which side of an interior edge we are taking the trace. Hence, given $v \in H^1(\mathcal{T}_h)$, we define the *averages* $\{v\} \in L^2(I_h)$ and *jumps* $[\![v]\!] \in \mathbf{L}^2(I_h)$ on the interior grid I_h by

$$\{v\}_e := \tfrac{1}{2}(v_K + v_{K'}) \quad \text{and} \quad [\![v]\!]_e := v_K\boldsymbol{\nu}_K + v_{K'}\boldsymbol{\nu}_{K'} \quad \forall\ e \in \mathcal{E}(K)\cap\mathcal{E}(K')\,.$$

Similarly, for vector valued functions $\boldsymbol{\tau} \in \mathbf{H}^1(\mathcal{T}_h)$, we define $\{\boldsymbol{\tau}\} \in \mathbf{L}^2(I_h)$ and $[\![\boldsymbol{\tau}]\!] \in L^2(I_h)$ by

$$\{\boldsymbol{\tau}\}_e := \tfrac{1}{2}(\boldsymbol{\tau}_K + \boldsymbol{\tau}_{K'}) \quad \text{and} \quad [\![\boldsymbol{\tau}]\!]_e := \boldsymbol{\tau}_K\cdot\boldsymbol{\nu}_K + \boldsymbol{\tau}_{K'}\cdot\boldsymbol{\nu}_{K'} \quad \forall\ e \in \mathcal{E}(K)\cap\mathcal{E}(K')\,.$$

In addition, let $\alpha \in P_0(I_h \cup \Gamma_h^0)$ and $\boldsymbol{\beta} \in \mathbf{P}_0(I_h)$ be given functions and assume that there exist C, c_0, $c_1 > 0$, independent of the grid, such that

$$\max_{e\in\mathcal{E}_h^{\text{int}}} |\boldsymbol{\beta}_e| \leq C \quad \text{and} \quad 0 < c_0 \leq h_{\mathcal{E}}\,\alpha \leq c_1\,, \tag{2.6}$$

where $h_{\mathcal{E}} \in P_0(I_h \cup \Gamma_h^0)$ is defined by $h_{\mathcal{E}}|_e := h_e \quad \forall\, e \in \mathcal{E}_h^{\mathrm{int}} \cup \mathcal{E}_h^{\Gamma_0}$.

We are now in a position to introduce the LDG scheme for the interior problem (1.2). As usual, we first define the gradient $\boldsymbol{\sigma} := \nabla u$ in Ω as an additional unknown where u is the exact solution of (1.2)–(1.3). Then, let $\lambda_h \in L^2(\Gamma)$ be a discrete approximation (to be defined below) of the normal derivative λ, and proceeding as in [10, 15] we arrive at the following global LDG formulation: Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ such that

$$\begin{aligned}
\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \;-\; \left\{ \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} - S(u_h, \boldsymbol{\tau}) \right\} \;&=\; 0 && \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \,, \\
\left\{ \int_\Omega \nabla_h v \cdot \boldsymbol{\sigma}_h \;-\; S(v, \boldsymbol{\sigma}_h) \right\} + \boldsymbol{\alpha}(u_h, v) \;&=\; \int_\Omega f\, v \;+\; \int_\Gamma \lambda_h\, v && \forall\, v \in V_h \,,
\end{aligned} \tag{2.7}$$

where ∇_h stands for the piecewise defined gradient, and $S : H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \to \mathbb{R}$ and $\boldsymbol{\alpha} : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ are the bilinear forms defined by:

$$S(w, \boldsymbol{\tau}) := \int_{I_h} [\![w]\!] \cdot (\{\boldsymbol{\tau}\} - [\![\boldsymbol{\tau}]\!]\, \boldsymbol{\beta}) \;+\; \int_{\Gamma_0} w\, (\boldsymbol{\tau} \cdot \boldsymbol{\nu}) \qquad \forall\, (w, \boldsymbol{\tau}) \,\in\, H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \,, \tag{2.8}$$

and

$$\boldsymbol{\alpha}(w, v) := \int_{I_h} \alpha\, [\![w]\!] \cdot [\![v]\!] \;+\; \int_{\Gamma_0} \alpha\, w\, v \qquad \forall\, (w, v) \,\in\, H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \,, \tag{2.9}$$

with the traces of w, v, and $\boldsymbol{\tau}$ on Γ_0 being defined elementwise.

2.3 The coupled LDG-BEM scheme

We now establish the coupled LDG-BEM scheme by combining a discrete form of (2.2) with the LDG formulation (2.7). This requires a subspace for λ_h and an approximant u_h of u which is continuous on Γ. For the discrete space approximating λ we take, for simplicity, the partition Γ_h of Γ and introduce

$$X_h := \{ \mu_h \in L^2(\Gamma) : \; \mu_h|_e \in P_{m-1}(e) \;\forall\, e \in \mathcal{E}_h^{\Gamma} \} \quad \text{and} \quad X_h^0 := \{ \mu_h \in X_h : \; \int_\Gamma \mu_h = 0 \} \,. \tag{2.10}$$

Then, we consider the subspace $\tilde{V}_h$ of V_h defined by

$$\tilde{V}_h := \{ v_h \in V_h : \quad v_h|_\Gamma \in C(\Gamma) \} \,.$$

Here, the trace $v_h|_\Gamma$ for $v_h \in V_h$ is defined in a piecewise manner on the edges of Γ_h and the condition $v_h|_\Gamma \in C(\Gamma)$ means that the function composed by the piecewise traces is continuous on Γ. Hence, substituting λ_h in (2.7) by a discrete version of the first equation in (2.2), in which u is replaced by its approximant u_h, and adding also a discrete formulation of the second equation in (2.2), we obtain the following coupled LDG-BEM scheme: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ such that

$$\begin{aligned}
\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \;-\; \boldsymbol{\rho}(u_h, \boldsymbol{\tau}) \quad & && &&= 0 \,, \\
\boldsymbol{\rho}(v, \boldsymbol{\sigma}_h) \;+\; \boldsymbol{\alpha}\,(u_h, v) + \; & \langle \mathcal{W} u_h, v \rangle && - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}') \lambda_h, v \rangle &&= \int_\Omega f\, v \,, \\
& \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K}) u_h \rangle && + \langle \mu, \mathcal{V} \lambda_h \rangle &&= 0
\end{aligned} \tag{2.11}$$

for all $(\boldsymbol{\tau}, v, \mu) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$, where $\boldsymbol{\rho} : H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \to \mathbb{R}$ is the bilinear form defined by

$$\boldsymbol{\rho}(v, \boldsymbol{\tau}) \;:=\; \int_\Omega \nabla_h v \cdot \boldsymbol{\tau} - S(v, \boldsymbol{\tau}) \qquad \forall\, (v, \boldsymbol{\tau}) \,\in\, H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \,. \tag{2.12}$$

This coupled LDG-BEM scheme is exactly of the skew-symmetric form known from the traditional coupling of finite elements and boundary elements, see [11, 17].

In order to compare the formulation (2.11) with the one from [15] we recall that the latter is given by: Find $(\boldsymbol{\sigma}_h, u_h, \lambda_{\tilde{h}}, \varphi_{\hat{h}}, \gamma_{\hat{h}}) \in \boldsymbol{\Sigma}_h \times V_h \times X^0_{\tilde{h}} \times Y^0_{\hat{h}} \times Z^0_{\hat{h}}$ such that

$$\begin{aligned}
&\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \;-\; \boldsymbol{\rho}(u_h, \boldsymbol{\tau}) && = 0\,,\\
&\boldsymbol{\rho}(v, \boldsymbol{\sigma}_h) \;+\; \boldsymbol{\alpha}\,(u_h, v) \;-\; \langle \lambda_{\tilde{h}}, v\rangle && = \int_\Omega f\,v\,,\\
&\qquad \langle \xi, u_h\rangle \qquad\qquad -\; \langle \xi, \varphi_{\hat{h}}\rangle && = 0\,,\\
&\qquad\qquad \langle \lambda_{\tilde{h}}, \psi\rangle \;+\; \langle \mathcal{W}\varphi_{\hat{h}}, \psi\rangle \qquad -\; \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\gamma_{\hat{h}}, \psi\rangle && = 0\,,\\
&\qquad\qquad\qquad \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})\varphi_{\hat{h}}\rangle \;+\; \langle \mu, \mathcal{V}\gamma_{\hat{h}}\rangle && = 0
\end{aligned} \tag{2.13}$$

for all $(\boldsymbol{\tau}, v, \xi, \psi, \mu) \in \boldsymbol{\Sigma}_h \times V_h \times X^0_{\tilde{h}} \times Y^0_{\hat{h}} \times Z^0_{\hat{h}}$, where $X^0_{\tilde{h}} \subseteq L^2(\Gamma) \cap H_0^{-1/2}(\Gamma)$, $Y^0_{\hat{h}} \subseteq C(\Gamma) \cap H_0^{1/2}(\Gamma)$, and $Z^0_{\hat{h}} \subseteq L^2(\Gamma) \cap H_0^{-1/2}(\Gamma)$ are boundary element subspaces, with independent meshsizes $\tilde{h}$ and $\hat{h}$, for the mortar-type auxiliary unknown $\lambda_{\tilde{h}}$ gluing the LDG and BEM modules, and for the Cauchy data $\varphi_{\hat{h}}$ and $\gamma_{\hat{h}}$, respectively. We observe that the computational implementation of (2.13) can be easily obtained by incorporating individual codes for each module, which constitutes the main advantage of this formulation, whereas the lower number of unknowns involved is the main strength of the present approach (2.11).

Now, for the solvability and stability of (2.11) we need an equivalent reduced formulation which is taken from [15]. To this end let $\mathbf{S}_h : H^1(\mathcal{T}_h) \to \boldsymbol{\Sigma}_h$ be the linear operator associated with the bilinear form S restricted to $H^1(\mathcal{T}_h) \times \boldsymbol{\Sigma}_h$. That is, given $w \in H^1(\mathcal{T}_h)$, $\mathbf{S}_h(w)$ is the unique element in $\boldsymbol{\Sigma}_h$ satisfying

$$\int_\Omega \mathbf{S}_h(w) \cdot \boldsymbol{\tau} \;=\; S(w, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,. \tag{2.14}$$

Next, let $B_h : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ be the bilinear form defined by

$$B_h(w, v) := \boldsymbol{\alpha}(w, v) + \int_\Omega (\nabla_h w - \mathbf{S}_h(w)) \cdot (\nabla_h v - \mathbf{S}_h(v)) \qquad \forall\, w,\, v \,\in\, H^1(\mathcal{T}_h). \tag{2.15}$$

The equivalence between (2.11) and a reduced problem involving B_h is established by the following lemma.

