The boundary element method with Lagrangian multipliers *

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Dedicated to Professor George C. Hsiao on the occasion of his 75th birthday.

Abstract

On open surfaces, the energy space of hypersingular operators is a fractional order Sobolev space of order 1/2 with homogeneous Dirichlet boundary condition (along the boundary curve of the surface) in a weak sense. We introduce a boundary element Galerkin method where this boundary condition is incorporated via the use of a Lagrangian multiplier. We prove the quasi-optimal convergence of this method (it is slightly inferior to the standard conforming method) and underline the theory by a numerical experiment.

The approach presented in this paper is not meant to be a competitive alternative to the conforming method but rather the basis for non-conforming techniques like the mortar method, to be developed.

Key words: boundary element method, non-conforming Galerkin method, Lagrangian multiplier
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1 Introduction

Lagrangian multipliers are very convenient to incorporate non-homogeneous essential boundary conditions into the finite element method (FEM). The basic setting goes back to the early paper by Babuška [1]. A recent approach in connection with mixed finite elements is analysed in [2]. There is no theory on the use of Lagrangian multipliers within the framework of the boundary element Galerkin method (BEM). There are two immediate reasons for this. Firstly, there are no boundary value problems with representation by boundary integral equations where non-homogeneous essential boundary conditions (for the solutions of the integral equations) appear.

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When dealing with hypersingular operators on open surfaces, boundary element functions of a conforming method (non-conforming methods have not yet been studied) need to vanish at the boundary of the surface [17]. This homogeneous condition can be easily implemented without Lagrangian multiplier. Secondly, in the case of hypersingular operators, the analysis of essential boundary conditions is problematic since there is no well-defined trace operator in $H^{1/2}$ (the critical space for hypersingular operators).

There are wide reaching implications of the unavailability of a weak treatment of boundary conditions in the BEM: non-conforming and discontinuous and mortar-type domain decomposition methods are all unknown for hypersingular integral operators. The principal reason for this is that one does not know how to deal with interface conditions in the space $H^{1/2}$.

The use of finite element technology for boundary integral equations of the first kind goes back to Hsiao and Wendland [12]. This required the use of fractional order Sobolev spaces. Later, Stephan [17] analysed the BEM for boundary integral equations of the first kind on open surfaces. Here, essential ingredients were fractional order Sobolev spaces consisting of functions which can be continuously extended by zero onto a larger surface. In this paper we go a step further in analysing and implementing this extendibility condition in the framework of Lagrangian multipliers. In this way we provide the missing analysis for weak interface conditions in $H^{1/2}$ and show that they can be incorporated into discrete subspaces of $H^{1/2}$ in a quasi-optimal way. Quasi-optimal means that the resulting method converges slightly slower than a conforming method (there is a logarithmical perturbation in the mesh size) and this is due to the non-existence of a trace operator. We consider the model situation of homogenous boundary conditions for the hypersingular operator (of the Laplacian) on an open surface. An extension of this analysis to other methods (like mortar-type domain decomposition) is under investigation.

The main procedure and result of this paper can be described as follows. We consider the hypersingular operator W on a flat open surface Γ in $\mathbb{R}^3$. The energy space is $\tilde{H}^{1/2}(\Gamma)$ (for an exact definition see below) and any conforming boundary element space must satisfy homogeneous boundary conditions along the boundary curve γ of Γ. We discretise, instead of $\tilde{H}^{1/2}(\Gamma)$, the space $H^{1/2}(\Gamma)$ and add a Lagrangian multiplier for the approximate implementation of the boundary condition. The space for the Lagrangian multiplier consists of piecewise constant functions on γ. Assuming a compatibility condition for quasi-uniform meshes on Γ and γ we prove that the scheme with Lagrangian multiplier converges like $O\Big(\log^{3/2}(1/h)h^{1/2-\epsilon}\Big)$ in $H^{1/2}(\Gamma)$ (see Theorem 3.1) whereas the conforming method converges like $O(h^{1/2})$ in $\tilde{H}^{1/2}(\Gamma)$ (see [4]). Here, $h < 1$ indicates the mesh size on Γ and ϵ is a positive and arbitrarily small but fixed number. We note that the parameter ϵ appears due to the unavailability of an appropriate regularity theory, and the term $\log^{3/2}(1/h)$ is due to the non-conformity of the method and the non-existence of a well-defined trace operator in $H^{1/2}(\Gamma)$.

In the next section we introduce the model problem along with Sobolev spaces and some technical results for surface differential operators. In Section 3 we present the BEM scheme with Lagrangian multiplier and state the main result (Theorem 3.1) on the convergence of the discrete scheme. Section 4 provides several technical results. Moreover, a Strang-type error estimate for the BEM with Lagrangian multiplier (Theorem 4.1) and a proof of Theorem 3.1 are given there. In Section 5 we discuss the evaluation of errors in $H^{1/2}$ and report on a numerical experiment which underlines the quasi-optimal convergence of our method.

Throughout the paper, C denotes a generic positive constant which is independent of mesh sizes and the appearing parameter $\epsilon > 0$.

2 Sobolev spaces and model problem

In this section we define our model problem, introduce a BEM formulation with Lagrangian multiplier for it and state the main result (Theorem 3.1) on the convergence of the BEM with Lagrangian multiplier. In order to present the discrete scheme we need to recall definitions of surface differential operators and an integration-by-parts formula. Also we need some technical results on the surface curl-operator. This is all done in this section.

First let us briefly define the needed Sobolev spaces. We consider standard Sobolev spaces where the following norms are used: For $\Omega \subset \mathbb{R}^n$ and $0 < s < 1$ we define

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|_{H^s(\Omega)} := \Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{2s+n}}\,dx\,dy\Big)^{1/2}.$$

For a Lipschitz domain Ω and $0 < s < 1$ the space $\tilde{H}^s(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\tilde{H}^s(\Omega)} := \Big(|u|^2_{H^s(\Omega)} + \int_\Omega \frac{u(x)^2}{(\mathrm{dist}(x,\partial\Omega))^{2s}}\,dx\Big)^{1/2}.$$

For $s \in (0, 1/2)$, $\|\cdot\|_{\tilde{H}^s(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$ are equivalent norms whereas for $s \in (1/2, 1)$ there holds $\tilde{H}^s(\Omega) = H_0^s(\Omega)$, the latter space being the completion of $C_0^\infty(\Omega)$ with norm in $H^s(\Omega)$. Also we note that functions from $\tilde{H}^s(\Omega)$ are continuously extendible by zero onto a larger domain. For all these results we refer to [13, 9]. For $s > 0$ the spaces $H^{-s}(\Omega)$ and $\tilde{H}^{-s}(\Omega)$ are the dual spaces of $\tilde{H}^s(\Omega)$ and $H^s(\Omega)$, respectively.

Now we introduce the model problem. For simplicity let Γ be the plane open surface $(0,1) \times (0,1) \times \{0\}$. We will identify it with the square $(0,1)^2 \subset \mathbb{R}^2$.

Our problem is: *For given* $f \in \tilde{H}^{-1/2}(\Gamma)$ *find* $\phi \in \tilde{H}^{1/2}(\Gamma)$ *such that*

$$W\phi(x) := -\frac{1}{4\pi}\frac{\partial}{\partial \mathbf{n}_x}\int_\Gamma \phi(y)\frac{\partial}{\partial \mathbf{n}_y}\frac{1}{|x-y|}\,dS_y = f(x), \quad x \in \Gamma. \tag{2.1}$$

Here, $\mathbf{n}$ is a normal unit vector on Γ, e.g. $\mathbf{n} = (0,0,1)^T$. Note that W maps $\tilde{H}^{1/2}(\Gamma)$ continuously on $H^{-1/2}(\Gamma)$ (see [7]). Nevertheless, we require f to be slightly more regular, $f \in \tilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$. This will be needed below when testing f with elements of $H^{1/2}(\Gamma)$.

The variational formulation of (2.1) is: *Find* $\phi \in \tilde{H}^{1/2}(\Gamma)$ *such that*

$$\langle W\phi, \psi\rangle_\Gamma = \langle f, \psi\rangle_\Gamma \quad \forall \psi \in \tilde{H}^{1/2}(\Gamma). \tag{2.2}$$

Here, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the $L^2(\Gamma)$ inner product and is also used for its generic extension to the duality between negative order Sobolev spaces and their dual spaces (in this case between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$). Later we will indicate just the support where the duality is taken (on Γ or its boundary γ).

