An iterative substructuring method for the *hp*-version of the BEM on quasi-uniform triangular meshes *

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Abstract

We study an additive Schwarz based preconditioner for the *hp*-version of the boundary element method with quasi-uniform triangular meshes and for hypersingular integral operators. The model problem is Laplace's equation exterior to an open surface and is generic for elliptic boundary value problems of second order in bounded and unbounded domains with closed or open boundary. The preconditioner is based on a non-overlapping subspace decomposition into a so-called wire basket space and interior functions for each element. We prove that the condition number of the preconditioned stiffness matrix has a bound which is independent of the mesh size h and which grows only polylogarithmically in p, the maximum polynomial degree. Numerical experiments confirm this result.

Key words: p- and hp-versions, boundary element method, preconditioner, domain decomposition, additive Schwarz method, iterative substructuring method
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1 Introduction

High order Galerkin methods like the p- and hp-versions are known to converge rapidly for smooth as well as for singular solutions. On the other hand the arising linear systems are highly ill-conditioned and their iterative solutions require efficient preconditioners. For piecewise polynomial spaces on meshes consisting of quadrilaterals or hexahedra, overlapping and non-overlapping (or iterative substructuring) methods define such optimal or quasi-optimal preconditioners, see [20, 21, 23, 10] for the finite element method (FEM) and [11, 12, 1, 15] for the boundary element method, for problems in three dimensions. For the p-version of the FEM

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with tetrahedral meshes and the p-version of the BEM with triangular meshes, however, there are only few results available. Numerical experiments for overlapping FEM preconditioners on triangular and tetrahedral meshes are reported in [22] and, recently, Schöberl *et al.* [26] provided the theory. Bică studies non-overlapping preconditioners for the p-version of the FEM with tetrahedra in his PhD-thesis [6] and also presents convincing numerical experiments (see also [7]). There are, however, unspecified constants (possibly depending on polynomial degrees) in some of his results, due to the application of an unproved extension theorem. In this paper we follow Bică's construction to define an iterative substructuring method for the hp-version of the BEM with triangular quasi-uniform meshes. Our preconditioned stiffnes matrix is quasi-optimal in the sense that the condition number is bounded by $\mathcal{O}((1+\log p)^4)$ with a constant that is independent of the mesh parameter h. In this way we improve the bound $\mathcal{O}((1+\log p)^7)$ by Cao & Guo [8] (for a slightly different construction) which, to our knowledge, is so far the only available theoretical result for additive Schwarz type preconditioners for the p-BEM with triangular meshes.
We give an analysis of our method for uniform polynomial degree distributions (the polynomial degree p is the same for all elements) but it is applicable to arbitrary degree distributions without any difficulty. The results are then valid by substituting p by the maximum polynomial degree. Our substructuring method uses the so-called wire basket space (consisting of nodal and side basis functions) with L^2-bilinear form or energy bilinear form (defined by the integral operator) and, for each triangle, the space of bubble functions on that element with energy bilinear form. Main technical details deal with traces and extensions for polynomials acting between L^2 on sides of triangles and $\tilde{H}^{1/2}$ (the energy space of the hypersingular operator) on triangles. Such traces and extensions, for tetrahedral meshes and the FEM, have been analysed by Muñoz-Sola in [18]. Essential tool is an appropriate extension operator. The counterpart of this operator in $\mathbb{R}^2$ for triangles, in combination with the discrete harmonic extension, has been used by Bică [6]. Recently, a logarithmical bound (in p) has been proved for this extension operator [14] and this forms an important part of our theory.
Our model problem is the hypersingular integral equation

$$a(u,v) := \langle Du, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma) \tag{1}$$

on a plane polygonal surface segment $\Gamma \subset \mathbb{R}^3$ where $f \in H^{-1/2}(\Gamma)$ is a given function. Here, D is the hypersingular integral operator

$$Du(x) = -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma,$$

which is a continuous and positive definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [27]. Hence, there holds the equivalence of norms

$$\langle Dv, v\rangle_\Gamma \simeq \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma). \tag{2}$$

The Galerkin scheme for (1) reads as follows. Given a finite dimensional subspace $\Psi \subset \tilde{H}^{1/2}(\Gamma)$ with $\dim \Psi = N$, find $u_N \in \Psi$ such that

$$\langle Du_N, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \text{for all} \quad v \in \Psi. \tag{3}$$