Lemma 2.1 *Let* $(\boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X^0_h$ *be a solution of* (2.11). *Then there holds*

$$\begin{aligned}
&B_h(u_h, v) + \;\langle \mathcal{W} u_h, v\rangle \;-\; \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle && = \int_\Omega f\,v\,,\\
&\langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K}) u_h\rangle \qquad +\; \langle \mu, \mathcal{V}\lambda_h\rangle && = 0
\end{aligned} \tag{2.16}$$

for any $(v, \mu) \in \tilde{V}_h \times X^0_h$. *Conversely, if* $(u_h, \lambda_h) \in \tilde{V}_h \times X^0_h$ *satisfies* (2.16) *and* $\boldsymbol{\sigma}_h := \nabla_h u_h - \mathbf{S}_h(u_h)$, *then* $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$ *is a solution of* (2.11). *If* $(u_h, \lambda_h) \in \tilde{V}_h \times X^0_h$ *is the only solution of* (2.16) *then* $(\boldsymbol{\sigma}_h, u_h, \lambda_h)$, *with* $\boldsymbol{\sigma}_h$ *defined as before, is the only solution of* (2.11).

Proof. This result is analogous to Lemma 2.2 in [15] and is based on the fact that the first equation in (2.11) can be written like

$$\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} \;-\; \int_\Omega (\,\nabla_h u_h - \mathbf{S}_h(u_h)\,) \cdot \boldsymbol{\tau} = 0 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h.$$

The fact that $r \geq m - 1$ guarantees that $\nabla_h u_h \in \boldsymbol{\Sigma}_h$, which yields $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$ and leads to the result. □

3 Unique solvability and stability

In this section we prove the unique solvability and stability of (2.11) through the corresponding analysis of the equivalent reduced formulation (2.16). We first introduce seminorms

$$|v|_{1,h}^2 := \|\nabla_h v\|_{0,\Omega}^2 , \qquad |v|_*^2 := \|h_{\mathcal{E}}^{-1/2}[\![v]\!]\|_{0,I_h}^2 + \|h_{\mathcal{E}}^{-1/2} v\|_{0,\Gamma_0}^2 \quad \forall\, v \in H^1(\mathcal{T}_h),$$

and the norm

$$|\!|\!| v |\!|\!|_h^2 := |v|_{1,h}^2 + |v|_*^2 \qquad \forall\, v \in H^1(\mathcal{T}_h).$$

Next, we let $\mathbf{B}_h$ denote the bilinear form defined by the left-hand side of (2.16), i.e.

$$\mathbf{B}_h(w,\eta;v,\mu) := B_h(w,v) + \langle \mathcal{W}w, v\rangle - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\eta, v\rangle + \langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K})w\rangle + \langle \mu, \mathcal{V}\eta\rangle$$

for

$$w, v \in H^1_{1/2}(\mathcal{T}_h) := \{w \in H^1(\mathcal{T}_h) : \quad w|_\Gamma \in H^{1/2}(\Gamma)\}$$

and $\eta, \mu \in H_0^{-1/2}(\Gamma)$. Analogously as before, the trace $w|_\Gamma$ for $w \in H^1(\mathcal{T}_h)$ is defined first on each edge of Γ_h and the condition $w|_\Gamma \in H^{1/2}(\Gamma)$ means that the function composed by the piecewise traces is in $H^{1/2}(\Gamma)$.

Essential ingredients of our analysis are the properties of the bilinear forms B_h and $\mathbf{B}_h$.

Lemma 3.1 [15, Lemma 3.2] *There exist positive constants c, C, independent of h, such that*

$$|B_h(w,v)| \le c\, |\!|\!| w |\!|\!|_h\, |\!|\!| v |\!|\!|_h \qquad \forall\, w,\, v \in H^1(\mathcal{T}_h) \tag{3.1}$$

and

$$B_h(v,v) \ge C\, |\!|\!| v |\!|\!|_h^2 \qquad \forall\, v \in H^1(\mathcal{T}_h). \tag{3.2}$$

Lemma 3.2 *There exist positive constants c, C, independent of h, such that*

$$|\mathbf{B}_h(w,\eta;v,\mu)| \le c\, \|(w,\eta)\|_{h,\Gamma}\, \|(v,\mu)\|_{h,\Gamma} \qquad \forall\, (w,\eta),\, (v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma) \tag{3.3}$$

and

$$\mathbf{B}_h(v,\mu;v,\mu) \ge C\, \|(v,\mu)\|_{h,\Gamma}^2 \qquad \forall\, (v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma) \tag{3.4}$$

where

$$\|(v,\mu)\|_{h,\Gamma} := \left\{ |\!|\!| v |\!|\!|_h^2 + \|v\|_{1/2,\Gamma,0}^2 + \|\mu\|_{-1/2,\Gamma}^2 \right\}^{1/2} \quad \forall\, (v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma).$$

Proof. According to the properties of the operators $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{K}$ (cf. Section 2.1), noting that $\mathcal{W}1 = 0$ and $\mathcal{K}1 = -\frac{1}{2}$ on Γ, and using the decomposition $H^{1/2}(\Gamma) = H_0^{1/2}(\Gamma) \oplus \mathbb{R}$ and the definition of the seminorm $\|\cdot\|_{1/2,\Gamma,0}$ (cf. (2.4)), we find that

$$|\langle \mu, \mathcal{V}\eta\rangle| \le C\, \|\mu\|_{-1/2,\Gamma}\, \|\eta\|_{-1/2,\Gamma} \quad \forall\, \mu,\, \eta \in H_0^{-1/2}(\Gamma),$$

$$|\langle \mathcal{W}w, v\rangle| \le C\, \|w\|_{1/2,\Gamma,0}\, \|v\|_{1/2,\Gamma,0} \quad \forall\, w,\, v \in H^1_{1/2}(\mathcal{T}_h),$$

and

$$|\langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K})w\rangle| \le C\, \|w\|_{1/2,\Gamma,0}\, \|\mu\|_{-1/2,\Gamma} \quad \forall\, (w,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma).$$

The above inequalities and Lemma 3.1 (cf. (3.1)) yield the continuity estimate (3.3) for $\mathbf{B}_h$. Next, we observe from the definition of $\mathbf{B}_h$ that

$$\mathbf{B}_h(v,\mu;v,\mu) = B_h(v,v) + \langle \mathcal{W}v, v\rangle + \langle \mu, \mathcal{V}\mu\rangle \quad \forall\,(v,\mu) \,\in\, H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)\,,$$

and hence, (2.3) and Lemma 3.1 (cf. (3.2)) imply the ellipticity estimate (3.4) for $\mathbf{B}_h$. □

We are now in a position to prove the unique solvability and stability of (2.11).

Theorem 3.1 *The coupled* LDG-BEM *scheme* (2.11) *is uniquely solvable and there holds the stability estimate:*

$$\|\boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|(u_h,\lambda_h)\|_{h,\Gamma} \,\le\, C\,\|f\|_{0,\Omega}\,.$$

Proof. By Lemma 2.1 it suffices to study the system (2.16) instead of (2.11). Indeed, the ellipticity of $\mathbf{B}_h$ (cf. Lemma 3.2) implies the unique solvability of (2.16), and using additionally that $\|v\|_{0,\Omega} \le C\,|\!|\!|v|\!|\!|_h \;\forall\, v \,\in\, V_h$ (see [1]), we deduce the stability estimate

$$\|(u_h,\lambda_h)\|_{h,\Gamma} \,\le\, C\,\|f\|_{0,\Omega}\,.$$

By Lemma 2.1 we then conclude the unique solvability of (2.11). By equation (3.11) in [15] there holds

$$\|\mathbf{S}_h(w)\|_{0,\Omega} \le C|w|_* \qquad \forall\, w \,\in\, H^1(\mathcal{T}_h). \tag{3.5}$$

Therefore, making use of the relation $\boldsymbol{\sigma}_h \,=\, \nabla_h u_h \,-\, \mathbf{S}_h(u_h)$, we find that

$$\|\boldsymbol{\sigma}_h\|_{0,\Omega} \,\le\, C\,|\!|\!|u_h|\!|\!|_h \,\le\, C\,\|f\|_{0,\Omega}\,,$$

which finishes the proof of the theorem. □

4 A priori error analysis

In order to derive the a priori error estimate of the coupled scheme some technical results are needed. In what follows let $\hat{K}$ denote the reference triangle

$$\hat{K} \,:=\, \{\,(x_1,x_2):\; 0 \,<\, x_1 \,<\, 1,\; 0 \,<\, x_2 \,<\, 1 - x_1\,\}\,.$$

We begin by recalling some local approximation properties from [4].