A standard boundary element method for the approximate solution of (2.2) is to select a piecewise polynomial subspace $\tilde{H}_h \subset \tilde{H}^{1/2}(\Gamma)$ and to define an approximant $\tilde{\phi}_h \in \tilde{H}_h$ by

$$\langle W\tilde{\phi}_h, \psi\rangle_\Gamma = \langle f, \psi\rangle_\Gamma \quad \forall \psi \in \tilde{H}_h.$$

The conformity condition $\tilde{H}_h \subset \tilde{H}^{1/2}(\Gamma)$ requires that any $\psi \in \tilde{H}_h$ vanishes on the boundary γ of Γ. In this paper we study, instead, a non-conforming discretisation of (2.2) by using subspaces H_h whose elements do not necessarily vanish on γ. Our subspaces will satisfy $H_h \subset H^{1/2}(\Gamma)$ but $H_h \not\subset \tilde{H}^{1/2}(\Gamma)$. The boundary condition is incorporated weakly by using a Lagrangian multiplier. Note that the natural domain of definition of the hypersingular operator W is $\tilde{H}^{1/2}(\Gamma)$ and not $H^{1/2}(\Gamma)$. We therefore need to deal with boundary data for an appropriate definition of W and this amounts to an integration-by-parts formula.

First let us consider the situation of the conforming continuous formulation (2.2). After that we will study a non-conforming discrete setting. There will be no non-conforming continuous formulation (i.e. a version of (2.2) with $\tilde{H}^{1/2}(\Gamma)$ replaced by $H^{1/2}(\Gamma)$). This is due to the fact that integration by parts involves the trace operator which is not well defined on $H^{1/2}(\Gamma)$. Integration by parts on Γ involves surface differential operators which will be introduced next.

We associate with any function φ on Γ a function Φ defined in $(0,1)^2\times(-1,1)$ by $\Phi(x_1,x_2,x_3) = \varphi(x_1,x_2)$. Then we define on Γ for a smooth function φ

$$\operatorname{grad}_\Gamma \varphi := (\operatorname{grad}\Phi)|_\Gamma, \qquad \mathbf{curl}_\Gamma\,\varphi := (\operatorname{grad}_\Gamma \varphi \times \mathbf{n})|_\Gamma.$$

Accordingly, we define for any sufficiently smooth tangential vector field $\boldsymbol{\varphi}$ on Γ

$$\operatorname{curl}_\Gamma \boldsymbol{\varphi} := \mathbf{n}\cdot(\operatorname{curl}\boldsymbol{\Phi})|_\Gamma.$$

Here, $\boldsymbol{\Phi}$ is the component-wise extension of $\boldsymbol{\varphi}$ as defined before. The definitions of $\operatorname{grad}_\Gamma$, $\mathbf{curl}_\Gamma$ and $\operatorname{curl}_\Gamma$ are appropriate for a non-flat smooth surface (using a coordinate direction normal to Γ instead of x_3 to define the extensions Φ and $\boldsymbol{\Phi}$) whereas, in our case of the flat surface Γ, they obviously reduce to

$$\operatorname{grad}_\Gamma \varphi = \big(\partial_{x_1}\varphi, \partial_{x_2}\varphi, 0\big), \quad \mathbf{curl}_\Gamma\,\varphi = \big(\partial_{x_2}\varphi, -\partial_{x_1}\varphi, 0\big), \quad \operatorname{curl}_\Gamma(\varphi_1,\varphi_2,0) = \partial_{x_1}\varphi_2 - \partial_{x_2}\varphi_1.$$

In the following we often extend Γ to a closed surface $\tilde{\Gamma}$. To distinguish between operators on different surfaces we add the notation of the corresponding surface as index to the operator, i.e. $\mathbf{curl}_\Gamma$ is defined on the plane open surface Γ and $\mathbf{curl}_{\tilde{\Gamma}}$ is defined on the (closed) surface $\tilde{\Gamma}$. For results in this paper it is enough to consider a smooth closed surface $\tilde{\Gamma}$. But in order to develop techniques that are applicable to polyhedral Lipschitz surfaces Γ we do not assume smoothness of $\tilde{\Gamma}$ but rather allow it to be a polyhedral Lipschitz surface. The surface differential operators can be defined in this case as well, but for details we refer to [6].

Following [6], $\mathbf{curl}_{\tilde{\Gamma}}$ can be extended to a continuous linear mapping from $H^{1/2}(\tilde{\Gamma})$ onto $\boldsymbol{H}_t^{-1/2}(\tilde{\Gamma})$ where $\tilde{\Gamma}$ is the boundary of a Lipschitz domain with $\Gamma \subset \tilde{\Gamma}$. Here, $\boldsymbol{H}_t^{-1/2}(\tilde{\Gamma})$ is a tangential subspace of $\left(H^{-1/2}(\tilde{\Gamma})\right)^3$, see [6] for a precise definition.

We use this extension to $H^{1/2}(\tilde{\Gamma})$ to define $\mathbf{curl}_\Gamma$ on $H^{1/2}(\Gamma)$:

$$\mathbf{curl}_\Gamma : \begin{cases} H^{1/2}(\Gamma) & \to \quad \boldsymbol{H}_t^{-1/2}(\Gamma) := \{\varphi \in \left(H^{-1/2}(\Gamma)\right)^3;\ \varphi \cdot \mathbf{n} = 0\} \\ \varphi & \mapsto \quad \left(\mathbf{curl}_{\tilde{\Gamma}}\, \tilde{\varphi}\right)|_\Gamma \end{cases} \tag{2.3}$$

where $\tilde{\varphi} \in H^{1/2}(\tilde{\Gamma})$ is an extension of φ. In Lemma 2.1 below we will show that $\mathbf{curl}_\Gamma$ is well-defined. To be precise, the definition of $\boldsymbol{H}_t^{-1/2}(\Gamma)$ in (2.3) is to be understood as the trace of $\boldsymbol{H}_t^{-1/2}(\tilde{\Gamma})$ onto Γ.

In the following we need the single layer potential operator V. It is defined by

$$V\boldsymbol{\varphi}(x) := \frac{1}{4\pi} \int_\Gamma \frac{\boldsymbol{\varphi}(y)}{|x-y|}\, dS_y, \quad \boldsymbol{\varphi} \in (\tilde{H}^{-1/2}(\Gamma))^3,\ x \in \Gamma.$$

It is well-known, and widely used in the boundary element literature, that weakly singular operators (e.g. the single layer potential operator V) can be used to represent hypersingular boundary integral operators (e.g. W), and that their bilinear forms relate like an integration-by-parts formula. This goes back to Maue [14] who studied the Helmholtz and Maxwell equations in three space dimensions, see also Nédélec [16]. The Lamé system in three dimensions has been dealt with by Nédélec [16], and Han [10] presented a simpler formula. All the mentioned authors studied closed smooth surfaces. Since we did not find a reference for open surfaces we recall this situation in Lemma 2.3 below. For its proof we need to study the mapping properties of $\mathbf{curl}_\Gamma$. This is done in the following two lemmas.