The space Ψ under consideration consists of continuous piecewise polynomials of degree p on regular quasi-uniform meshes formed by triangles. Our iterative substructuring method defines a preconditioner for the stiffness matrix A of system (3). Equivalently, the method results in a preconditioned stiffness matrix which can be considered as the additive Schwarz operator P corresponding to the underlying subspace decomposition with given bilinear forms. The main result of this paper states that the condition number of the preconditioned matrix P is bounded by $\mathcal{O}((1+\log p)^4)$ with a constant that is independent of the mesh parameter h.
An outline of this paper is as follows. In §2 we introduce the needed Sobolev spaces. Furthermore we define basis functions for the p-version and a decomposition of the ansatz space, and state the main result (Theorem 3). For the definition of the basis functions we need special extension operators which are also presented. In §3 we prove several technical lemmas. The proof of the main result is given in §4. Finally, in §5 we present some numerical experiments which underline the asymptotic behaviour of the preconditioner. For the convenience of the reader we collect some technical results from other authors in an appendix (Section A). In particular, we indicate proofs of some of Bică's results which are used in this paper.
Throughout the paper C denotes a generic positive number which is independent of p and the characteristic mesh size h, if not otherwise stated.

2 Sobolev spaces, basis functions and preconditioners

On an open surface segment Γ we introduce the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ where the latter space is most often denoted by $H_{00}^{1/2}(\Gamma)$ in the finite element literature. Let $\tilde{\Gamma}$ be a closed surface (in our case a polyhedral surface) with $\Gamma \subset \tilde{\Gamma}$. We define

$$H^{1/2}(\tilde{\Gamma}) := \{\phi|_{\tilde{\Gamma}};\ \phi \in H^1(\mathbb{R}^3)\}, \quad H^{1/2}(\Gamma) := \{\phi|_{\Gamma};\ \phi \in H^{1/2}(\tilde{\Gamma})\},$$

and

$$\tilde{H}^{1/2}(\Gamma) := \{\phi \in H^{1/2}(\Gamma);\ \tilde{\phi} \in H^{1/2}(\tilde{\Gamma})\},$$

where $\tilde{\phi}$ denotes the extension of ϕ by 0 from Γ onto $\tilde{\Gamma}$. A norm in $H^{1/2}(\Gamma)$ is given by (see [16])

$$\|\cdot\|^2_{H^{1/2}(\Gamma)} = \|\cdot\|^2_{L^2(\Gamma)} + |\cdot|^2_{H^{1/2}(\Gamma)}$$

with semi-norm

$$|v|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{|v(x)-v(y)|^2}{|x-y|^3}\, dS_x\, dS_y.$$

To calculate the $H^{1/2}$-norm over two adjacent elements Γ_i and Γ_j we consider the following equivalent norm, compare Grisvard [9]

$$\|v\|^2_{H^{1/2}(\Gamma_i \cup \Gamma_j)} = \|v\|^2_{H^{1/2}(\Gamma_i)} + \|v\|^2_{H^{1/2}(\Gamma_j)} + \int_{\Gamma_i} \int_{\Gamma_j} \frac{(v(x)-v(y))^2}{|x-y|^3}\, dy\, dx.$$

Finally, in $\tilde{H}^{1/2}$ on Γ and correspondingly on local portions $\Gamma_j \subset \Gamma$, we use the norm

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} = |v|^2_{H^{1/2}(\Gamma)} + \int_\Gamma \frac{|v(x)|^2}{\operatorname{dist}(x,\partial\Gamma)}\, dS_x. \tag{4}$$

We also note that the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ can be equivalently defined as intermediate spaces between $L^2(\Gamma)$ and $H^1(\Gamma)$ or $H^1_0(\Gamma)$ ($H^1_0(\Gamma)$ is the completion of $C^\infty_0(\Gamma)$ within $H^1(\Gamma)$).

Furthermore, for a triangle T, we consider special subspaces of $H^{1/2}(T)$ related to one or two edges of T. Let λ_i be the barycentric function related to the edge I_i of T. More precisely, λ_i is linear, vanishes on I_i and has value 1 in the vertex opposite to I_i. Then $\tilde{H}^{1/2}(T, I_i)$ consists of functions $u \in H^{1/2}(T)$ which vanish on the edge I_i and satisfy $\lambda_i^{-1/2} \cdot u \in L^2(T)$, with norm

$$\|u\|^2_{\tilde{H}^{1/2}(T,I_i)} = |u|^2_{H^{1/2}(T)} + \|\lambda_i^{-1/2} u\|^2_{L^2(T)}, \tag{5}$$

and for $i \neq j$ let $\tilde{H}^{1/2}(T, I_i, I_j) = \tilde{H}^{1/2}(T, I_i) \cap \tilde{H}^{1/2}(T, I_j)$ with norm

$$\|u\|^2_{\tilde{H}^{1/2}(T,I_i,I_j)} = |u|^2_{H^{1/2}(T)} + \|\lambda_i^{-1/2} u\|^2_{L^2(T)} + \|\lambda_j^{-1/2} u\|^2_{L^2(T)}. \tag{6}$$

Next we consider the construction of basis functions for the p-version. To this end we will use extension operators as described below.
Extensions can be defined locally on patches of elements. For the extension of basis functions associated with edges (so-called edge basis functions) the situation is as indicated in Figure 1(a). A polynomial f defined on the edge I vanishes at the endpoints of I and needs to be extended to a piecewise polynomial U on $K := T_1 \cup T_2$ such that it can be extended continuously by zero onto an enlarged patch $\tilde{K}$ which contains K.
For functions associated with nodes (nodal functions) the situation is analogous. Namely, for a given patch as in Figure 1(b), values on the skeleton of the edges of the patch are given including 1 in the center node and 0 on the boundary. As for edge functions these values are extended locally onto the triangles, see the construction on the reference triangle below.