Lemma 4.1 *Let* $K \in \mathcal{T}_h$ *and let* $\hat{e}$ *be a side of* $\hat{K}$*. Suppose that* $u \in H^k(K)$ *and let* $\hat{u} := u \circ M_K$ *where* M_K *is an invertible affine mapping from* $\hat{K}$ *onto* K*. Then, given an integer* $m \ge 1$*, there exists an operator* $\hat{\boldsymbol{\pi}} :\, H^k(\hat{K}) \to P_m(\hat{K})$ *such that*

$$\|\hat{u} - \hat{\boldsymbol{\pi}}\,\hat{u}\|_{H^q(\hat{K})} \,\le\, C_1\,h^\mu\,\|u\|_{H^k(K)}, \quad k \ge 0, \quad 0 \le q \le k\,, \tag{4.1}$$

$$|(\hat{u} - \hat{\boldsymbol{\pi}}\,\hat{u})(\hat{x})| \,\le\, C_2\,h^\mu\,\|u\|_{H^k(K)}, \quad k > 1, \quad \hat{x} \in \hat{K}\,, \tag{4.2}$$

$$\|\hat{u} - \hat{\boldsymbol{\pi}}\,\hat{u}\|_{H^s(\hat{e})} \,\le\, C_3\,h^\mu\,\|u\|_{H^k(K)}, \quad k > 3/2, \quad s \in \{0,1\}\,, \tag{4.3}$$

where $\mu \,=\, \min\{k-1, m\}$*, and the positive constants* C_1*,* C_2*,* C_3 *are independent of* u *and* h *but depend on* m*,* k*,* q*, and* s*, as indicated below using a generic positive constant* C*:*

$$C_1 \,=\, C\,m^{-(k-q)}\,, \quad C_2 \,=\, C\,m^{-(k-1)}\,, \quad C_3 \,=\, C\,m^{-(k-s-1/2)}\,.$$

Proof. This is Lemma 4.1 in [4] which is proved by collecting several results from [2, 3]. $\square$

The following lemma, whose proof below makes extensive use of the estimates (4.1) - (4.3), provides a global approximation property of the subspace $\tilde{V}_h$.

Lemma 4.2 *Assume that $u \in H^{1+\delta}(\Omega)$ for some $\delta > 1/2$. Then there exists $v_h \in \tilde{V}_h$ such that*

$$|\!|\!| u - v_h |\!|\!|_h \;+\; \|u - v_h\|_{1/2,\Gamma,0} \;\leq\; C\, h^{\min\{\delta,m\}}\, \|u\|_{1+\delta,\Omega}\,. \tag{4.4}$$

Here, $C > 0$ is a constant independent of h.

Proof. We begin by defining $\bar{v}_h \in V_h$ such that

$$|\!|\!| u - \bar{v}_h |\!|\!|_h \leq C\, h^{\min\{\delta,m\}}\, \|u\|_{1+\delta,\Omega}. \tag{4.5}$$

To this end we consider any element $K \in \mathcal{T}_h$ with generic invertible affine mapping $M_K : \hat{K} \to K$ and construct (by using Lemma 4.1) an approximation $\hat{\boldsymbol{\pi}}\, \hat{u}_K$ of $\hat{u}_K := u \circ M_K$. This piecewise construction delivers an approximation $\bar{v}_h$ of u given elementwise by $\bar{v}_h|_K := \hat{\boldsymbol{\pi}}\, \hat{u}_K \circ M_K^{-1}$. Taking into account the scaling properties of the norms involved and applying (4.1) we obtain

$$\begin{aligned} |u - \bar{v}_h|_{1,h}^2 \;&=\; \sum_{K \in \mathcal{T}_h} |u - \bar{v}_h|_{1,K}^2 \leq C \sum_{K \in \mathcal{T}_h} |\hat{u}_K - \hat{\boldsymbol{\pi}}\, \hat{u}_K|_{1,\hat{K}}^2 \\ &\leq\; C \sum_{K \in \mathcal{T}_h} h^{2\min\{\delta,m\}}\, \|u_K\|_{1+\delta,K}^2 \;\leq\; C\, h^{2\min\{\delta,m\}}\, \|u\|_{1+\delta,\Omega}^2. \end{aligned} \tag{4.6}$$

Also, (4.3) yields for any $e \in I_h$ with $e = K \cap K'$ the estimate

$$\begin{aligned} &\|h_{\mathcal{E}}^{-1/2} [\![u - \bar{v}_h]\!] \|_{0,e}^2 \\ &\quad\leq\; 2\, \|h_{\mathcal{E}}^{-1/2} (u - \bar{v}_h)|_K\|_{0,e}^2 + 2\, \|h_{\mathcal{E}}^{-1/2} (u - \bar{v}_h)|_{K'}\|_{0,e}^2 \leq C\, h^{2\min\{\delta,m\}}\, \|u\|_{1+\delta,K\cup K'}^2\,. \end{aligned} \tag{4.7}$$

Edges of Γ_h and Γ_h^0 are dealt with analogously. In this way, (4.6) and (4.7) prove (4.5).

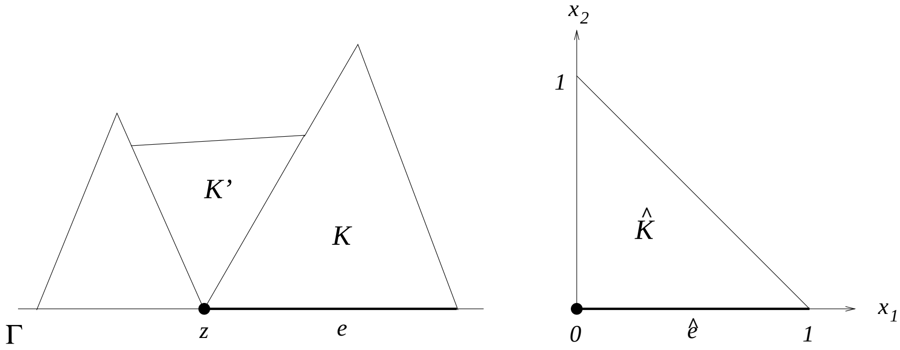

Figure 1: Adjusting to continuity at the boundary.

Now, in order to find an approximant $v_h \in \tilde{V}_h$ which is continuous on Γ and satisfies (4.4) we have to adjust $\bar{v}_h$ at the nodes of the mesh that lie on Γ. Actually, this technique is standard in finite element analysis. Note, however, that we only need to deal with boundary nodes and therefore, one simple construction works for any polynomial degree m. We adapt $\bar{v}_h$ to a function v_h such that its trace on Γ coincides with $u|_\Gamma$ in any node on Γ. This means in particular that v_h is continuous on

Γ, i.e. $v_h \in \tilde{V}_h$. A generic situation is given in Figure 1. We consider the approximation of u on the triangle K that has the edge e with Γ in common. One of the nodes of e is denoted by $\boldsymbol{z}$. In general $u(\boldsymbol{z})$ is different from $\bar{v}_h|_K(\boldsymbol{z})$. Note that u is continuous since $u \in H^{1+\delta}(\Omega)$ with $\delta > 1/2$. As before, let $\hat{u}_K$ denote the linearly transformed function on $\hat{K}$, that is $\hat{u}_K := u \circ M_K$. By construction of $\bar{v}_h$ there holds $\bar{v}_h|_K \circ M_K = \boldsymbol{\pi}\,\hat{u}_K$. Let us assume that e is mapped onto $\hat{e} := (0,1) \times \{0\}$ by M_K^{-1} and that $M_K^{-1}(\boldsymbol{z}) = (0,0)$. We then approximate $\hat{u}_K$ by

$$\hat{\Pi}\,\hat{u}_K(x_1,x_2) \;:=\; \boldsymbol{\pi}\,\hat{u}_K(x_1,x_2) \;+\; \Big(\hat{u}_K(0,0) - \boldsymbol{\pi}\,\hat{u}_K(0,0)\Big)\,(1 - x_1 - x_2). \tag{4.8}$$

It is clear from (4.8) that the new approximant $\hat{\Pi}\,\hat{u}_K$ coincides with the previous one $\boldsymbol{\pi}\,\hat{u}_K$ at the other two vertices of $\hat{K}$. Then, transforming back to K we obtain an approximant v_h given by $v_h|_K := \hat{\Pi}\,\hat{u}_K \circ M_K^{-1} \in P_m(K)$ which, according to (4.8), satisfies

$$v_h(\boldsymbol{z}) \;=\; (\hat{\Pi}\,\hat{u}_K \circ M_K^{-1})(\boldsymbol{z}) \;=\; \hat{\Pi}\,\hat{u}_K(0,0) \;=\; \hat{u}_K(0,0) \;=\; (u \circ M_k)(0,0) \;=\; u(\boldsymbol{z})\,.$$

Moreover, by (4.1) and (4.2) there holds

$$\begin{aligned}
|u - v_h|_{1,K}^2 \;&\le\; C\,|\hat{u}_K - \hat{\Pi}\,\hat{u}_K|_{1,\hat{K}}^2 \\
&\le\; C\,\Big(|\hat{u}_k - \boldsymbol{\pi}\,\hat{u}_k|_{1,\hat{K}}^2 + \big|(\hat{u}_k - \boldsymbol{\pi}\,\hat{u}_K)(0,0)\big|^2\Big) \\
&\le\; C\,h^{2\min\{\delta,m\}}\,\|u\|_{1+\delta,K}^2\,.
\end{aligned} \tag{4.9}$$

Analogously, we find by using (4.3) and (4.2)

$$\begin{aligned}
&\|h_{\mathcal{E}}^{-1/2}[\![u - v_h]\!]\|_{0,e}^2 \le C\,\|\hat{u}_K - \hat{\Pi}\,\hat{u}_K\|_{0,\hat{e}}^2 \\
&\le\; C\,\Big(\|\hat{u}_K - \boldsymbol{\pi}\,\hat{u}_k)\|_{0,\hat{e}}^2 + \big|(\hat{u}_k - \boldsymbol{\pi}\,\hat{u}_K)(0,0)\big|^2\Big) \le C\,h^{2\min\{\delta,m\}}\,\|u\|_{1+\delta,K}^2\,.
\end{aligned} \tag{4.10}$$

In the latter estimate we used the fact that $\delta > 1/2$. The estimate for the other edge of K which has $\boldsymbol{z}$ as a node is analogous. From (4.8) it follows that the approximant v_h coincides with $\bar{v}_h$ on the third edge of K, and in particular in the second node of e. Therefore, this method to adapt $\bar{v}_h$ on Γ is a local procedure and can be applied to any element and any node independently. Note that we do not alter the approximant on the element K' which has only a single vertex ($\boldsymbol{z}$ in this case) on Γ. The estimates (4.9) and (4.10) yield

$$|\!|\!|u - v_h|\!|\!|_h \;\le\; C\,h^{\min\{\delta,m\}}\,\|u\|_{1+\delta,\Omega}\,. \tag{4.11}$$

In order to conclude (4.4) it just remains to show that

$$\|u - v_h\|_{1/2,\Gamma,0} \;\le\; C\,h^{\min\{\delta,m\}}\,\|u\|_{1+\delta,\Omega}\,. \tag{4.12}$$