Lemma 2.1. *The operator* $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \boldsymbol{H}_t^{-1/2}(\Gamma)$ *defined by* (2.3) *is continuous.*

Proof. The continuity of $\mathbf{curl}_\Gamma$ holds by the existence of an extension operator $H^{1/2}(\Gamma) \to H^{1/2}(\tilde{\Gamma})$, the continuity of $\mathbf{curl}_{\tilde{\Gamma}} : H^{1/2}(\tilde{\Gamma}) \to \boldsymbol{H}_t^{-1/2}(\tilde{\Gamma})$ (see [6, Proposition 3.6]) and the continuity of the restriction $\boldsymbol{H}_t^{-1/2}(\tilde{\Gamma}) \to \boldsymbol{H}_t^{-1/2}(\Gamma)$. The definition of $\mathbf{curl}_\Gamma$ on $H^{1/2}(\Gamma)$ is independent of the particular extension since, for given $\varphi \in H^{1/2}(\Gamma)$ and two extensions $\tilde{\varphi}_1$, $\tilde{\varphi}_2 \in H^{1/2}(\tilde{\Gamma})$, there holds (with ψ^0 denoting the extension by 0 onto $\tilde{\Gamma}$ of ψ defined on Γ)

$$\langle \mathbf{curl}_{\tilde{\Gamma}}(\tilde{\varphi}_1 - \tilde{\varphi}_2), \psi^0 \rangle_{\tilde{\Gamma}} = \langle \tilde{\varphi}_1 - \tilde{\varphi}_2, \mathrm{curl}_{\tilde{\Gamma}}\, \psi^0 \rangle_{\tilde{\Gamma}} = \langle \tilde{\varphi}_1 - \tilde{\varphi}_2, \mathrm{curl}_\Gamma\, \psi \rangle_\Gamma = 0 \quad \forall \psi \in \boldsymbol{C}_{0,t}^\infty(\Gamma)$$

where

$$\boldsymbol{C}_{0,t}^\infty(\Gamma) := \{\psi \in \left(C_0^\infty(\Gamma)\right)^3;\ \psi \cdot \mathbf{n} = 0\}$$

which is dense in the dual space $\left(\boldsymbol{H}_t^{-1/2}(\Gamma)\right)'$. ∎

Lemma 2.2. *The restriction* $\mathbf{curl}_\Gamma|_{\tilde{H}^{1/2}(\Gamma)}$ *is continuous as a mapping* $\tilde{H}^{1/2}(\Gamma) \to \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)$ *where*

$$\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma) := \{\psi \in \left(\tilde{H}^{-1/2}(\Gamma)\right)^3;\ \psi \cdot \mathbf{n} = 0\}.$$

Proof. Let us introduce the space

$$\boldsymbol{H}_t^{1/2}(\Gamma) := \{\psi \in \left(H^{1/2}(\Gamma)\right)^3;\ \psi \cdot \mathbf{n} = 0\}$$

and for a function ψ on Γ let ψ^0 denote its extension onto $\tilde{\Gamma}$ by 0. For $\varphi \in C_0^\infty(\Gamma)$ there holds $\left(\mathbf{curl}_\Gamma\, \varphi\right)^0 = \mathbf{curl}_{\tilde{\Gamma}}\, \varphi^0$. Therefore, using the continuity of $\mathbf{curl}_{\tilde{\Gamma}}:\ H^{1/2}(\tilde{\Gamma}) \to \boldsymbol{H}_t^{-1/2}(\tilde{\Gamma})$, we obtain

$$\begin{aligned}\|\,\mathbf{curl}_\Gamma\, \varphi\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} &= \sup_{0\neq\psi\in\boldsymbol{H}_t^{1/2}(\Gamma)} \frac{\langle \mathbf{curl}_\Gamma\, \varphi, \psi\rangle_\Gamma}{\|\psi\|_{\boldsymbol{H}_t^{1/2}(\Gamma)}} \simeq \sup_{0\neq\tilde{\psi}\in\boldsymbol{H}_t^{1/2}(\tilde{\Gamma})} \frac{\langle \left(\mathbf{curl}_\Gamma\, \varphi\right)^0, \tilde{\psi}\rangle_{\tilde{\Gamma}}}{\|\tilde{\psi}\|_{\boldsymbol{H}_t^{1/2}(\tilde{\Gamma})}} \\ &= \|\,\mathbf{curl}_{\tilde{\Gamma}}\, \varphi^0\|_{\boldsymbol{H}_t^{-1/2}(\tilde{\Gamma})} \leq C\, \|\varphi^0\|_{H^{1/2}(\tilde{\Gamma})} \simeq \|\varphi\|_{\tilde{H}^{1/2}(\Gamma)} \quad \forall \varphi \in C_0^\infty(\Gamma).\end{aligned}$$

Here, $\simeq$ denotes the equivalence of norms. The assertion follows by the density of $C_0^\infty(\Gamma)$ in $\tilde{H}^{1/2}(\Gamma)$. ☐

Lemma 2.3. *There holds*

$$W = \mathrm{curl}_\Gamma\, V\, \mathbf{curl}_\Gamma \quad \textit{in} \quad \mathcal{L}(\tilde{H}^{1/2}(\Gamma), H^{-1/2}(\Gamma)). \tag{2.4}$$

Moreover

$$\langle W\phi, \psi\rangle_\Gamma = \langle \mathbf{curl}_\Gamma\, \psi, V\, \mathbf{curl}_\Gamma\, \phi\rangle_\Gamma \quad \forall \phi, \psi \in \tilde{H}^{1/2}(\Gamma). \tag{2.5}$$

Proof. Using the surface differential operators introduced before, there holds in the distributional sense $W\phi = \mathrm{curl}_\Gamma\, V\, \mathbf{curl}_\Gamma\, \phi$, see [14, 16]. This formula extends from $C_0^\infty(\Gamma)$ to $\phi \in \tilde{H}^{1/2}(\Gamma)$ since $\mathbf{curl}_\Gamma:\ \tilde{H}^{1/2}(\Gamma) \to \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)$ by Lemma 2.2, $V:\ \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma) \to \boldsymbol{H}_t^{1/2}(\Gamma)$ by [7], and $\mathrm{curl}_\Gamma:\ \boldsymbol{H}_t^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ since it is the adjoint operator of $\mathbf{curl}_\Gamma$, cf. [6]. The relation (2.5) follows by integration by parts. ☐

An immediate consequence of Lemma 2.3 is that an equivalent variational formulation of (2.1) is: *Find* $\phi \in \tilde{H}^{1/2}(\Gamma)$ *such that*

$$\langle \mathbf{curl}_\Gamma\, \psi, V\, \mathbf{curl}_\Gamma\, \phi\rangle_\Gamma = \langle f, \psi\rangle_\Gamma \quad \forall \psi \in \tilde{H}^{1/2}(\Gamma). \tag{2.6}$$

This formulation forms the basis of our boundary element method with Lagrangian multiplier.

3 Discrete variational formulation with Lagrangian multiplier

In this section we formulate a discretisation with Lagrangian multiplier of the continuous problem (2.1) and state its quasi-optimal convergence (Theorem 3.1 below).

The weak formulation (2.6) is the appropriate basis for the non-conforming method we have in mind. Note that we will consider discrete subspaces of $H^{1/2}(\Gamma)$ where the hypersingular operator W is not well defined. The formulation (2.6) does make sense for continuous discrete functions which do not vanish on γ (the boundary of Γ). Nevertheless, for the error analysis of our scheme we must relate a discrete version of (2.6) with the original problem (2.1), i.e. with (2.2). This requires an integration-by-parts formula that corresponds to (2.5) but is valid for functions which do not vanish on γ. This will be studied in Section 4 (see formula (4.1) and Lemma 4.2).

In order to introduce the discrete scheme let us define a regular, quasi-uniform mesh $\mathcal{T}_h$ of shape regular elements $T \in \mathcal{T}_h$ such that $\bar{\Gamma} = \cup_{T\in\mathcal{T}_h} \bar{T}$. As usual, h denotes the mesh size (being proportional to the diameters of the elements). Throughout this paper we assume that $h < 1$. This is no restriction of generality and is just needed to simplify the writing of logarithmic terms. Elements can be triangles or quadrilaterals. Using this mesh we define the boundary element space

$$H_h := \{\varphi \in C^0(\Gamma);\ \varphi|_T \text{ is a polynomial of degree one } \forall T \in \mathcal{T}_h\}.$$

Note that $H_h \subset H^{1/2}(\Gamma)$, but $H_h \not\subset \tilde{H}^{1/2}(\Gamma)$.