For our analysis we explicitly consider the situation on the reference triangle $T := \{(x,y);\, x \geq 0,\, y \geq 0,\, x + y \leq 1\}$. The vertices and edges of T are denoted by V_i and I_i, $i = 1,2,3$, respectively, see Figure 2. The edges I_1 and I_3 will be identified with the Interval $I := (0,1)$, and $I = I_1$ will be used without further notice. We also need the polynomial spaces

$$P^p(I) := \operatorname{span}\{x^i,\, 0 \leq i \leq p\}, \qquad P^p(T) := \operatorname{span}\{x^i y^j,\, 0 \leq i + j \leq p\}.$$

For the construction of our basis functions we need two extension operators, the operator F frequently used in finite element analysis (see [3, 2]), and the operator E used for problems in three dimensions (see [17, 18]).

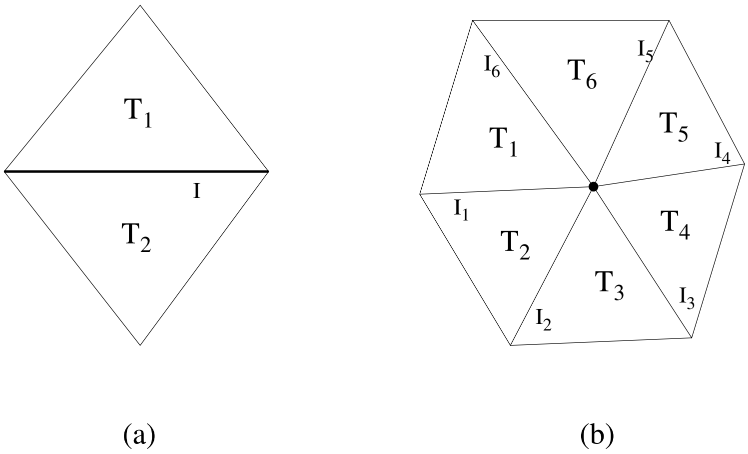

Figure 1: Constructing edge and nodal basis functions by extension.

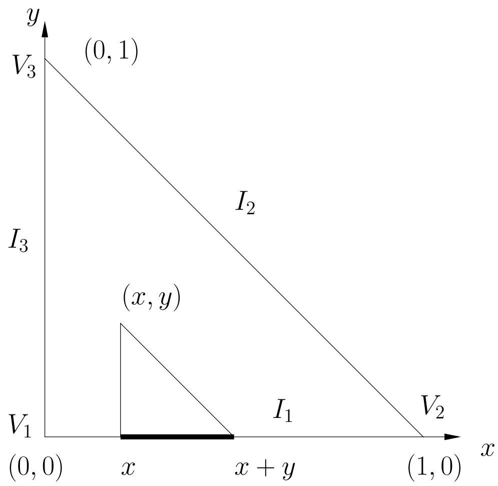

Figure 2: The reference triangle T.

The operator F is defined by

$$F(f)(x,y) := \frac{1}{y}\int_x^{x+y} f(t)\,dt.$$

It extends polynomials of degree p on I to polynomials of total degree p on T. It cannot be used to construct the extension needed for the vertex and edge functions since, e.g., a root of f in 0 does not extend to a zero trace of $F(f)$ on I_3. This is precisely the property of E which is defined by

$$E(f)(x,y) := \frac{x}{y}\int_x^{x+y}\frac{f(t)}{t}\,dt \quad (f(0)=0).$$

More generally, for $f \in P^p(I)$ we define extension operators from I_1 by

$$\begin{aligned}
E_1^1(f)(x,y) :=& E(f)(x,y) = \frac{x}{y}\int_x^{x+y}\frac{f(t)}{t}\,dt \qquad \text{if } f(0)=0,\\
E_2^1(f)(x,y) :=& \frac{1-x-y}{y}\int_x^{x+y}\frac{f(t)}{1-t}\,dt \qquad \text{if } f(1)=0,\\
E^1(f)(x,y) :=& \frac{x(1-x-y)}{y}\int_x^{x+y}\frac{f(t)}{t(1-t)}\,dt \qquad \text{if } f(0)=f(1)=0.
\end{aligned}$$