For $e \in \mathcal{E}_h^\Gamma$ let K_e denote the element which has e as an edge. When transforming K_e onto $\hat{K}$ assume that e is mapped onto $\hat{e} = (0,1) \times \{0\}$. Hence, using (4.2) and (4.3) with $s = 0$ we then find that there holds

$$\begin{aligned}
\|u - v_h\|_{0,\Gamma}^2 \;=\; \sum_{e\in\mathcal{E}_h^\Gamma} \|u - v_h\|_{0,e}^2 \;&\le\; C \sum_{e\in\mathcal{E}_h^\Gamma} h_K\,\|\hat{u}_K - \hat{\Pi}\,\hat{u}_K\|_{0,\hat{e}}^2 \\
&\le\; C\,h^{2\min\{\delta,m\}+1} \sum_{e\in\mathcal{E}_h^\Gamma} \|u\|_{1+\delta,K_e}^2\,,
\end{aligned}$$

and for $s = 1$ we obtain

$$
\begin{aligned}
\|u - v_h\|_{1,\Gamma}^2 = \sum_{e\in\mathcal{E}_h^\Gamma} \|u - v_h\|_{1,e}^2 \quad &\le \quad C \sum_{e\in\mathcal{E}_h^\Gamma} h_K^{-1}\, \|\hat{u}_K - \hat{\Pi}\,\hat{u}_K\|_{1,\hat{e}}^2 \\
&\le \quad C\, h^{2\min\{\delta,m\}-1} \sum_{e\in\mathcal{E}_h^\Gamma} \|u\|_{1+\delta,K_e}^2 \,.
\end{aligned}
$$

Interpolation between the last two estimates proves (4.12). This completes the proof. □

We note that defining $\bar{v}_h$ as the $L^2(\Omega)$-orthogonal projection of u onto V_h would also yield the estimate (4.5) (see Lemmas 4.2 and 4.4 in [15] for details). However, this choice of $\bar{v}_h$ does not allow the further construction of $v_h \in \tilde{V}_h$ satisfying the approximation property (4.4). This is the reason why we proceed differently and employ the local approximant provided by Lemma 4.1.

Next, we derive an approximation property for the subspace X_h^0. To this end, we now let $\{\Gamma_1, \dots, \Gamma_N\}$ denote the edges of the polygon Γ and recall that the Sobolev space $\tilde{H}^{-1/2}(\Gamma_j)$ is the dual of $H^{1/2}(\Gamma_j) := \{\, \xi|_{\Gamma_j} :\ \xi \in H^{1/2}(\Gamma)\,\}$. Similarly, $H^{-1/2}(\Gamma_j)$ is the dual of $\tilde{H}^{1/2}(\Gamma_j)$, the 1/2-interpolation space between $L^2(\Gamma_j)$ and $H_0^1(\Gamma_j)$. The norms of $\tilde{H}^{-1/2}(\Gamma_j)$ and $\tilde{H}^{1/2}(\Gamma_j)$ are denoted, respectively, by $\|\cdot\|_{-1/2,\tilde{\Gamma}_j}$ and $\|\cdot\|_{1/2,\tilde{\Gamma}_j}$. In particular, it is well-known (see, e.g., [24, 20]) that there holds

$$
\|\mu\|_{-1/2,\Gamma}^2 \;\le\; C \sum_{j=1}^N \|\mu\|_{-1/2,\tilde{\Gamma}_j}^2 \qquad \forall\, \mu \,\in\, H^{-1/2}(\Gamma)\,. \tag{4.13}
$$

Then we have the following result.

Lemma 4.3 *Assume that* $\lambda \in H_0^{-1/2}(\Gamma) \cap H^t(\Gamma)$ *for some* $t > 0$*. Then there exists* $\mu_h \in X_h^0$ *such that*

$$
\|\lambda - \mu_h\|_{-1/2,\Gamma} \;\le\; C\, h^{\min\{t,m\}+1/2}\, \|\lambda\|_{t,\Gamma}\,. \tag{4.14}
$$

Here, $C > 0$ *is a constant independent of* h*.*

Proof. We follow the strategy in [16]. Let us consider a fixed edge Γ_j and identify it with the interval $(0, a)$, $a = |\Gamma_j|$. Defining

$$
\Lambda(x) := \int_0^x (\lambda(s) - \bar{\lambda})\, ds\,, \quad \text{with} \quad \bar{\lambda} \,:=\, \frac{1}{a}\int_0^a \lambda(s)\, ds\,,
$$

there holds $\Lambda \in \tilde{H}^{1/2}(0, a)$. Then, applying Theorem 3.1 from [23] one finds an element $w_h \in \tilde{V}_h|_\Gamma$ which coincides with Λ in the endpoints of Γ_j and which satisfies

$$
\|\Lambda - w_h\|_{1/2,\tilde{\Gamma}_j} \;\le\; C\, h^{\min\{k-1/2,m+1/2\}} \|\Lambda\|_{k,\Gamma_j} \qquad \forall\, k \,>\, 1/2\,.
$$

Now we define $\bar{\mu}_h := w_h' + \bar{\lambda}$ on Γ_j. Obviously $\bar{\mu}_h \in X_h|_{\Gamma_j}$ (cf. (2.10)). Differentiation with respect to the arc length maps $\tilde{H}^{1/2}(\Gamma_j)$ continuously onto $\tilde{H}^{-1/2}(\Gamma_j)$, see Lemma 3.4 in [23]. Moreover, the antiderivative operator is continuous as a mapping $H^{k-1}(\Gamma_j) \to H^k(\Gamma_j)$ for $k \ge 0$. Therefore, using the previous estimate we find that there holds

$$
\begin{aligned}
\|\lambda - \bar{\mu}_h\|_{-1/2,\tilde{\Gamma}_j} \quad &= \quad \|\Lambda' - w_h'\|_{-1/2,\tilde{\Gamma}_j} \;\le\; C\, \|\Lambda - w_h\|_{1/2,\tilde{\Gamma}_j} \\
&\le \quad C\, h^{\min\{k-1/2,m+1/2\}}\, \|\Lambda\|_{k,\Gamma_j} \\
&\le \quad C\, h^{\min\{k-1/2,m+1/2\}} \left\{ \|\lambda\|_{k-1,\Gamma_j} \,+\, \|\bar{\lambda}\|_{k-1,\Gamma_j} \right\} \\
&\le \quad C\, h^{\min\{k-1/2,m+1/2\}}\, \|\lambda\|_{k-1,\Gamma_j}\,.
\end{aligned}
$$

In particular, taking $k = t+1$ we deduce that

$$\|\lambda - \bar{\mu}_h\|_{-1/2,\tilde{\Gamma}_j} \le C\, h^{\min\{t,m\}+1/2} \, \|\lambda\|_{t,\Gamma_j} .$$

Hence, repeating this procedure for every edge of the polygon Γ, making use of estimate (4.13), and noting that $\sum_{j=1}^{N} \|\lambda\|^2_{t,\Gamma_j} \le C\,\|\lambda\|^2_{t,\Gamma}$, we conclude that

$$\|\lambda - \bar{\mu}_h\|_{-1/2,\Gamma} \le C\, h^{\min\{t,m\}+1/2} \, \|\lambda\|_{t,\Gamma} .$$

Finally, the stability of the decomposition $H^{-1/2}(\Gamma) = H_0^{-1/2}(\Gamma) \oplus \mathbb{R}$ yields

$$\|\lambda - \mu_h\|_{-1/2,\Gamma} \le C\,\|\lambda - \bar{\mu}_h\|_{-1/2,\Gamma} \le C\, h^{\min\{t,m\}+1/2} \, \|\lambda\|_{t,\Gamma} ,$$

where $\bar{\mu}_h = \mu_h + c$, $\mu_h \in X_h^0 \subseteq H_0^{-1/2}(\Gamma)$, $c \in \mathbb{R}$, which completes the proof. $\square$

The a priori error estimate for the coupled LDG-BEM scheme (2.11) can be established now.

Theorem 4.1 *Assume that $u \in H^{1+\delta}(\Omega)$ with $\delta > 1/2$. Then there exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \le C\, h^{\min\{\delta,m\}} \, \|u\|_{1+\delta,\Omega} . \tag{4.15}$$

Proof. We first note that $\lambda := \partial_{\boldsymbol{\nu}} u \in H^{\delta-1/2}(\Gamma)$ and there holds

$$\|\lambda\|_{\delta-1/2,\Gamma} \le C\,\|u\|_{1+\delta,\Omega} . \tag{4.16}$$

In fact, for $\delta > 1/2$, $\nabla : H^{1+\delta}(\Omega) \to \mathbf{H}^{\delta}(\Omega)$ is bounded and the normal trace operator $(\cdot)|_{\Gamma} \cdot \boldsymbol{\nu}$ maps $\mathbf{H}^{\delta}(\Omega)$ continuously onto $H^{\delta-1/2}(\Gamma)$. In addition, it is not difficult to see that u and λ satisfy

$$\begin{aligned} B_h(u,v) + \langle \mathcal{W} u, v\rangle \; - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda, v\rangle \; &= \int_\Omega f\,v , \\ \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})u\rangle \; + \langle \mu, \mathcal{V}\lambda\rangle \; &= 0 \end{aligned}$$

for any $(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$. Using the bilinear form $\mathbf{B}_h$, the above means that

$$\mathbf{B}_h(u,\lambda; v,\mu) = \int_\Omega f\,v \qquad \forall\,(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H^{-1/2}(\Gamma) .$$

On the other hand, the discrete system (2.16) renders like

$$\mathbf{B}_h(u_h,\lambda_h; v,\mu) = \int_\Omega f\,v \qquad \forall\,(v,\mu) \in \tilde{V}_h \times X_h^0 .$$

Hence, the ellipticity and continuity of the bilinear form $\mathbf{B}_h$ (cf. Lemma 3.2) imply the quasi-optimality

$$\|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \le C\,\|(u,\lambda) - (v_h,\mu_h)\|_{h,\Gamma} \qquad \forall\,(v_h,\mu_h) \in \tilde{V}_h \times X_h^0 . \tag{4.17}$$

Also, since $\boldsymbol{\sigma} = \nabla u = \nabla u - \mathbf{S}_h(u)$ and $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$ (cf. Lemma 2.1), we obtain with (3.5) the upper bound

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le C\, |\!|\!| u - u_h |\!|\!|_h , \tag{4.18}$$

which, together with (4.17), gives

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \,\leq\, C\,\|(u,\lambda) - (v_h,\mu_h)\|_{h,\Gamma} \qquad \forall\,(v_h,\mu_h) \,\in\, \tilde{V}_h \times X_h^0 . \tag{4.19}$$

Finally, applying the approximation properties from Lemmas 4.2 and 4.3 (with $t = \delta - 1/2$), using (4.16) in the latter one, and combining the resulting estimates with (4.19) we arrive at (4.15). This finishes the proof. $\square$

We remark that the a priori error estimate (4.15) is independent of the polynomial degree r that defines the subspace $\boldsymbol{\Sigma}_h$ (cf. (2.5)). Hence, since the restriction $r \geq m-1$ is required only to deduce that $\boldsymbol{\sigma}_h = \nabla_h u_h - \mathbf{S}_h(u_h)$ (cf. Lemma 2.1), for practical computations it suffices to take $r = m-1$.