For the definition of a discrete Lagrangian multiplier space we introduce a quasi-uniform mesh $\mathcal{G}_k$ on $\gamma = \partial\Gamma$ that consists of straight line pieces $J \in \mathcal{G}_k$: $\gamma = \cup_{J\in\mathcal{G}_k} \bar{J}$. The parameter k refers to the mesh size of $\mathcal{G}_k$ (being proportional to the lengths of the elements). The discrete space for the Lagrangian multiplier is

$$M_k := \{q \in L^2(\gamma);\ q|_J \text{ is constant } \forall J \in \mathcal{G}_k\}.$$

We also define the bilinear forms

$$\begin{aligned} a(\varphi,\psi) &:= \langle \mathbf{curl}_\Gamma\,\psi, V\,\mathbf{curl}_\Gamma\,\varphi\rangle_\Gamma \qquad && (\varphi,\psi \in \tilde{H}^{1/2}(\Gamma) \cup H_h),\\ b(\varphi,q) &:= \langle \varphi, q\rangle_\gamma := \int_\gamma \varphi q\,ds && (\varphi|_\gamma, q \in L_2(\gamma)), \end{aligned}$$

the linear form

$$L(\varphi) := \langle f, \varphi\rangle_\Gamma \qquad (\varphi \in H^{1/2}(\Gamma))$$

and the space

$$V_h := \{\varphi \in H_h;\ b(\varphi,q) = 0\ \forall q \in M_k\}.$$

The boundary element scheme with Lagrangian multiplier for the approximate solution of (2.6) then is: *Find* $(\phi_h, \lambda_k) \in H_h \times M_k$ *such that*

$$\begin{array}{lcll} a(\phi_h,\psi) + b(\psi,\lambda_k) &=& L(\psi) & \forall \psi \in H_h,\\ b(\phi_h,q) &=& 0 & \forall q \in M_k. \end{array} \tag{3.1}$$

This scheme is equivalent to

$$\phi_h \in V_h:\ a(\phi_h,\psi) = L(\psi) \quad \forall \psi \in V_h.$$

Scheme (3.1) has the typical saddle point structure of a variational formulation with Lagrangian multiplier. In the finite element case such a scheme is usually analysed by the Babuška-Brezzi theory, see also [1]. Our situation, however, is not standard in the sense that there is no corresponding continuous saddle point formulation. Also, we do not intend to derive an error estimate for the Lagrangian multiplier since the corresponding continuous unknown has no physical meaning or relevance in applications. Moreover, this unknown (it is $\mathbf{t} \cdot V\, \mathbf{curl}_\Gamma\, \phi$ with $\mathbf{t}$ being the unit tangential vector along γ in mathematically positive sense, see (4.1)) is not well-defined for $\phi \in \tilde{H}^{1/2}(\Gamma)$ in general.

Our main result is the following quasi-optimal error estimate for the approximation of $\phi \in \tilde{H}^{1/2}(\Gamma)$ by $\phi_h \in V_h$.

Theorem 3.1. *Let $\mathcal{T}_h|_\gamma$ be a refinement of $\mathcal{G}_k$ (i.e. any node of $\mathcal{G}_k$ is a boundary node of $\mathcal{T}_h$ and any element of $\mathcal{G}_k$ has a node of $\mathcal{T}_h$ in its interior) and let k be sufficiently small. Then system* (3.1) *is uniquely solvable and there exists a constant C, independent of h, such that for any small $\epsilon > 0$ there holds the error estimate*

$$\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} \le C \left(\log^{1/2}(1/h) h^{1/2-\epsilon} + \log^{3/2}(1/h) k^{1/2-\epsilon}\right) \|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)}.$$

In particular, selecting k to be proportional to h, there holds for any small $\epsilon > 0$

$$\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} \le C \log^{3/2}(1/h) h^{1/2-\epsilon} \|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)}.$$

A proof of this result will be given at the end of Section 4.

Remark 3.1. *The solution ϕ of* (2.1) *has strong corner and corner-edge singularities such that $\phi \notin H^1(\Gamma)$ in general, see* [18]*. A refined error analysis for the conforming BEM yields for quasi-uniform meshes an optimal error estimate*

$$\|\phi - \phi_h\|_{\tilde{H}^{1/2}(\Gamma)} \le C\, h^{1/2},$$

see [4]*. Such an error analysis makes use of an explicit knowledge of the appearing singularities. When using only the Sobolev regularity $\phi \in \tilde{H}^{1-\epsilon}(\Gamma)$ with $\epsilon > 0$, standard approximation theory proves*

$$\|\phi - \phi_h\|_{\tilde{H}^{1/2}(\Gamma)} \le C\, h^{1/2-\epsilon} \|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)}.$$

Our proof of Theorem 3.1 *makes use of standard Sobolev regularity and, thus, cannot be optimal. Without taking into account specific knowledge of the solution ϕ, the appearing parameter ϵ perturbing the rates of convergence $O(h^{1/2})$ and $O(k^{1/2})$ is unavoidable. The logarithmical perturbation in $1/h$ in the error estimate is due to the non-conformity of the method. In particular, the non-existence of a trace operator $H^{1/2}(\Gamma) \to L^2(\gamma)$ leads to such a perturbation.*

4 Technical results and the proof of the main theorem

In this section we prove various technical results. In particular we study an integration-by-parts formula for $\mathbf{curl}_\Gamma$ and the continuity and ellipticity of the bilinear form a. We finish this section with the Strang estimate by Theorem 4.1 and the proof of Theorem 3.1.

Lemma 4.1. *There exists a positive constant C such that*

$$|\varphi|_{H^{1/2}(\Gamma)} \le C \, \| \mathbf{curl}_\Gamma \varphi \|_{\boldsymbol{H}_t^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{1/2}(\Gamma).$$

Proof. By Lemma 2.1, $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \boldsymbol{H}_t^{-1/2}(\Gamma)$ is continuous. Moreover, the range of $\mathbf{curl}_\Gamma$ is closed in $\boldsymbol{H}_t^{-1/2}(\Gamma)$. This follows from the closedness of the range of $\mathbf{curl}_{\tilde\Gamma} : H^{1/2}(\tilde\Gamma) \to \boldsymbol{H}_t^{-1/2}(\tilde\Gamma)$ (see [6, Remark 5.2]) where $\tilde\Gamma$ is a closed Lipschitz surface containing Γ. Since the range of $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \to \boldsymbol{H}_t^{-1/2}(\Gamma)$ is the restriction onto Γ of the range of $\mathbf{curl}_{\tilde\Gamma} : H^{1/2}(\tilde\Gamma) \to \boldsymbol{H}_t^{-1/2}(\tilde\Gamma)$, the closedness of the range of $\mathbf{curl}_\Gamma$ follows, see, e.g., [15, pp. 76f.]. Further we note that the kernel of $\mathbf{curl}_\Gamma$ in $H^{1/2}(\Gamma)$ consists of constant functions. This follows by noting that any $\varphi \in H^{1/2}(\Gamma)$ with $\mathbf{curl}_\Gamma \varphi = 0$ satisfies $\varphi \in H^1(\Gamma)$ such that $\mathbf{curl}_\Gamma \varphi$ is defined in the usual weak sense. The kernel of $\mathbf{curl}_\Gamma |_{H^1(\Gamma)}$ is given by the constant functions. Therefore, an application of the closed graph theorem yields the estimate

$$\inf_{c \in \mathbb{R}} \|\varphi - c\|_{H^{1/2}(\Gamma)} \le C \, \| \mathbf{curl}_\Gamma \varphi \|_{\boldsymbol{H}_t^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{1/2}(\Gamma),$$

and the assertion follows by the Poincaré-Friedrichs inequality. ☐

We now turn our attention to an integration-by-parts formula. For a smooth scalar function v and a smooth tangential vector field $\boldsymbol{\varphi}$, integration by parts gives

$$\langle \mathbf{curl}_\Gamma v, \boldsymbol{\varphi} \rangle_\Gamma = \langle \mathrm{curl}_\Gamma \boldsymbol{\varphi}, v \rangle_\Gamma - \langle \boldsymbol{\varphi} \cdot \mathbf{t}, v \rangle_\gamma.$$

Here, $\mathbf{t}$ is the unit tangential vector on γ (in mathematically positive orientation when identifying Γ with $(0,1)^2$). Applying this formula to $v = \psi$, $\boldsymbol{\varphi} = V \, \mathbf{curl}_\Gamma \phi$, and using (2.4) and (2.5) we obtain, for sufficiently smooth ϕ and ψ,

$$\langle \mathbf{t} \cdot V \, \mathbf{curl}_\Gamma \phi, \psi \rangle_\gamma = \langle W\phi, \psi \rangle_\Gamma - \langle \mathbf{curl}_\Gamma \psi, V \, \mathbf{curl}_\Gamma \phi \rangle_\Gamma. \tag{4.1}$$

This formula does not extend to $\psi \in H^{1/2}(\Gamma)$ since the trace of such a ψ onto γ is not well defined in general. However, there holds the following lemma.