We note that there holds

$$E^1(f)(x,y) = (1-x-y)E_1^1(f)(x,y) + xE_2^1(f)(x,y).$$

Moreover, $E_2^1(f)=0$ on I_2 and $E^1(f)=0$ on $I_2\cup I_3$.
Extension operators E_3^3 (for $f\in P^p(I_3)$ with $f(1)=0$), E_1^3 (if $f(0)=0$) and E^3 (if $f(0)=f(1)=0$) from I_3 onto T are defined analogously.
For a polynomial $f\in P^p(I_2)$ we define

$$\begin{aligned}
E_2^2(f)(x,y) :=& \frac{y}{1-x-y}\int_x^{1-y}\frac{f(t,1-t)}{(1-t)}\,dt \qquad \text{if } f(1,0)=0,\\
E_3^2(f)(x,y) :=& \frac{x}{1-x-y}\int_x^{1-y}\frac{f(t,1-t)}{t}\,dt \qquad \text{if } f(0,1)=0,\\
E^2 f(x,y) :=& \frac{xy}{1-x-y}\int_x^{1-y}\frac{f(t,1-t)}{t(1-t)}\,dt \qquad \text{if } f(1,0)=f(0,1)=0.
\end{aligned}$$

There holds

$$E^2 f(x,y) = xE_2^2(f) + yE_3^2(f)$$

and $E_2^2(f)=0$ on I_1, $E_3^2(f)=0$ on I_3, $E^2(f)=0$ on $I_1\cup I_3$.
It is easy to see that all the extensions are polynomials of degree p on T. Furthermore, all the operators which deal with polynomials that vanish in only one vertex are linear transformations of the operator $E=E_1^1$. The main results concerning this operator are given in the next theorem. For the proof see [14].

Theorem 1. *For $f\in P^p(I)$ with $f(0)=0$ there holds*

$$\|E(f)\|_{\tilde H^{1/2}(T,I_3)} \le C\,(1+\log p)^{1/2}\,\|f\|_{L^2(I)}. \tag{7}$$

For $f\in P^p(I)$ with $f(1)=0$ there holds

$$\|E_2^1(f)\|_{\tilde H^{1/2}(T,I_2)} \le C\,(1+\log p)^{1/2}\,\|f\|_{L^2(I)}. \tag{8}$$

For $f \in P^p(I)$ with $f(0) = f(1) = 0$ there holds

$$\|E^1(f)\|_{\tilde{H}^{1/2}(T, I_2 \cup I_3)} \le C\,(1 + \log p)^{1/2}\,\|f\|_{L^2(I)}. \tag{9}$$

The constant C above is independent of p and f.

In [14] we then prove the following extension theorem.

Theorem 2. *Let $\tilde{T}$ be a triangle and let Γ be one of its sides or the union of two. Then, for a given continuous function f on $\partial\tilde{T}$ which is a polynomial of degree up to p on each of the sides and which vanishes on Γ, there exists an extension U on $\tilde{T}$ such that U is a polynomial of total degree up to p, $U = f$ on $\partial\tilde{T}$ and*

$$\|U\|_{\tilde{H}^{1/2}(\tilde{T}, \Gamma)} \le C\,(1 + \log p)^{1/2}\,\|f\|_{L^2(\partial\tilde{T})}. \tag{10}$$

Here, the constant $C > 0$ is independent of f and p.

For the construction of vertex basis functions we consider special low energy functions, cf. Pavarino & Widlund [23]. Let ϕ_0 be the polynomial of degree p that minimises the $L^2(0,1)$-norm and satisfies $\phi_0(0) = 1$ and $\phi_0(1) = 0$. The corresponding polynomial satisfying $\phi_0(0) = 0$ and $\phi_0(1) = 1$ is denoted by $\phi_0^-(x) = \phi_0(1-x)$.
These polynomials are L^2-orthogonal to $P_0^p(0,1)$ (the polynomials with homogeneous boundary values), and there holds

$$\|\phi_0\|^2_{L^2(0,1)} = 1/(p^2 + p) \text{ and } (\phi_0, \phi_0^-)_{L^2(0,1)} = \frac{(-1)^{p+1}}{2(p+1)}\|\phi_0\|^2_{L^2(0,1)}, \tag{11}$$

see [23]. The expansion of such polynomials as a linear combination of Legendre polynomials is also given in [23]. For illustration see Figure 3 where ϕ_0 for $p = 10$ is given.
A vertex basis function $\tilde{\phi}_{V_1}$, e.g. for vertex V_1, is defined as follows. Set $\tilde{\phi}_{V_1} = \phi_0$ on I_1 and I_3, and $\tilde{\phi}_{V_1} = 0$ on I_2. Extend $\tilde{\phi}_{V_1}$ from I_1 onto T by using the extension operator E_2^1, $\psi_1 := E_2^1\tilde{\phi}_{V_1} = E_2^1\phi_0$. Let g_3 be the trace of ψ_1 on I_3 and define $\psi_3 := E^3(g_3 - \tilde{\phi}_{V_1})$, the extension of $g_3 - \tilde{\phi}_{V_1}$ from I_3 onto T with $\psi_3 = 0$ on I_1 and I_2. Eventually we set $\tilde{\phi}_{V_1} := \psi_1 - \psi_3$. The other vertex functions are defined analogously.