5 The coupled LDG-BEM scheme with Lagrangian multiplier

To implement the discrete scheme (2.11) one has to deal with the continuity condition of the space $\tilde{V}_h$. A direct implementation is possible without any difficulty. However, in order to maintain the full flexibility of the discontinuous method one can use a Lagrangian multiplier instead and work with V_h rather than $\tilde{V}_h$. The needed multiplier is simply a vector constant. In addition, the zero mean value condition of the unknown $\lambda_h \in X_h^0$ can be dealt with similarly, whence the resulting formulation employs the subspace X_h instead of X_h^0. This strategy is described in this section.

We first notice that the bilinear form of the coupled system (2.11), which is given by

$$\begin{aligned}\mathbf{A}_h(\boldsymbol{\zeta},w,\xi;\boldsymbol{\tau},v,\mu) \;&:=\; \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \,-\, \boldsymbol{\rho}(w,\boldsymbol{\tau}) \,+\, \boldsymbol{\rho}(v,\boldsymbol{\zeta}) \,+\, \boldsymbol{\alpha}\,(w,v) \,+\, \langle \mathcal{W}w, v\rangle \\ &\quad - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\xi, v\rangle \,+\, \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})w\rangle \,+\, \langle \mu, \mathcal{V}\xi\rangle\,,\end{aligned}$$

is not well defined on $\boldsymbol{\Sigma}_h \times V_h \times X_h$. For instance, the well-posedness of the bilinear form $\langle \mathcal{W}w, v\rangle$ requires that $w|_\Gamma, v|_\Gamma \in H^{1/2}(\Gamma)$. This is in general not true for $w, v \in V_h$. Therefore, we consider instead the bilinear form

$$\begin{aligned}\tilde{\mathbf{A}}_h(\boldsymbol{\zeta},w,\xi;\boldsymbol{\tau},v,\mu) \;&:=\; \int_\Omega \boldsymbol{\zeta}\cdot\boldsymbol{\tau} \,-\, \boldsymbol{\rho}(w,\boldsymbol{\tau}) \,+\, \boldsymbol{\rho}(v,\boldsymbol{\zeta}) \,+\, \boldsymbol{\alpha}\,(w,v) \,+\, \langle \partial_h w, \mathcal{V}\partial_h v\rangle \\ &\quad - \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\xi, v\rangle \,+\, \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\mu, w\rangle \,+\, \langle \mu, \mathcal{V}\xi\rangle\,.\end{aligned}$$

Here, $\partial_h w$ is defined piecewise by $\partial_h w|_e = (w|_e)'$ for any edge $e \in \Gamma_h$ and $(w|_e)'$ denotes the derivative of w on e with respect to the arc length. Note that $\partial_h w \in L^2(\Gamma)$ for any $w \in V_h$. Then the updated bilinear forms $\langle \partial_h w, \mathcal{V}\partial_h v\rangle$ and $\langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\mu, w\rangle$ are well defined for $w, v \in V_h$ and $\mu \in X_h$. Moreover, there holds

$$\langle \mathcal{W}w, v\rangle \;=\; \langle \partial_h w, \mathcal{V}\partial_h v\rangle \qquad \forall\, w, v \in \tilde{V}_h$$

(see [22]) and

$$\langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K})w\rangle \;=\; \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\mu, w\rangle \qquad \forall\,(w,\mu) \,\in\, \tilde{V}_h \times X_h$$

so that

$$\mathbf{A}_h(\boldsymbol{\zeta},w,\xi;\boldsymbol{\tau},v,\mu) \;=\; \tilde{\mathbf{A}}_h(\boldsymbol{\zeta},w,\xi;\boldsymbol{\tau},v,\mu) \quad \forall\,(\boldsymbol{\zeta},w,\xi),\,(\boldsymbol{\tau},v,\mu) \,\in\, \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h\,.$$

Now, let $\{\boldsymbol{z}_1,\ldots,\boldsymbol{z}_n\}$ denote the nodes of $\mathcal{T}_h$ on Γ, and let e_i^- and e_i^+ denote the two elements of Γ_h which have $\boldsymbol{z}_i$ as a common node. We then define the bilinear form

$$b_h((v,\mu),\vec{\mathbf{y}}) := \sum_{i=1}^n \Big(v|_{e_i^+}(\boldsymbol{z}_i) - v|_{e_i^-}(\boldsymbol{z}_i)\Big)\, y_i \;+\; y_{n+1}\int_\Gamma \mu$$

for $(v,\mu)\in V_h\times X_h$, $\vec{\mathbf{y}} = (y_1,\dots,y_{n+1})\in\mathbb{R}^{n+1}$, and consider the following LDG-BEM scheme with Lagrangian multiplier $\vec{\boldsymbol{x}}$: Find $(\boldsymbol{\sigma}_h,u_h,\lambda_h,\vec{\boldsymbol{x}}) \in \boldsymbol{\Sigma}_h\times V_h\times X_h\times\mathbb{R}^{n+1}$ such that

$$\begin{aligned}\tilde{\mathbf{A}}_h(\boldsymbol{\sigma}_h,u_h,\lambda_h;\boldsymbol{\tau},v,\mu) \quad + b_h((v,\mu),\vec{\boldsymbol{x}}) \;&= \int_\Omega f\,v\,,\\ b_h((u_h,\lambda_h),\vec{\mathbf{y}}) \;&= 0\end{aligned}\tag{5.1}$$

for any $(\boldsymbol{\tau},v,\mu,\vec{\mathbf{y}}) \in \boldsymbol{\Sigma}_h\times V_h\times X_h\times\mathbb{R}^{n+1}$. Then, we have the following result.

Theorem 5.1 *There exists a unique solution* $(\boldsymbol{\sigma}_h,u_h,\lambda_h,\vec{\boldsymbol{x}}) \in \boldsymbol{\Sigma}_h\times V_h\times X_h\times\mathbb{R}^{n+1}$ *of* (5.1) *and* $(\boldsymbol{\sigma}_h,u_h,\lambda_h)$ *solves* (2.11). *In particular the error estimate from Theorem* 4.1 *holds.*

Proof. It is immediate that there holds a (non-uniform) inf-sup condition for b_h:

$$\sup_{(v,\mu)\in V_h\times X_h} b_h((v,\mu),\vec{\mathbf{y}}) \,>\, 0 \qquad \forall\,\vec{\mathbf{y}}\,\in\,\mathbb{R}^{n+1}\,.$$

We also have that the discrete null space of b_h is given by

$$\tilde{V}_h\times X_h^0 \,=\, \{(v,\mu)\in V_h\times X_h:\quad b_h((v,\mu),\vec{\mathbf{y}}) \,=\, 0 \quad \forall\,\vec{\mathbf{y}}\,\in\,\mathbb{R}^{n+1}\}\,.$$

Therefore, Theorem 3.1 and the Babuška-Brezzi theory for discrete problems ensure the unique solvability of (5.1) and then $(\boldsymbol{\sigma}_h,u_h,\lambda_h)\in\boldsymbol{\Sigma}_h\times\tilde{V}_h\times X_h^0$ becomes the unique solution of (2.11), whence the error estimate of Theorem 4.1 holds. □

6 Extension to nonlinear problems

In this section we extend the present LDG-BEM approach to the class of nonlinear exterior transmission problems studied in [7], [8], and [9]. In order to describe the model problem let Ω_0 be a simply connected and bounded domain in $\mathbb{R}^2$ with polygonal boundary Γ_0. Then, let Ω_1 be an annular and simply connected domain surrounded by Γ_0 and another polygonal boundary Γ_1. In addition, let $\mathbf{a}:\Omega_1\times\mathbb{R}^2\to\mathbb{R}^2$ be a nonlinear function satisfying the conditions specified in [5] (see also [7]) which, in particular, imply that the associated operator becomes Lipschitz continuous and strongly monotone. Thus, given $f\in L^2(\mathbb{R}^2\setminus\bar{\Omega}_0)$ with compact support, $g_0\in H^{1/2}(\Gamma_0)$, $g_1\in H^{1/2}(\Gamma_1)$, and $g_2\in L^2(\Gamma_1)$, we consider the nonlinear exterior transmission problem:

$$\begin{gathered}-\mathrm{div}\;\mathbf{a}(\cdot,\nabla u_1) \,=\, f \quad\text{in}\quad \Omega_1,\quad u_1\,=\,g_0\quad\text{on}\quad\Gamma_0,\\ -\Delta u_2 \,=\, f\quad\text{in}\quad \mathbb{R}^2\setminus(\bar{\Omega}_0\cup\bar{\Omega}_1),\quad u_1-u_2\,=\,g_1\quad\text{on}\quad\Gamma_1,\\ \mathbf{a}(\cdot,\nabla u_1)\cdot\boldsymbol{\nu}_1-\nabla u_2\cdot\boldsymbol{\nu}_1\,=\,g_2\quad\text{on}\quad\Gamma_1,\quad\text{and}\quad u_2(\boldsymbol{x})=\mathcal{O}(1)\quad\text{as}\quad|\boldsymbol{x}|\to\infty\,.\end{gathered}\tag{6.1}$$

Here, $\boldsymbol{\nu}_1$ stands for the unit outward normal to Γ_1. This kind of problems appears in the computation of magnetic fields of electromagnetic devices (see, e.g. [18], [19]), in some subsonic flow and fluid mechanics problems (see, e.g. [13], [14]), and also in steady state heat conduction. For instance, in the latter case, one has $\mathbf{a}(\boldsymbol{x},\nabla u(\boldsymbol{x})) = k(\boldsymbol{x},\nabla u(\boldsymbol{x}))\,\nabla u$, where u is the temperature and $k:\Omega_1\times\mathbb{R}^2\to\mathbb{R}$ is the heat conductivity.