Lemma 4.2. *For $\phi \in \tilde H^{1/2}(\Gamma)$ with $W\phi = f \in \tilde H^{-1/2}(\Gamma)$, (4.1) defines $\mathbf{t} \cdot V \, \mathbf{curl}_\Gamma \phi \in H^{-1/2}(\gamma)$.*

Proof. Let us denote

$$\boldsymbol{L}_t^2(\Gamma) := \{\psi \in \left(L^2(\Gamma)\right)^3; \ \psi \cdot \mathbf{n} = 0\}.$$

Extending $\psi \in H^{1/2}(\gamma)$ to an element of $H^1(\Gamma)$ (using the same name) there holds

$$\begin{aligned} |\langle \mathbf{t} \cdot V \, \mathbf{curl}_\Gamma \phi, \psi \rangle_\gamma| &= |\langle f, \psi \rangle_\Gamma - \langle \mathbf{curl}_\Gamma \psi, V \, \mathbf{curl}_\Gamma \phi \rangle_\Gamma| \\ &\le \|f\|_{\tilde H^{-1/2}(\Gamma)} \|\psi\|_{H^{1/2}(\Gamma)} + \|V \, \mathbf{curl}_\Gamma \phi\|_{\boldsymbol{L}_t^2(\Gamma)} \| \mathbf{curl}_\Gamma \psi \|_{\boldsymbol{L}_t^2(\Gamma)} \\ &\le (\|f\|_{\tilde H^{-1/2}(\Gamma)} + \|V \, \mathbf{curl}_\Gamma \phi\|_{\boldsymbol{H}_t^{1/2}(\Gamma)}) \|\psi\|_{H^1(\Gamma)}. \end{aligned}$$

The result follows by using the continuity of $\mathbf{curl}_\Gamma : \tilde H^{1/2}(\Gamma) \to \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)$ (see Lemma 2.2) and $V : \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma) \to \boldsymbol{H}_t^{1/2}(\Gamma)$ (cf. the proof of Lemma 2.3), and by the continuity of the extension of $\psi \in H^{1/2}(\gamma)$ to $\psi \in H^1(\Gamma)$. ☐

Obviously, continuity and V_h-ellipticity of $a(\cdot,\cdot)$ are critical for the error analysis of (3.1). Lemma 4.4 below shows the continuity and then Lemma 4.6 proves that $a(\cdot,\cdot)$ is almost uniformly V_h-elliptic. For its proof we need the following result on the trace operator.

Lemma 4.3. *There exists $C>0$ such that, for any $\epsilon\in(0,1/2)$, there holds*

$$\|v\|_{H^\epsilon(\gamma)} \le \frac{C}{\epsilon^{1/2}}\|v\|_{H^{1/2+\epsilon}(\Gamma)} \quad \forall v\in H^{1/2+\epsilon}(\Gamma).$$

Proof. The trace theorem is usually proved by applying local mappings onto the half-space case where the Fourier transformation is used. This yields the estimate

$$\|v\|^2_{H^{s-1/2}(\gamma)} \le C\,M_s\,\|v\|^2_{H^s(\Gamma)} \quad \forall v\in H^s(\Gamma),\ 1/2<s\le 1$$

with C depending only on Γ and with

$$M_s = \int_{\mathbb{R}} (1+t^2)^{-s}\,dt,$$

see, e.g., [15, Lemma 3.35, Theorem 3.37]. Noting that $M_{1/2+\epsilon}=O(\epsilon^{-1})$ finishes the proof. □

Lemma 4.4. *There exists a constant $C>0$ such that there hold*

$$a(v,w)\le C\,\log(1/h)\|v\|_{H^{1/2}(\Gamma)}\|\,\mathbf{curl}_\Gamma\,w\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \quad \forall v,w\in H_h$$

and

$$a(v,w)\le C\,\log^2(1/h)\|v\|_{H^{1/2}(\Gamma)}\|w\|_{H^{1/2}(\Gamma)} \quad \forall v,w\in H_h.$$

Proof. By the continuity of $V:\ \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)\to \boldsymbol{H}_t^{1/2}(\Gamma)$ there holds

$$a(v,w)\le C\,\|\,\mathbf{curl}_\Gamma\,v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}\|\,\mathbf{curl}_\Gamma\,w\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}.$$

Then we use [11, Lemma 6] which says that for any piecewise polynomial on a quasi-uniform mesh the energy norm of V can be bounded by

$$\|\varphi\|_{\tilde{H}^{-1/2}(\Gamma)}\le C\,\log(1/h)\|\varphi\|_{H^{-1/2}(\Gamma)}.$$

Here the constant C is independent of h. Therefore,

$$\|\,\mathbf{curl}_\Gamma\,v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}\le C\,\log(1/h)\|\,\mathbf{curl}_\Gamma\,v\|_{\boldsymbol{H}_t^{-1/2}(\Gamma)} \quad \forall v\in V_h, \tag{4.2}$$

and the continuity of $\mathbf{curl}_\Gamma:\ H^{1/2}(\Gamma)\to \boldsymbol{H}_t^{-1/2}(\Gamma)$ proves both assertions. □

The next result will be needed to prove ellipticity of the bilinear form a.

Lemma 4.5. *There exists $C > 0$ such that for any $h \in (0,1)$ and for sufficiently small k there holds*

$$|v|_{H^{1/2}(\Gamma)} \geq C \log^{-1/2}(1/h) \|v\|_{H^{1/2}(\Gamma)} \quad \forall v \in V_h.$$

If k is proportional to a positive power of h then k satisfies the assumption if h is sufficiently small.

Proof. We decompose any $v \in V_h$ into $v = v_0 + d$ with $\int_\Gamma v_0(x)\, dS_x = 0$ and $d = |\Gamma|^{-1} \int_\Gamma v(x)\, dS_x$. In order to estimate d we note that there holds $\|d\|_{\tilde{H}^{-1}(\gamma)} = |d|\, \|1\|_{\tilde{H}^{-1}(\gamma)}$, that is

$$|d| = \|1\|_{\tilde{H}^{-1}(\gamma)}^{-1} \sup_{q \in H^1(\gamma)\setminus\{0\}} \frac{\langle d, q\rangle_\gamma}{\|q\|_{H^1(\gamma)}}. \tag{4.3}$$

Since $v \in V_h$, i.e. $b(v, q_k) = 0$ for any $q_k \in M_k$, we find

$$\begin{aligned} |\langle d, q\rangle_\gamma| &= |\langle v - v_0, q\rangle_\gamma| = |\langle v, q - q_k\rangle_\gamma - \langle v_0, q\rangle_\gamma| \\ &\leq \|v\|_{L^2(\gamma)} \|q - q_k\|_{L^2(\gamma)} + \|v_0\|_{L^2(\gamma)} \|q\|_{L^2(\gamma)} \quad \forall q_k \in M_k. \end{aligned}$$

For $\epsilon > 0$ we embed $L^2(\gamma)$ into $H^\epsilon(\gamma)$ and apply Lemma 4.3. Together with the error bound

$$\inf_{q_k \in M_k} \|q - q_k\|_{L^2(\gamma)} \leq Ck \|q\|_{H^1(\gamma)}$$

the previous estimate then yields

$$|\langle d, q\rangle_\gamma| \leq \frac{C}{\epsilon^{1/2}} k \|v\|_{H^{1/2+\epsilon}(\Gamma)} \|q\|_{H^1(\gamma)} + \frac{C}{\epsilon^{1/2}} \|v_0\|_{H^{1/2+\epsilon}(\Gamma)} \|q\|_{H^1(\gamma)}.$$