As basis for the edges we use affine images of antiderivatives of Legendre polynomials that vanish in the corners. The antiderivatives of the Legendre polynomials are defined on the interval $[-1,1]$ by

$$\mathcal{L}_0(x) := \frac{1-x}{2}, \quad \mathcal{L}_1(x) := \frac{1+x}{2}, \quad \mathcal{L}_n(x) := \frac{L_n(x) - L_{n-2}(x)}{2n-1} = \int_{-1}^{x} L_{n-1}(y)\,dy,$$

where L_n denotes the Legendre polynomial of degree n. These basis functions are extended onto the triangle using the extension operators E^i, $i = 1, 2, 3$. There are $p-1$ basis functions on each edge.

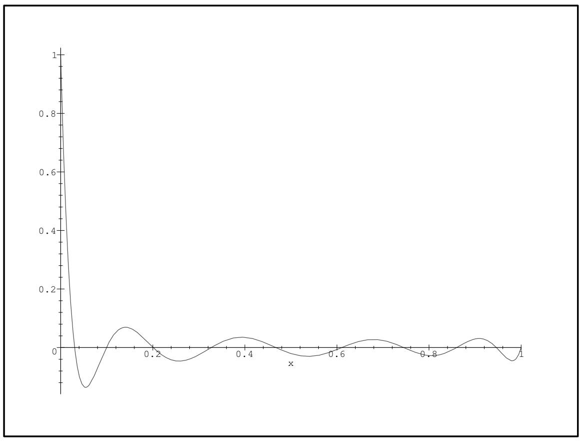

Figure 3: Low energy function ϕ_0 for $p = 10$.

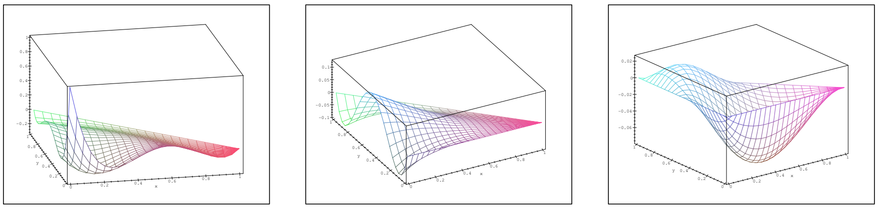

Figure 4: Basis functions, $p = 4$

As interior (or bubble) functions we use tensor products of antiderivatives of Legendre polynomials. On the reference triangle T these are the functions

$$\phi_{k,l}(x,y) = \frac{\mathcal{L}_{k+1}(2x-1)}{1-x}\frac{\mathcal{L}_l(2y-1)}{1-y}(1-x-y), \quad 1 \le k,\ 2 \le l,\ k+l \le p.$$

There are $(p-1)(p-2)/2$ interior functions per triangle.

For a sample set of nodal, edge and interior basis functions see Figure 4.

For a given triangle $\tilde{T}$, affine transformations of the basis functions defined above are used to span the polynomial space $P^p(\tilde{T})$. Given $u \in P^p(\tilde{T})$ this function has the unique representation $u = \sum_{i=1}^3 \tilde{u}_{V_i} + \sum_{i=1}^3 \tilde{u}_{I_i} + \tilde{u}_{\tilde{T}}$ where $\tilde{u}_{V_i}$ $\tilde{u}_{I_i}$ and $\tilde{u}_{\tilde{T}}$ are the vertex, edge and interior components, respectively. An interpolation operator $\tilde{I}^W$ onto the space of wire basket functions is defined by

$$\tilde{I}^W u := \sum_{i=1}^3 \tilde{u}_{V_i} + \sum_{i=1}^3 \tilde{u}_{I_i}. \tag{12}$$

Since the space of wire basket functions does not contain constants on $\tilde{T}$ we redefine the vertex and edge functions as follows. Let $\mathcal{F}$ denote the part of the expansion of the constant function 1 which belongs to the interior functions, i.e. $\mathcal{F} := 1 - \tilde{I}^W 1$. Then we define a new interpolation operator by

$$I^W u := \tilde{I}^W u + \mathcal{F}\overline{u}_W, \tag{13}$$

where $\overline{u}_W := \frac{\int_{\partial\tilde{T}} u}{\int_{\partial\tilde{T}} 1}$. This operator maps a constant function onto itself. The new vertex and edge components of u for the changed basis functions are denoted by u_{V_i} and u_{E_i}, $i = 1, \ldots, 3$. They are images under I^W of the preliminary components $\tilde{u}_{V_i}$ and $\tilde{u}_{E_i}$. The interior basis functions are unchanged.