Next, we introduce a closed polygonal curve Γ such that its interior contains the support of f. Then, let Ω_2 be the annular domain bounded by Γ_1 and Γ and set $\Omega_e := \mathbb{R}^2\setminus(\bar{\Omega}_0\cup\bar{\Omega}_1\cup\bar{\Omega}_2)$ (see Figure 2 below). It follows that (6.1) can be equivalently rewritten as the nonlinear boundary value problem in Ω_1:

$$-\mathrm{div}\,\mathbf{a}(\cdot,\nabla u_1) \,=\, f \quad\text{in}\quad \Omega_1,\quad u_1\,=\,g_0\quad\text{on}\quad\Gamma_0\,,\tag{6.2}$$

the Poisson equation in Ω_2:

$$-\Delta u_2 \,=\, f \quad \text{in} \quad \Omega_2\,, \tag{6.3}$$

and the Laplace equation in the exterior unbounded region Ω_e:

$$-\Delta u_2 \,=\, 0 \quad \text{in} \quad \Omega_e, \quad u_2(\boldsymbol{x}) = \mathcal{O}(1) \quad \text{as} \quad |\boldsymbol{x}| \to \infty\,, \tag{6.4}$$

coupled with the transmission conditions on Γ_1 and Γ, respectively,

$$u_1 \,-\, u_2 \,=\, g_1 \quad \text{and} \quad \mathbf{a}(\cdot, \nabla u_1) \cdot \boldsymbol{\nu}_1 \,-\, \nabla u_2 \cdot \boldsymbol{\nu}_1 \,=\, g_2 \quad \text{on} \quad \Gamma_1\,, \tag{6.5}$$

and

$$\lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_2}} u_2(\boldsymbol{x}) \,=\, \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_e}} u_2(\boldsymbol{x}) \quad \text{and} \quad \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_2}} \nabla u_2(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}_0) \,=\, \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_e}} \nabla u_2(\boldsymbol{x}) \cdot \boldsymbol{\nu}(\boldsymbol{x}_0) \tag{6.6}$$

for almost all $\boldsymbol{x}_0 \in \Gamma$, where $\boldsymbol{\nu}(\boldsymbol{x}_0)$ denotes the unit outward normal to $\boldsymbol{x}_0$.

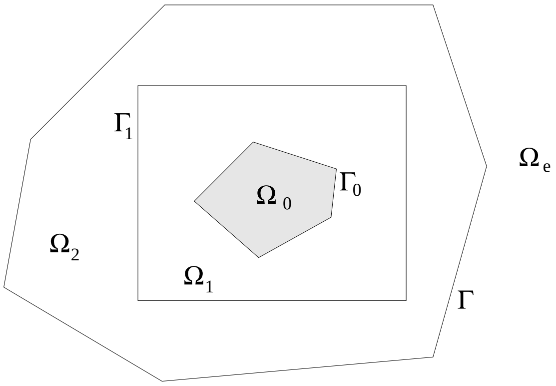

Figure 2: Geometry of the transmission problem.

We now follow [15] and [7] and introduce the gradients $\boldsymbol{\theta}_1 := \nabla u_1$ in Ω_1 and $\boldsymbol{\theta}_2 := \nabla u_2$ in Ω_2, and the fluxes $\boldsymbol{\sigma}_1 := \mathbf{a}(\cdot, \boldsymbol{\theta}_1)$ in Ω_1 and $\boldsymbol{\sigma}_2 := \boldsymbol{\theta}_2$ in Ω_2, as additional unknowns. Also, as in Section 2, let $\lambda_h \in X_h^0$ be a discrete approximation of the normal derivative $\lambda := \partial_{\boldsymbol{\nu}} u_2$ on Γ, and proceeding in the usual way (see [7] for details). We arrive at the following global LDG formulation in $\Omega := \Omega_1 \cup \Gamma_1 \cup \Omega_2$: *Find* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h$ *such that*

$$\begin{array}{lcll}
\displaystyle\int_\Omega \bar{\mathbf{a}}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta} - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta} & = & 0 & \forall\, \boldsymbol{\zeta} \in \boldsymbol{\Sigma}_h\,, \\
\displaystyle\int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau} \,-\, \left\{ \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} - S(u_h, \boldsymbol{\tau}) \right\} & = & G_h(\boldsymbol{\tau}) & \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,, \\
\displaystyle\left\{ \int_\Omega \nabla_h v \cdot \boldsymbol{\sigma}_h \,-\, S(v, \boldsymbol{\sigma}_h) \right\} + \boldsymbol{\alpha}(u_h, v) & = & \displaystyle F_h(v) + \int_\Gamma \lambda_h\, v & \forall\, v \in \tilde{V}_h\,,
\end{array} \tag{6.7}$$

where

$$\bar{\mathbf{a}}(\cdot, \boldsymbol{\zeta}) \,:=\, \left\{ \begin{array}{ll} \mathbf{a}(\cdot, \boldsymbol{\zeta}) & \text{in } \Omega_1 \\ \boldsymbol{\zeta} & \text{in } \Omega_2 \end{array} \right. \qquad \forall\, \boldsymbol{\zeta} \in [L^2(\Omega)]^2\,,$$

and the bilinear forms $S : H^1(\mathcal{T}_h) \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ and $\boldsymbol{\alpha} : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ as well as the linear operators $G_h : \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ and $F_h : H^1(\mathcal{T}_h) \to \mathbb{R}$ are defined by

$$S(w, \boldsymbol{\tau}) := \int_{I_h} [\![w]\!] \cdot (\{\boldsymbol{\tau}\} - [\![\boldsymbol{\tau}]\!]\, \boldsymbol{\beta}) \,+\, \int_{\Gamma_0} w\,(\boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}) \,+\, \int_{\Gamma_1} (w_1 - w_2)\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_1\,,$$

$$\boldsymbol{\alpha}(w, v) := \int_{I_h} \alpha\, [\![w]\!] \cdot [\![v]\!] \,+\, \int_{\Gamma_0} \alpha\, w\, v \,+\, \int_{\Gamma_1} \alpha\,(w_1 - w_2)\,(v_1 - v_2)\,,$$

$$G_h(\boldsymbol{\tau}) := \int_{\Gamma_0} g_0\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu} \,+\, \int_{\Gamma_1} g_1\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_1\,,$$

and

$$F_h(v) := \int_\Omega f\, v \,+\, \int_{\Gamma_0} \alpha\, g_0\, v_1 \,+\, \int_{\Gamma_1} \alpha\, g_1\,(v_1 - v_2) \,+\, \int_{\Gamma_1} g_2\, v_2$$

for all $w\,, v \in H^1(\mathcal{T}_h)$, $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega)$, with $w_i := w|_{\Omega_i}$, $v_i := v|_{\Omega_i}$, and $\boldsymbol{\tau}_i := \boldsymbol{\tau}|_{\Omega_i}$, for each $i \in \{1, 2\}$. Hereafter, $\mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$, where $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$ are shape regular triangulations of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, respectively, which satisfy the same properties and assumptions as indicated in Section 2.2.

Next, introducing the boundary integral formulation in Ω_e, exactly as in Section 2.1, substituting λ_h in (6.7) by a discrete version of the first equation in (2.2), in which u is replaced by its approximant u_h, and adding a discrete formulation of the second equation in (2.2), we obtain the following coupled LDG-BEM scheme: Find $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ such that

$$\begin{aligned}
\int_\Omega \bar{\mathbf{a}}(\cdot, \boldsymbol{\theta}_h) \cdot \boldsymbol{\zeta} - \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\zeta} \quad &= \quad 0\,,\\
\int_\Omega \boldsymbol{\theta}_h \cdot \boldsymbol{\tau} \,-\, \boldsymbol{\rho}(u_h, \boldsymbol{\tau}) \quad &= \quad G_h(\boldsymbol{\tau})\,,\\
\boldsymbol{\rho}(v, \boldsymbol{\sigma}_h) \,+\, \boldsymbol{\alpha}(u_h, v) \,+\, \langle \mathcal{W} u_h, v\rangle \,-\, \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle \quad &= \quad F_h(v)\,,\\
\langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K}) u_h\rangle \,+\, \langle \mu, \mathcal{V}\lambda_h\rangle \quad &= \quad 0
\end{aligned} \tag{6.8}$$

for all $(\boldsymbol{\zeta}, \boldsymbol{\tau}, v, \mu) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$, where $\boldsymbol{\rho} : H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h) \to \mathbb{R}$ is the analogue of the bilinear form defined by (2.12), that is

$$\boldsymbol{\rho}(v, \boldsymbol{\tau}) \,:=\, \int_\Omega \nabla_h v \cdot \boldsymbol{\tau} - S(v, \boldsymbol{\tau}) \qquad \forall\, (v, \boldsymbol{\tau}) \,\in\, H^1(\mathcal{T}_h) \times \mathbf{H}^1(\mathcal{T}_h)\,.$$

In what follows we proceed as in Section 2.3 (see also Section 2.4 of [15]) and derive an equivalent formulation to (6.8). We begin by defining a linear operator $\mathbf{S}_h : H^1(\mathcal{T}_h) \to \boldsymbol{\Sigma}_h$ as in (2.14), where, given $v \in H^1(\mathcal{T}_h)$, $\mathbf{S}_h(v)$ is the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathbf{S}_h(v) \cdot \boldsymbol{\tau} \,=\, S(v, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,. \tag{6.9}$$