Applying the inverse property $\|v\|_{H^{1/2+\epsilon}(\Gamma)} \leq Ch^{-\epsilon} \|v\|_{H^{1/2}(\Gamma)}$ (see, e.g., [11, Lemma 4]) and selecting $\epsilon := 1/\log(1/h)$ for $h < 1$, we conclude that there holds

$$|\langle d, q\rangle_\gamma| \leq C \log^{1/2}(1/h) \left(k \|v\|_{H^{1/2}(\Gamma)} + \|v_0\|_{H^{1/2}(\Gamma)} \right) \|q\|_{H^1(\gamma)}.$$

Making use of (4.3) this yields

$$|d| \leq C \log^{1/2}(1/h) \left(k \|v\|_{H^{1/2}(\Gamma)} + \|v_0\|_{H^{1/2}(\Gamma)} \right). \tag{4.4}$$

Therefore, using the triangle inequality, we estimate

$$\|v\|_{H^{1/2}(\Gamma)}^2 \leq C(\|v_0\|_{H^{1/2}(\Gamma)}^2 + d^2) \leq C\, \log(1/h) \|v_0\|_{H^{1/2}(\Gamma)}^2 + C\, \log(1/h) k^2 \|v\|_{H^{1/2}(\Gamma)}^2,$$

so that

$$\begin{aligned} (1 - C\, k^2 \log(1/h)) \|v\|_{H^{1/2}(\Gamma)}^2 &\leq C\, \log(1/h) \|v_0\|_{H^{1/2}(\Gamma)}^2 \\ &= C\, \log(1/h) \Big(|v_0|_{H^{1/2}(\Gamma)}^2 + \inf_{c \in \mathbb{R}} \|v - c\|_{L^2(\Gamma)}^2 \Big) \\ &\leq C\, \log(1/h) |v|_{H^{1/2}(\Gamma)}^2. \end{aligned} \tag{4.5}$$

In the last step we made use of the Poincaré-Friedrichs inequality. Selecting k small enough such that $1 - C\, k^2 \log(1/h) > 0$ this proves the lemma. ☐

The next lemma states three variants of ellipticity for the bilinear form a. Part (i) will be used to establish our computable error bound (5.1) needed for the numerical experiment, and part (ii) is essential for proving the Strang-type error estimate by Theorem 4.1. Part (iii) proves the V_h-ellipticity with $H^{1/2}(\Gamma)$-norm needed for the Babuška-Brezzi theory. We note that part (iii) would be enough to prove a Strang estimate and eventually an a priori error bound. But the ellipticity provided by (ii) leads to a sharper result.

Lemma 4.6. (i) *There exists a constant $C > 0$ such that*

$$a(v,v) \geq C\,|v|^2_{H^{1/2}(\Gamma)} \quad \forall v \in \tilde{H}^{1/2}(\Gamma) \cup H^1(\Gamma).$$

(ii) *There exists $C > 0$ such that for any $h \in (0,1)$ and for sufficiently small k there holds*

$$a(v,v) \geq C \log^{-1/2}(1/h) \|v\|_{H^{1/2}(\Gamma)} \|\,\mathbf{curl}_\Gamma\, v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \quad \forall v \in V_h.$$

If k is proportional to a positive power of h then k satisfies the assumption if h is sufficiently small.
(iii) *Under the conditions of part* (ii) *there holds*

$$a(v,v) \geq C \log^{-1}(1/h) \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in V_h.$$

Proof. First we note that $a(v,v)$ is well-defined for any $v \in \tilde{H}^{1/2}(\Gamma) \cup H^1(\Gamma)$. This follows from the mapping properties of V and the continuities $\mathbf{curl}_\Gamma : \tilde{H}^{1/2}(\Gamma) \to \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)$ by Lemma 2.2 and $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \boldsymbol{L}^2_t(\Gamma)$. (In fact, one can show that the bilinear form is well-defined on $H^{1/2+\epsilon}(\Gamma)$ for any $\epsilon > 0$.)

The ellipticity of V and Lemma 4.1 prove that there holds

$$a(v,v) \geq C\,\|\,\mathbf{curl}_\Gamma\, v\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \geq C\,\|\,\mathbf{curl}_\Gamma\, v\|^2_{\boldsymbol{H}_t^{-1/2}(\Gamma)} \geq C\,|v|^2_{H^{1/2}(\Gamma)} \quad \forall v \in \tilde{H}^{1/2}(\Gamma) \cup H^1(\Gamma).$$

This is the first assertion. Now, to prove the second assertion, we conclude as before

$$\begin{aligned} a(v,v) \geq C\,\|\,\mathbf{curl}_\Gamma\, v\|^2_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} &\geq C\,\|\,\mathbf{curl}_\Gamma\, v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \|\,\mathbf{curl}_\Gamma\, v\|_{\boldsymbol{H}_t^{-1/2}(\Gamma)} \\ &\geq C\,\|\,\mathbf{curl}_\Gamma\, v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} |v|_{H^{1/2}(\Gamma)} \quad \forall v \in V_h \subset H^1(\Gamma). \end{aligned}$$

An application of Lemma 4.5 finishes the proof of part (ii). The proof of part (iii) is analogous. □

We are now ready to establish the following Strang-type result.

Theorem 4.1. *Let $\mathcal{T}_h|_\gamma$ be a refinement of $\mathcal{G}_k$ (i.e. any node of $\mathcal{G}_k$ is a boundary node of $\mathcal{T}_h$ and any element of $\mathcal{G}_k$ has a node of $\mathcal{T}_h$ in its interior) and let k be sufficiently small. Then system* (3.1) *is uniquely solvable and there holds*

$$\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} \leq C\,\log^{1/2}(1/h) \Big(\inf_{v \in V_h \cap \tilde{H}^{1/2}(\Gamma)} \|\phi - v\|_{\tilde{H}^{1/2}(\Gamma)} + \sup_{v \in V_h \setminus \{0\}} \frac{|a(\phi - \phi_h, v)|}{\|\,\mathbf{curl}_\Gamma\,\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}} \Big).$$

Proof. The existence and uniqueness of $(\phi_h, \lambda_k) \in H_h \times M_k$ follows from the Babuška-Brezzi theory. The bilinear form a is continuous on H_h by Lemma 4.4 and V_h-elliptic by Lemma 4.6(iii). (The continuity and ellipticity numbers depend on h but this does not matter here.) Moreover, the bilinear form b satisfies an inf-sup condition (not necessarily uniformly) since there holds the implication

$$\Big(q \in M_k: \quad b(\varphi, q) = \langle \varphi, q\rangle_\gamma = 0 \quad \forall \varphi \in H_h\Big) \qquad \Rightarrow \qquad q = 0$$

(for a given interval $J \in \mathcal{G}_k$ select the nodal hat function associated with a node of $\mathcal{T}_h$ interior to J leading to $q = 0$ on J). Therefore, there exists a unique solution $(\phi_h, \lambda_k) \in H_h \times M_k$ of (3.1).

To prove the Strang-type error estimate we follow the standard procedure, see, e.g., [5]. Using the triangle inequality and the quasi-uniform V_h-ellipticity of a (see Lemma 4.6(ii)) we find that for any $\psi \in V_h$ there holds

$$\begin{aligned}
&\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} \le \|\phi - \psi\|_{H^{1/2}(\Gamma)} + \|\psi - \phi_h\|_{H^{1/2}(\Gamma)} \\
&\le \|\phi - \psi\|_{H^{1/2}(\Gamma)} + C \log^{1/2}(1/h) \sup_{v \in V_h \setminus\{0\}} \frac{|a(\psi - \phi_h, v)|}{\|\,\mathbf{curl}_\Gamma v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}} \\
&\le \|\phi - \psi\|_{H^{1/2}(\Gamma)} + \\
&\qquad C \log^{1/2}(1/h) \left(\sup_{v \in V_h \setminus\{0\}} \frac{|a(\psi - \phi, v)|}{\|\,\mathbf{curl}_\Gamma v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}} + \sup_{v \in V_h \setminus\{0\}} \frac{|a(\phi - \phi_h, v)|}{\|\,\mathbf{curl}_\Gamma v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}} \right).
\end{aligned} \tag{4.6}$$