Now, in order to define the boundary element space Ψ we introduce a quasi-uniform mesh $\overline{\Gamma} = \cup_{i=1}^n \overline{\Gamma}_i$ consisting of triangles Γ_i and define

$$\Psi := S_h^p := \{u \in \mathcal{C}^0(\Gamma); u|_{\Gamma_i} \in P^p(\Gamma_i)\} \subset \tilde{H}^{1/2}(\Gamma).$$

Here, h denotes the maximum diameter of the elements of the mesh. In a standard way we utilise the local basis functions defined above to generate a basis for S_h^p. In particular we use the notation for components in (12) and the wire basket interpolation operator in (13) for the global setting. Additionally, W denotes the wire basket of the mesh, i.e. the union of nodes and edges.

Next we introduce a preconditioner in the additive Schwarz framework. For simplicity we consider the situation that $\Gamma \subset \mathbb{R}^3$ is a surface piece. In fact, the case of a closed surface Γ is implicitly covered by our theory without any complication. The analysis for open surfaces is more involved since in this case the energy space of hypersingular operators must incorporate homogeneous boundary conditions.

The additive Schwarz preconditioner is based upon a subspace decomposition

$$S_h^p(\Gamma) = H_0 + H_1 + \cdots + H_n.$$

For our method we choose $H_0 := \Psi_W(\Gamma)$ being the space of wire basket functions and H_j consisting of the interior functions on Γ_j, $j = 1, \ldots, n$. Accordingly any $u \in S_n^p$ has a unique representation

$$u = u_W + \sum_{i=1}^n u_{\Gamma_i}, \tag{14}$$

where $u_W \in \Psi_W(\Gamma)$ and u_{Γ_i} are the interior functions with support in Γ_i.

Then the **additive Schwarz method** reads: Solve

$$Pu_N := (P_0 + P_1 + \cdots + P_n)u_N = f_N$$

where $P_j : S_h^p(\Gamma) \to H_j$, $j = 0, \ldots, n$, are projection operators defined by

$$b_j(P_j v, \varphi) = \langle Dv, \varphi \rangle_\Gamma \qquad \forall\, \varphi \in H_j.$$

For the interior spaces $H_1, \ldots, H_n$, b_0 is the energy bilinear form

$$b_j(v, w) := a(v, w) = \langle Dv, w \rangle_\Gamma, \qquad v, w \in H_j,\ j = 1, \ldots, n, \tag{15}$$

and for the wire basket space Ψ_W we consider two bilinear forms b_0. For our first method we choose

$$b_0(u, u) := \hat{a}_W(u, u) := (1 + \log p)^3 \sum_{i=1}^{n} \inf_{c_i \in \mathbb{R}} \|u - c_i\|^2_{L^2(W_i)}, \tag{16}$$

where W_i denotes the boundary of Γ_i. The corresponding additive Schwarz operator P will be denoted by P_W.

For the second method we use the energy bilinear form,

$$b_0(v, w) := a(v, w) = \langle Dv, w \rangle_\Gamma, \qquad v, w \in \Psi_W(\Gamma). \tag{17}$$

In this case we denote the additive Schwarz operator by $P = P_D$.

The main result of this paper is the following theorem.

Theorem 3. *Let b_0 denote one of the bilinear forms $\hat{a}_W$ or a. Then, for any $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p(\Gamma)$, there holds*

$$\begin{aligned} C_0\,(1 + \log p)^{-4} \Big(b_0(u_W, u_W) + \sum_{i=1}^{n} a(u_{\Gamma_i}, u_{\Gamma_i}) \Big) \\ \le a(u, u) \\ \le C_1 \Big(b_0(u_W, u_W) + \sum_{i=1}^{n} a(u_{\Gamma_i}, u_{\Gamma_i}) \Big). \end{aligned} \tag{18}$$

Here the constants $C_0, C_1 > 0$ are independent of h, p and u. Therefore, the minimum and maximum eigenvalues of the additive Schwarz operator P ($P = P_W$ if $b_0 = \hat{a}_W$ or $P = P_D$ if $b_0 = a$) are bounded like

$$\lambda_{min}(P) \ge C_0\,(1 + \log p)^{-4}, \quad \lambda_{max}(P) \le C_1,$$

and the condition number satisfies with a constant $C > 0$, independent of h and p,

$$\kappa(P) = \frac{\lambda_{max}(P)}{\lambda_{min}(P)} \le C\,(1 + \log p)^4.$$

The bounds on the eigenvalues of P are immediate implications of the inequalities (18), see, e.g., Zhang [28]. The inequalities are proved by Theorems 4, 5, 6 in Section 4.