Next, let $\mathcal{G}_h$ be the unique element in $\boldsymbol{\Sigma}_h$ such that

$$\int_\Omega \mathcal{G}_h \cdot \boldsymbol{\tau} \,=\, G_h(\boldsymbol{\tau}) \,:=\, \int_{\Gamma_0} g_0\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu} \,+\, \int_{\Gamma_1} g_1\, \boldsymbol{\tau}_1 \cdot \boldsymbol{\nu}_1 \qquad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,. \tag{6.10}$$

It is easy to see that $\mathcal{G}_h\big|_{\Omega_2} = \mathbf{0}$. From now on we set $u \,:=\, \begin{cases} u_1 \text{ in } \Omega_1 \\ u_2 \text{ in } \Omega_2 \end{cases}$. Then, if the solution of problem (6.1) satisfies $u_1 \in H^t(\Omega_1)$ and $u_2 \in H^s(\Omega_2)$, with $t\,, s > 1$, we find that $\mathbf{S}_h(u) = \mathcal{G}_h$. In addition, it follows from the first two equations in (6.8) that, whenever this system is solvable, there holds

$$\boldsymbol{\theta}_h \,=\, \nabla_h u_h - \mathbf{S}_h(u_h) \,+\, \mathcal{G}_h \quad \text{and} \quad \boldsymbol{\sigma}_h \,=\, \Pi_{\boldsymbol{\Sigma}_h} \bar{\mathbf{a}}(\cdot, \boldsymbol{\theta}_h)\,, \tag{6.11}$$

where $\Pi_{\boldsymbol{\Sigma}_h}$ denotes the $\boldsymbol{L}^2(\Omega)-$orthogonal projection onto $\boldsymbol{\Sigma}_h$. We observe here, as in the proof of Lemma 2.1, that the fact that $r \geq m-1$ guarantees that $\nabla_h u_h \in \boldsymbol{\Sigma}_h$, which yields the above expression for $\boldsymbol{\theta}_h$. Then, replacing the unknown $\boldsymbol{\sigma}_h$ by

$$\Pi_{\boldsymbol{\Sigma}_h} \bar{\mathbf{a}}(\cdot, \nabla_h u_h - \mathbf{S}_h(u_h) \,+\, \mathcal{G}_h)$$

in the third equation of (6.8), we are led to the semilinear form $A_h : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ defined by

$$A_h(w,v) \,:=\, \boldsymbol{\alpha}(w,v) \,+\, \int_\Omega \bar{\mathbf{a}}(\cdot, \nabla_h w - \mathbf{S}_h(w) + \mathcal{G}_h) \cdot (\nabla_h v - \mathbf{S}_h(v)) \qquad \forall\, w, v \in H^1(\mathcal{T}_h)\,.$$

Moreover, we can establish the following equivalence result which constitutes the nonlinear analogue of Lemma 2.1.

Lemma 6.1 *Let* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Sigma}_h \times \tilde{V}_h \times X_h^0$ *be a solution of* (6.8). *Then there holds*

$$\begin{aligned} A_h(u_h, v) \,+\, \langle \mathcal{W} u_h, v\rangle \,-\, \langle (\tfrac{1}{2}\mathcal{I} - \mathcal{K}')\lambda_h, v\rangle \;&=\; F_h(v)\,, \\ \langle \mu, (\tfrac{1}{2}\mathcal{I} - \mathcal{K}) u_h\rangle \,+\, \langle \mu, \mathcal{V}\lambda_h\rangle \;&=\; 0 \end{aligned} \tag{6.12}$$

for any $(v,\mu) \in \tilde{V}_h \times X_h^0$. *Conversely, if* $(u_h, \lambda_h) \in \tilde{V}_h \times X_h^0$ *satisfies* (6.12) *and* $\boldsymbol{\theta}_h$ *and* $\boldsymbol{\sigma}_h$ *are defined by* (6.11)*, then* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h)$ *is a solution of* (6.8). *If* $(u_h, \lambda_h) \in \tilde{V}_h \times X_h^0$ *is the only solution of* (6.12) *then* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h)$*, with* $\boldsymbol{\theta}_h$ *and* $\boldsymbol{\sigma}_h$ *defined as indicated above, is the only solution of* (6.8).

Proof. It is similar to the proof of Lemma 2.1 (see also Lemma 2.2 in [15]) and is based on the identities (6.11). ☐

We now introduce seminorms

$$|v|_{1,h}^2 := \|\nabla_h v\|_{0,\Omega}^2\,, \quad |v|_*^2 := \|h_{\mathcal{E}}^{-1/2}[\![v]\!]\|_{0,I_h}^2 + \|h_{\mathcal{E}}^{-1/2} v\|_{0,\Gamma_0}^2 \,+\, \|h_{\mathcal{E}}^{-1/2}(v_1 - v_2)\|_{0,\Gamma_1}^2 \quad \forall\, v \,\in\, H^1(\mathcal{T}_h),$$

and the norms

$$|\!|\!| v |\!|\!|_h^2 := |v|_{1,h}^2 + |v|_*^2 \qquad \forall\, v \,\in\, H^1(\mathcal{T}_h)\,,$$

$$\|(v,\mu)\|_{h,\Gamma}^2 \,:=\, |\!|\!| v |\!|\!|_h^2 \,+\, \|v\|_{1/2,\Gamma,0}^2 \,+\, \|\mu\|_{-1/2,\Gamma}^2 \qquad \forall\,(v,\mu) \,\in\, H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)\,.$$

Next, let $\mathbf{A}_h$ be the semilinear form defined by the left-hand side of (6.12), i.e.

$$\mathbf{A}_h(w,\eta; v,\mu) \,:=\, A_h(w,v) \,+\, \langle \mathcal{W}w, v\rangle - \langle (\frac{1}{2}\mathcal{I} - \mathcal{K}')\eta, v\rangle \,+\, \langle \mu, (\frac{1}{2}\mathcal{I} - \mathcal{K})w\rangle \,+\, \langle \mu, \mathcal{V}\eta\rangle$$

for any (w,η), $(v,\mu) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$. The following result shows that $\mathbf{A}_h$ is Lipschitz continuous and strongly monotone with respect to $\|\cdot\|_{h,\Gamma}$. This is crucial for the analysis of (6.12) (and hence of (6.8)).

Lemma 6.2 *There exist positive constants* C_{LM} *and* C_{SM}*, independent of* h*, such that*

$$|\mathbf{A}_h(w,\eta; z,\xi) \,-\, \mathbf{A}_h(v,\mu; z,\xi)| \,\leq\, C_{LM}\, \|(w,\eta) - (v,\mu)\|_{h,\Gamma}\, \|(z,\xi)\|_{h,\Gamma} \tag{6.13}$$

and

$$\mathbf{A}_h(w,\eta; (w,\eta) - (v,\mu)) \,-\, \mathbf{A}_h(v,\mu; (w,\eta) - (v,\mu)) \,\geq\, C_{SM}\, \|(w,\eta) - (v,\mu)\|_{h,\Gamma}^2 \tag{6.14}$$

for any (w,η), (v,μ), $(z,\xi) \in H^1_{1/2}(\mathcal{T}_h) \times H_0^{-1/2}(\Gamma)$.

Proof. The Lipschitz continuity and strong monotonicity of the semilinear form A_h with respect to the norm $\|\!|\cdot|\!\|_h$ are provided by Lemmas 4.1 and 4.2 in [5]. The estimates required for the remaining boundary integral terms of $\mathbf{A}_h$ follow exactly as in the proof of Lemma 3.2. We omit further details. □

The unique solvability of (6.8) is established now.

Theorem 6.1 *There exists a unique* $(\boldsymbol{\theta}_h, \boldsymbol{\sigma}_h, u_h, \lambda_h) \in \mathbf{\Sigma}_h \times \mathbf{\Sigma}_h \times \tilde{V}_h \times X_h^0$ *solution to the coupled* LDG-BEM *scheme* (6.8). *In addition, there exists* $C > 0$, *independent of* h, *such that*

$$\|\boldsymbol{\theta}_h\|_{0,\Omega} + \|\boldsymbol{\sigma}_h\|_{0,\Omega} + \|(u_h,\lambda_h)\|_{h,\Gamma} \le C\Big\{\mathcal{N}(f,g_0,g_1,g_2) + \|\bar{\mathbf{a}}(\cdot,0)\|_{0,\Omega}\Big\} \tag{6.15}$$

where

$$\mathcal{N}(f,g_0,g_1,g_2) := \Big\{||f||_{0,\Omega}^2 + ||\alpha^{1/2}\,g_0||_{0,\Gamma_0}^2 + ||\alpha^{1/2}\,g_1||_{0,\Gamma_1}^2 + ||\alpha^{1/2}\,g_2||_{0,\Gamma_1}^2\Big\}^{1/2}.$$

Proof. By Lemma 6.1 it suffices to analyze the reduced system (6.12) instead of (6.8). It is clear that (6.12) can be equivalently formulated as: Find $(u_h, \lambda_h) \in \tilde{V}_h \times X_h^0$ such that

$$\mathbf{A}_h(u_h,\lambda_h; v,\mu) := F_h(v) \qquad \forall\,(v,\mu) \in \tilde{V}_h \times X_h^0.$$

Now, proceeding as in the proof of Lemma 4.4 in [5], we find $C > 0$, independent of h, such that

$$|F_h(v)| \le C\,\mathcal{N}(f,g_0,g_1,g_2)\ \|\!|v|\!\|_h, \quad \forall\, v \in \tilde{V}_h. \tag{6.16}$$

Hence, Lemma 6.2 and a classical result of nonlinear functional analysis imply the unique solvability of (6.12). The rest of the proof follows very closely the proof of Theorem 3.2 in [7]. In fact, using again the strong monotonicity of $\mathbf{A}_h$, estimate (6.16), the fact that

$$\mathbf{A}_h((0,0),(v,\mu)) = A_h(0,v) = \int_\Omega \bar{\mathbf{a}}(\cdot,\mathcal{G}_h)\cdot(\nabla_h v - \boldsymbol{S}_h v) \qquad \forall\,(v,\mu) \in \tilde{V}_h \times X_h^0,$$

the boundedness of $\mathbf{S}_h$ (cf. (3.5)), and the Lipschitz continuity of the nonlinear operator induced by $\bar{\mathbf{a}}$, one deduces that

$$\|(u_h,\lambda_h)\|_{h,\Gamma} \le C\Big\{\mathcal{N}(f,g_0,g_1,g_2) + \|\bar{\mathbf{a}}(\cdot,0)\|_{0,\Omega} + \|\mathcal{G}_h\|_{0,\Omega}\Big\}. \tag{6.17}$$