Selecting $\psi \in V_h \cap \tilde{H}^{1/2}(\Gamma)$ (i.e. $\psi = 0$ on γ) we can use the boundedness $\mathbf{curl}_\Gamma : \tilde{H}^{1/2}(\Gamma) \to \tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)$ (see Lemma 2.2) and obtain (by also using the continuity of V)

$$\begin{aligned}
|a(\psi - \phi, v)| &\le C \,\|\,\mathbf{curl}_\Gamma(\psi - \phi)\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \|\,\mathbf{curl}_\Gamma v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)} \\
&\le C \,\|\psi - \phi\|_{\tilde{H}^{1/2}(\Gamma)} \|\,\mathbf{curl}_\Gamma v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}.
\end{aligned} \tag{4.7}$$

A combination of (4.6) and (4.7) finishes the proof. □

Proof of Theorem 3.1. We apply Theorem 4.1. By assumption, the given function f is in $\tilde{H}^{-1/2}(\Gamma)$. Therefore, by Lemma 4.2, (4.1) holds for any $\psi \in H^{1/2}(\gamma)$. In particular, we can apply (4.1) to elements of $H_h \subset H^1(\Gamma)$. This yields for any $q_k \in M_k$ and $v \in V_h$

$$\begin{aligned}
|a(\phi - \phi_h, v)| &= |a(\phi, v) - L(v)| = |\langle \mathbf{t} \cdot V\, \mathbf{curl}_\Gamma\, \phi, v\rangle_\gamma| \\
&= |\langle \mathbf{t} \cdot V\, \mathbf{curl}_\Gamma\, \phi - q_k, v\rangle_\gamma| \le \|\mathbf{t} \cdot V\, \mathbf{curl}_\Gamma\, \phi - q_k\|_{L^2(\gamma)} \|v\|_{L^2(\gamma)}.
\end{aligned} \tag{4.8}$$

By Lemma 4.3 and the inverse property one obtains, as in the proof of Lemma 4.5,

$$\|v\|_{L^2(\gamma)} \le C \log^{1/2}(1/h) \|v\|_{H^{1/2}(\Gamma)}.$$

Assuming that the mesh size k is small enough we thus obtain with (4.5) and Lemma 4.1 the bound

$$\|v\|_{L^2(\gamma)} \le C\,\log(1/h)|v|_{H^{1/2}(\Gamma)} \le C\,\log(1/h)\|\,\mathbf{curl}_\Gamma\, v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}. \tag{4.9}$$

Noting that $\phi \in \tilde{H}^{1-\epsilon}(\Gamma)$ for any $\epsilon > 0$ (see [18]) we conclude that $\mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\,\phi \in H^{1/2-\epsilon}(\gamma)$ for any $\epsilon \in (0, 1/2)$. This follows from the continuity of $\mathbf{curl}_\Gamma :\ \tilde{H}^{1-\epsilon}(\Gamma) \to \tilde{\boldsymbol{H}}_t^{-\epsilon}(\Gamma)$ (see [3, Lemma 3.1]), $V :\ \tilde{H}^{-\epsilon}(\Gamma) \to H^{1-\epsilon}(\Gamma)$, and the tangential trace $\boldsymbol{H}_t^{1-\epsilon}(\Gamma) \to H^{1/2-\epsilon}(\gamma)$. Therefore, a standard approximation estimate gives

$$\inf_{q_k\in M_k} \|\mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\,\phi - q_k\|_{L^2(\gamma)} \le C\,k^{1/2-\epsilon}\|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)}. \tag{4.10}$$

We conclude from (4.8), by using (4.9) and (4.10), that

$$|a(\phi-\phi_h, v)| \le C\,k^{1/2-\epsilon}\log(1/h)\|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)}\|\,\mathbf{curl}_\Gamma\,v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)},$$

that is

$$\sup_{v\in V_h\setminus\{0\}} \frac{|a(\phi-\phi_h, v)|}{\|\,\mathbf{curl}_\Gamma\,v\|_{\tilde{\boldsymbol{H}}_t^{-1/2}(\Gamma)}} \le C\,k^{1/2-\epsilon}\log(1/h)\|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)} \quad \forall\epsilon\in(0,1/2).$$

On the other hand, a standard approximation estimate yields

$$\inf_{v\in V_h\cap\tilde{H}^{1/2}(\Gamma)} \|\phi - v\|_{\tilde{H}^{1/2}(\Gamma)} \le C\,h^{1/2-\epsilon}\|\phi\|_{\tilde{H}^{1-\epsilon}(\Gamma)}.$$

Therefore, the theorem is proved by application of Theorem 4.1. □

5 Numerical results

We consider the model problem (2.1) with $\Gamma = (0,1)\times(0,1)$ and $f = 1$. The meshes $\mathcal{T}_h$ are uniform consisting of squares of side-length h and for each $\mathcal{T}_h$ we select the mesh $\mathcal{G}_k$ which is compatible with $\mathcal{T}_h|_\gamma$ (as required by Theorem 3.1) with $k = 2h$, see Figure 5.1 where the bullets indicate the nodes of $\mathcal{G}_k$.

These meshes define the boundary element space H_h and the space M_k for the Lagrangian multiplier. Implementing the scheme (3.1) we calculate the approximation $\phi_h \in V_h \subset H_h$ to the exact solution ϕ of (2.1). Since ϕ is unknown there is no direct way to calculate the error $\|\phi-\phi_h\|_{H^{1/2}(\Gamma)}$. Instead, we approximate an upper bound to the semi-norm $|\phi-\phi_h|_{H^{1/2}(\Gamma)}$. By Lemma 4.6(i) there holds

$$a(\phi-\phi_h, \phi-\phi_h) \ge C|\phi-\phi_h|^2_{H^{1/2}(\Gamma)}. \tag{5.1}$$

Moreover, since ϕ solves (2.1) and ϕ_h solves (3.1), one finds that there holds

$$\begin{aligned} a(\phi-\phi_h, \phi-\phi_h) &= a(\phi,\phi) - 2a(\phi,\phi_h) + a(\phi_h,\phi_h) \\ &= \langle W\phi,\phi\rangle_\Gamma - 2a(\phi,\phi_h) + \langle f,\phi_h\rangle_\Gamma - b(\phi_h,\lambda_k) \\ &= \langle W\phi,\phi\rangle_\Gamma + \langle f,\phi_h\rangle_\Gamma - 2a(\phi,\phi_h). \end{aligned}$$

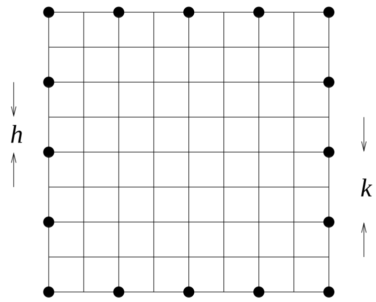

Figure 5.1: Uniform meshes $\mathcal{T}_h$ and $\mathcal{G}_k$.

As in (4.8) we can apply (4.1) and obtain

$$a(\phi, \phi_h) = \langle f, \phi_h \rangle_\Gamma - \langle \mathbf{t} \cdot V \,\mathbf{curl}_\Gamma\, \phi, \phi_h \rangle_\gamma$$

so that, with the previous relation,

$$\begin{aligned} a(\phi - \phi_h, \phi - \phi_h) &= \langle W\phi, \phi \rangle_\Gamma - \langle f, \phi_h \rangle_\Gamma + 2\langle \mathbf{t} \cdot V \,\mathbf{curl}_\Gamma\, \phi, \phi_h \rangle_\gamma && (5.2)\\ &\le |\langle W\phi, \phi \rangle_\Gamma - \langle f, \phi_h \rangle_\Gamma| + 2\|\mathbf{t} \cdot V \,\mathbf{curl}_\Gamma\, \phi\|_{L^2(\gamma)} \|\phi_h\|_{L^2(\gamma)}. && (5.3) \end{aligned}$$

As in the proof of Theorem 3.1 one establishes that $\|\mathbf{t} \cdot V \,\mathbf{curl}_\Gamma\, \phi\|_{L^2(\gamma)}$ is bounded by a constant depending on ϕ. Therefore, by (5.1) there exists a constant $C > 0$, independent of h, such that

$$|\phi - \phi_h|^2_{H^{1/2}(\Gamma)} \le C \Big(|\langle W\phi, \phi \rangle_\Gamma - \langle f, \phi_h \rangle_\Gamma| + \|\phi_h\|_{L^2(\gamma)} \Big).$$