3 Technical tools

In this section we collect some technical lemmas which are needed to prove our main result (Theorem 3).

Lemma 1. *Let* $0 \le x \le 1$ *and* $f \in L^2(x,1)$. *Then there holds*

$$\int_0^{1-x} \frac{1}{y^2} \left(\int_x^{x+y} f(t)\, dt \right)^2 dy \le 4 \int_x^1 f^2(t)\, dt. \tag{19}$$

Proof. See Lemma 1 in [14]. □

Lemma 2. *Let* I *be one side of the reference triangle* T. *Then for any polynomial* v *of degree* p *on* T *there holds*

$$\|v\|^2_{L^2(I)} \le C(1+\log p)\|v\|^2_{H^{1/2}(T)}.$$

Let $\overline{v}_{W_T} := \frac{1}{|\partial T|}\int_{\partial T} v\, ds$. *Then there holds*

$$\|v - \overline{v}_{W_T}\|^2_{L^2(\partial T)} \le C(1+\log p)|v|^2_{H^{1/2}(T)}. \tag{20}$$

Proof. Let Q denote the reference square $(0,1)\times(0,1)$ and $I=(0,1)$. For a polynomial u of degree p there holds

$$\begin{aligned} \|u\|^2_{L^2(I)} &= \int_0^1 u(x,y=0)^2\, dx \le \int_0^1 \|u(x,\cdot)\|^2_{L^\infty(0,1)}\, dx \\ &\le C(1+\log p) \int_0^1 \|u(x,\cdot)\|^2_{H^{1/2}(0,1)}\, dx. \end{aligned} \tag{21}$$

The last estimate is due to Theorem 6.2 in Babuška *et al.* [2].
For the special case of a square we use an equivalent definition for the $H^{1/2}$-semi-norm (see Lemma 5.3, Chap. 2, in Nečas [19])

$$\begin{aligned} |u|^2_{H^{1/2}(Q)} \cong & \int_0^1\int_0^1 \frac{\|u(s_1,\cdot)-u(t_1,\cdot)\|^2_{L^2(0,1)}}{(s_1-t_1)^2}\, ds_1\, dt_1 \\ & + \int_0^1\int_0^1 \frac{\|u(\cdot,s_2)-u(\cdot,t_2)\|^2_{L^2(0,1)}}{(s_2-t_2)^2}\, ds_2\, dt_2. \end{aligned}$$

Therefore, we can estimate

$$\begin{aligned} \int_0^1 |u(x,\cdot)|^2_{H^{1/2}(0,1)}\, dx &= \int_0^1\int_0^1\int_0^1 \frac{(u(x,y_1)-u(x,y_2))^2}{(y_1-y_2)^2}\, dy_1\, dy_2\, dx \\ &= \int_0^1\int_0^1 \frac{\|u(\cdot,y_1)-u(\cdot,y_2)\|^2_{L^2(0,1)}}{(y_1-y_2)^2}\, dy_1\, dy_2 \\ &\le C|u|^2_{H^{1/2}(Q)}. \end{aligned} \tag{22}$$

Furthermore it is

$$\int_0^1 \|u(x,\cdot)\|^2_{L^2(0,1)}\,dx = \|u\|^2_{L^2(Q)}.$$

Combining this relation with (21) and (22) we obtain

$$\|u\|^2_{L^2(I)} \le C(1+\log p)\|u\|^2_{H^{1/2}(Q)}.$$

Now for the reference triangle T we extend the function v from T onto the reference square Q by reflecting it at I_2. The reflected function on the reflected triangle $\tilde{T}$ is denoted by $\tilde{v}$. By symmetry the coupling term between v and $\tilde{v}$ in the $H^{1/2}$-norm vanishes. Therefore we deduce that there holds

$$\begin{aligned}\|v\|^2_{L^2(I)} &\le C(1+\log p)\|v\|^2_{H^{1/2}(Q)}\\ &\le C(1+\log p)\left(|v|^2_{H^{1/2}(T)} + |\tilde{v}|^2_{H^{1/2}(\tilde{T})} + \|v\|^2_{L^2(T)} + \|\tilde{v}\|^2_{L^2(\tilde{T})}\right)\\ &= 2C(1+\log p)\|v\|^2_{H^{1/2}(T)}.\end{aligned}$$

To prove (20) we use the minimising property of $\overline{v}_{W_T}$ and a quotient space argument as follows:

$$\begin{aligned}\|v-\overline{v}_{W_T}\|^2_{L^2(\partial T)} &\le \|v-c\|^2_{L^2(\partial T)} = \sum_{i=1}^{3}\|v-c\|^2_{L^2(I_i)}\\ &\le C(1+\log p)\|v-c\|^2_{H^{1/2}(T)} \qquad \forall\, c\in\mathbb{R}.\end{aligned}$$