Also, using the expressions for $\boldsymbol{\theta}_h$ and $\boldsymbol{\sigma}_h$ given by (6.11) , and applying again the boundedness of $\mathbf{S}_h$ and the Lipschitz continuity of $\bar{\mathbf{a}}$, we obtain

$$\|\boldsymbol{\theta}_h\|_{0,\Omega} \le C\Big\{\|\!|u_h|\!\|_h + \|\mathcal{G}_h\|_{0,\Omega}\Big\} \quad \text{and} \quad \|\boldsymbol{\sigma}_h\|_{0,\Omega} \le C\Big\{\|\boldsymbol{\theta}_h\|_{0,\Omega} + \|\bar{\mathbf{a}}(\cdot,0)\|_{0,\Omega}\Big\}. \tag{6.18}$$

Then, it is easy to show, as in the proof of Lemma 3.4 in [5], that (cf. (6.10))

$$\|\mathcal{G}_h\|_{0,\Omega} \le C\Big\{||\alpha^{1/2}\,g_0||_{0,\Gamma_0} + ||\alpha^{1/2}\,g_1||_{0,\Gamma_1}\Big\}. \tag{6.19}$$

In this way, (6.15) follows directly from (6.17), (6.18), and (6.19), which ends the proof. □

Finally, we prove the a priori error estimate for the coupled LDG-BEM scheme (6.8).

Theorem 6.2 *Define the additional continuous unknowns*

$$\boldsymbol{\theta} \,=\, \begin{cases} \boldsymbol{\theta}_1 \,:=\, \nabla u_1 \ \text{ in } \ \Omega_1 \\ \boldsymbol{\theta}_2 \,:=\, \nabla u_2 \ \text{ in } \ \Omega_2 \end{cases} , \ \boldsymbol{\sigma} \,=\, \begin{cases} \boldsymbol{\sigma}_1 \,:=\, \mathbf{a}(\cdot, \boldsymbol{\theta}_1) \ \text{ in } \ \Omega_1 \\ \boldsymbol{\sigma}_2 \,:=\, \boldsymbol{\theta}_2 \ \text{ in } \ \Omega_2 \end{cases} , \ \text{and} \ \lambda \,=\, \partial_{\boldsymbol{\nu}} u_2 \quad on \quad \Gamma \,.$$

Assume that there exist δ_1, $\delta_2 > 1/2$ *such that* $u_1 \in H^{1+\delta_1}(\Omega_1)$, $u_2 \in H^{1+\delta_2}(\Omega_2)$, *and* $\boldsymbol{\sigma}_1 \in [H^{\delta_1}(\Omega_1)]^2$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$\begin{aligned} & \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \\ & \le\, C \left\{ h^{\min\{\delta_1,m\}} \, \|u_1\|_{1+\delta_1,\Omega_1} \,+\, h^{\min\{\delta_1,m\}} \, \|\boldsymbol{\sigma}_1\|_{\delta_1,\Omega_1} \,+\, h^{\min\{\delta_2,m\}} \, \|u_2\|_{1+\delta_2,\Omega_2} \right\} . \end{aligned} \tag{6.20}$$

Proof. We observe, similarly as in the linear case (cf. Theorem 4.1), that $\lambda \,\in\, H^{\delta_2 - 1/2}(\Gamma)$ and $\|\lambda\|_{\delta_2-1/2,\Gamma} \le C\,\|u_2\|_{1+\delta_2,\Omega_2}$. Also, according to the definitions of the semilinear form $\mathbf{A}_h$ and the linear operator F_h, and taking into account the equations, the boundary conditions, and the transmission conditions satisfied by u, one can prove that u and λ satisfy

$$\mathbf{A}_h(u,\lambda; v,\mu) \,=\, F_h(v) \qquad \forall\,(v,\mu) \,\in\, H^1_{1/2}(\mathcal{T}_h) \times H^{-1/2}(\Gamma) \,.$$

In addition, it is clear that the discrete system (6.12) renders like

$$\mathbf{A}_h(u_h,\lambda_h; v,\mu) \,=\, F_h(v) \qquad \forall\,(v,\mu) \,\in\, \tilde{V}_h \times X_h^0 \,.$$

Then, the Lipschitz continuity and strong monotonicity of $\mathbf{A}_h$ also yield the quasi-optimal estimate (4.17), that is

$$\|(u,\lambda) - (u_h,\lambda_h)\|_{h,\Gamma} \,\le\, C\,\|(u,\lambda) - (v_h,\mu_h)\|_{h,\Gamma} \qquad \forall\,(v_h,\mu_h) \,\in\, \tilde{V}_h \times X_h^0 \,. \tag{6.21}$$

Now, using that $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}_h(u_h) + \mathcal{G}_h$ (cf. (6.11)), $\boldsymbol{\theta} \,=\, \nabla\, u$ in Ω, $\mathbf{S}_h(u) \,=\, \mathcal{G}_h$, and applying the boundedness of $\mathbf{S}_h$, we obtain

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{0,\Omega} \,\le\, C\,|\!|\!| u - u_h |\!|\!|_h \,. \tag{6.22}$$

It remains to estimate $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$. Using that $\boldsymbol{\sigma}_h \,=\, \Pi_{\boldsymbol{\Sigma}_h}\, \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta}_h)$ (cf. (6.11)) and $\boldsymbol{\sigma} \,=\, \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta})$, and applying the triangle inequality and the Lipschitz-continuity of the nonlinear operator induced by $\bar{\mathbf{a}}$, we deduce that

$$\begin{aligned} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \quad & \le\, \|\boldsymbol{\sigma} \,-\, \Pi_{\boldsymbol{\Sigma}_h} \boldsymbol{\sigma}\|_{0,\Omega} \,+\, \|\Pi_{\boldsymbol{\Sigma}_h} \left\{ \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta}) \,-\, \bar{\mathbf{a}}(\cdot,\boldsymbol{\theta}_h) \right\}\|_{0,\Omega} \\ & \le\, \|\boldsymbol{\sigma} \,-\, \Pi_{\boldsymbol{\Sigma}_h} \boldsymbol{\sigma}\|_{0,\Omega} \,+\, C\,\|\boldsymbol{\theta} \,-\, \boldsymbol{\theta}_h\|_{0,\Omega} \,. \end{aligned} \tag{6.23}$$

Then, applying local approximation properties of piecewise polynomials (see, e.g. Lemma 4.2 in [15]), recalling from (2.5) that on $K \in \mathcal{T}_h$, $\Pi_{\boldsymbol{\Sigma}_h}$ reduces to the $\boldsymbol{L}^2(K)$-orthogonal projection onto $\mathbf{P}_r(K)$, which is denoted by Π_K^r, and noting that $r + 1 \ge m$, we find that

$$\begin{aligned} \|\boldsymbol{\sigma} \,-\, \Pi_{\boldsymbol{\Sigma}_h} \boldsymbol{\sigma}\|_{0,\Omega_1} \,=\, & \sum_{K\in\mathcal{T}_{h,1}} \|\boldsymbol{\sigma}_1 \,-\, \Pi_K^r\, \boldsymbol{\sigma}_1\|_{0,K}^2 \le C \sum_{K\in\mathcal{T}_{h,1}} h_K^{2\,\min\{\delta_1,r+1\}} \, \|\boldsymbol{\sigma}_1\|_{\delta_1,K}^2 \\ \le\, C\,h^{2\,\min\{\delta_1,r+1\}} \, \|\boldsymbol{\sigma}_1\|_{\delta_1,\Omega_1}^2 \,\le\, & C\,h^{2\,\min\{\delta_1,m\}} \, \|\boldsymbol{\sigma}_1\|_{\delta_1,\Omega_1}^2 \,, \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} \|\boldsymbol{\sigma} \,-\, \Pi_{\boldsymbol{\Sigma}_h} \boldsymbol{\sigma}\|_{0,\Omega_2}^2 \,&=\, \sum_{K\in\mathcal{T}_{h,2}} \|\boldsymbol{\theta}_2 \,-\, \Pi_K^r\, \boldsymbol{\theta}_2\|_{0,K}^2 \\ &=\, \sum_{K\in\mathcal{T}_{h,2}} \|\nabla u_2 \,-\, \Pi_K^r\, \nabla u_2\|_{0,K}^2 \le C \sum_{K\in\mathcal{T}_{h,2}} h_K^{2\,\min\{\delta_2,r+1\}} \, \|\nabla u_2\|_{\delta_2,K}^2 \\ &\le\, C\,h^{2\,\min\{\delta_2,r+1\}} \, \|u_2\|_{1+\delta_2,\Omega_2}^2 \,\le\, C\,h^{2\,\min\{\delta_2,m\}} \, \|u_2\|_{1+\delta_2,\Omega_2}^2 \,. \end{aligned} \tag{6.25}$$

In this way, the approximation properties from Lemmas 4.2 and 4.3 (with $t = \delta_2 - 1/2$), together with the bound $\|\lambda\|_{\delta_2-1/2,\Gamma} \le C\, \|u_2\|_{1+\delta_2,\Omega_2}$, and inequalities (6.21), (6.22), (6.23), (6.24), and (6.25), imply the required a priori error estimate and finish the proof. □

We end this section by remarking, as we did for the linear case at the end of Section 4, that the a priori error estimate (6.20) is also independent of the polynomial degree r that defines the subspace $\mathbf{\Sigma}_h$ (cf. (2.5)). Therefore, since the restriction $r \ge m-1$ is required only to deduce that $\boldsymbol{\theta}_h = \nabla_h u_h - \mathbf{S}_h(u_h) + \mathcal{G}_h$ (cf. (6.11)), it suffices also to take $r = m-1$ in the present nonlinear case.

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