The terms $\langle f, \phi_h \rangle_\Gamma$ and $\|\phi_h\|_{L^2(\gamma)}$ can be easily calculated and $\langle W\phi, \phi \rangle_\Gamma$ can be approximated by an extrapolated value that we denote by $\|\phi\|^2_{\mathrm{ex}}$ (cf. [8]). Therefore, instead of the relative error $\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} / \|\phi\|_{H^{1/2}(\Gamma)}$, we present results for the expression

$$\Big(\big| \|\phi\|^2_{\mathrm{ex}} - \langle f, \phi_h \rangle_\Gamma \big| + \|\phi_h\|_{L^2(\gamma)} \Big)^{1/2} / \|\phi\|_{\mathrm{ex}} \tag{5.4}$$

which is, up to a constant factor, an approximative upper bound for $|\phi - \phi_h|_{H^{1/2}(\Gamma)}$ normalised by $\langle W\phi, \phi \rangle^{1/2}_\Gamma \simeq \|\phi\|_{\tilde{H}^{1/2}(\Gamma)}$. The corresponding curve is indicated by (2) in Figure 5.2. Here a double logarithmic scale is used and all numbers are plotted versus the dimension of H_h. As can be seen, curve (2) is parallel to the curve indicated by (1) which gives the errors in $\tilde{H}^{1/2}(\Gamma)$ for the corresponding conforming method (i.e. for the case $V_h \subset \tilde{H}^{1/2}(\Gamma)$ using the same meshes). Actually, for $\varphi \in \tilde{H}^{1/2}(\Gamma)$, $\|\varphi\|^2_{\tilde{H}^{1/2}(\Gamma)}$ and $a(\varphi, \varphi)$ are equivalent and the latter expression is used. Both errors, (1) and (2), behave like $O(h^{1/2})$ whose curve is also given. In this way the result of Theorem 3.1 is confirmed. Note, however, that we do not observe an ϵ-perturbation (reduced convergence $O(h^{1/2-\epsilon})$) nor a logarithmic perturbation in $1/h$. The absence of an ϵ-perturbation is expected, cf. Remark 3.1. The absence of a logarithmic perturbation in $1/h$ in this range of unknowns is likely influenced by the fact that we did not present upper bounds for the full norm of the error but rather for a semi-norm, cf. Lemma 4.6.

Considering the results in Figure 5.2 it appears that the errors of the BEM with Lagrangian multiplier are much larger than those of the conforming method. This is not true for our model problem. Our substitution (5.4) for $|\phi - \phi_h|_{H^{1/2}(\Gamma)}/\langle W\phi, \phi\rangle_\Gamma^{1/2}$ is far from being precise. Not only did we replace the term $2\|\mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\,\phi\|_{L^2(\gamma)}$ in (5.3) by 1 (and the constant C in (5.1) by 1) but indeed the term $\langle \mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\,\phi, \phi_h\rangle_\gamma$ is of higher order than $\|\phi_h\|_{L^2(\gamma)}$. According to the proof of Theorem 3.1 (see (4.8) and (4.10)) there holds

$$\begin{aligned}|\langle \mathbf{t}\cdot V\,\mathbf{curl}_\Gamma\,\phi, \phi_h\rangle_\gamma| &\le C\,k^{1/2-\epsilon}\|\phi\|_{\tilde H^{1-\epsilon}(\Gamma)}\|\phi_h\|_{L^2(\gamma)}\\ &= C\,(2h)^{1/2-\epsilon}\|\phi\|_{\tilde H^{1-\epsilon}(\Gamma)}\|\phi_h\|_{L^2(\gamma)} \quad (\epsilon > 0).\end{aligned}$$

Therefore, by (5.2) the term

$$\Big(\big|\langle W\phi,\phi\rangle_\Gamma - \langle f,\phi_h\rangle_\Gamma\big|\Big)^{1/2}/\langle W\phi,\phi\rangle_\Gamma^{1/2} \approx \Big(\big|\|\phi\|_{\mathrm{ex}}^2 - \langle f,\phi_h\rangle_\Gamma\big|\Big)^{1/2}/\|\phi\|_{\mathrm{ex}} \qquad \text{“part 1 of error”}$$

is asymptotically equal to

$$a(\phi-\phi_h,\phi-\phi_h)^{1/2}/\langle W\phi,\phi\rangle_\Gamma^{1/2}.$$

The former numbers are the ones given by curve (3) (“part 1 of error”) and curve (1) (“conforming BEM”) presents the numbers $a(\phi-\tilde\phi_h,\phi-\tilde\phi_h)^{1/2}/\|\phi\|_{\mathrm{ex}}$ where $\tilde\phi_h$ is the conforming BE-approximation of ϕ. Both curves seem to coincide asymptotically indicating the good performance of the BEM with Lagrangian multiplier. We also include the values of $\|\phi_h\|_{L^2(\gamma)}^{1/2}/\|\phi\|_{\mathrm{ex}}$ (curve (4) “part 2 of error”) to confirm that they dominate our expression (5.4). The behaviour of this curve, indicating $\|\phi_h\|_{L^2(\gamma)} = O(h)$, demonstrates that our non-conforming BEM very efficiently approximates the conformity condition that (conforming) ansatz functions must vanish on γ.

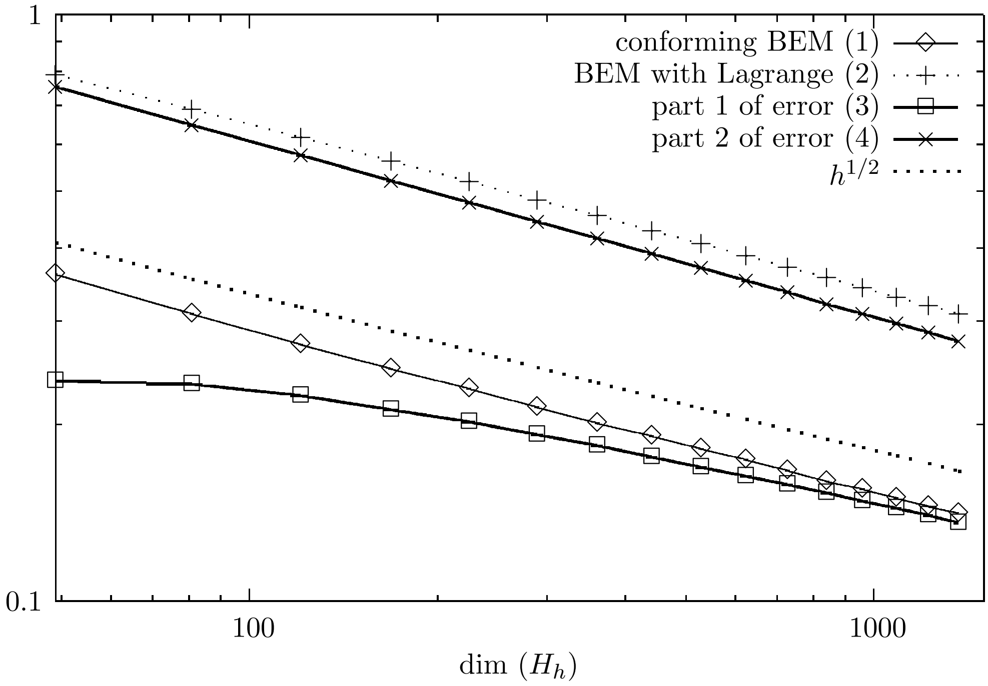

Figure 5.2: Relative error curves (normalised by the extrapolated value $\|\phi\|_{\rm ex}$): (1) error in $\tilde H^{1/2}(\Gamma)$ for conforming BEM, (2) $\Big(\big|\|\phi\|_{\rm ex} - \langle f, \phi_h\rangle_\Gamma\big| + \|\phi_h\|_{L^2(\gamma)}\Big)^{1/2}$, (3) $\Big|\|\phi\|_{\rm ex} - \langle f, \phi_h\rangle_\Gamma\Big|^{1/2}$, (4) $\|\phi_h\|^{1/2}_{L^2(\gamma)}$.

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