Therefore,

$$\|v-\overline{v}_{W_T}\|^2_{L^2(\partial T)} \le C(1+\log p)\inf_{c\in\mathbb{R}}\|v-c\|^2_{H^{1/2}(T)} \le C(1+\log p)|v|^2_{H^{1/2}(T)}.$$

□

Lemma 3. *Let $\tilde{T}$ be a triangle of diameter h and $u\in H^{1/2}(\tilde{T})$. Then the mean value $\overline{u}_{W_{\tilde{T}}} = \frac{1}{|\partial\tilde{T}|}\int_{\partial\tilde{T}} u\,ds$ of u on the boundary of $\tilde{T}$ can be bounded by*

$$\overline{u}^2_{W_{\tilde{T}}} \le C\,h^{-1}\|u\|^2_{L^2(\partial\tilde{T})}.$$

On the reference triangle T there holds for $u\in P^p(T)$

$$\overline{u}^2_{W_T} \le C(1+\log p)\|u\|^2_{H^{1/2}(T)}.$$

Proof. Using the Cauchy-Schwarz inequality we get

$$\overline{u}^2_{W_{\tilde{T}}} \le \frac{1}{|\partial\tilde{T}|^2}\int_{\partial\tilde{T}} 1\,ds\int_{\partial\tilde{T}} u^2\,ds \cong C\,h^{-1}\|u\|^2_{L^2(\partial\tilde{T})},$$

which is the first assertion of the lemma. Analogously on the reference triangle T we have

$$\overline{u}^2_{W_T} \le C\,\|u\|^2_{L^2(\partial T)}$$

and using Lemma 2 we obtain the second assertion. □

Lemma 4. *For a polynomial f of degree p which vanishes on the boundary of T there holds*

$$\|f\|_{\tilde{H}^{1/2}(T)} \le C(1+\log p)\|f\|_{H^{1/2}(T)}. \tag{23}$$

Proof. See Lemma 6 in [13].

□

Lemma 5. *Let $u \in P^p(T)$ with representation $u = u_W + u_T$. Then there holds*

$$|u_W|^2_{H^{1/2}(T)} \le C(1+\log p)^2 \|u\|^2_{H^{1/2}(T)}.$$

Proof. Using the definition of the interpolation operator I^W in (13) we get

$$\begin{aligned} |u_W|^2_{H^{1/2}(T)} &= \left| \sum_{i=1}^3 \tilde{u}_{V_i} + \sum_{i=1}^3 \tilde{u}_{I_i} + \mathcal{F}\overline{u}_{W_T} \right|^2_{H^{1/2}(T)} \\ &\le C \left(\sum_{i=1}^3 |\tilde{u}_{V_i}|^2_{H^{1/2}(T)} + \sum_{i=1}^3 |\tilde{u}_{I_i}|^2_{H^{1/2}(T)} + \overline{u}^2_{W_T} |\mathcal{F}|^2_{H^{1/2}(T)} \right). \end{aligned}$$

First let us consider the vertex function for the vertex V_1.
It is constructed using the extensions $\psi_1 = E^1_2 \phi_0$ and $\psi_3 = E^3(\psi_1|_{I_3} - \phi_0)$ with ϕ_0 defined in §2. The vertex function associated with V_1 is $\tilde{u}_{V_1} = c_1(\psi_1 - \psi_3)$ (here, $c_1 = \tilde{u}_{V_1}(V_1)$). Then we can estimate using Theorem 1 as follows.

$$|\tfrac{1}{c_1}\tilde{u}_{V_1}|_{H^{1/2}(T)} \le |\psi_1|_{H^{1/2}(T)} + |\psi_3|_{H^{1/2}(T)} \le (1+\log p)^{1/2} \left(\|\phi_0\|_{L^2(I_1)} + \|\psi_1|_{I_3} - \phi_0\|_{L^2(I_3)} \right) \tag{24}$$

By definition of ψ_1 we have

$$\|\psi_1|_{I_3}\|^2_{L^2(I_3)} = \int_0^1 \left(\frac{1-y}{y}\right)^2 \left(\int_0^y \frac{\phi_0(t)}{1-t}\, dt \right)^2 dy \le \int_0^1 \frac{1}{y^2} \left(\int_0^y |\phi_0(t)|\, dt \right)^2 dy.$$

Using Lemma 1 this yields

$$\|\psi_1|_{I_3}\|^2_{L^2(I_3)} \le C \|\phi_0\|^2_{L^2(I_3)}, \tag{25}$$

and with (24)

$$|\tilde{u}_{V_1}|_{H^{1/2}(T)} \le C(1+\log p)^{1/2} \|c_1\phi_0\|_{L^2(I_1)} = C(1+\log p)^{1/2} \|\tilde{u}_{V_1}\|_{L^2(I_1)}. \tag{26}$$

To bound the edge component of u we use (9) and obtain

$$|\tilde{u}_{I_1}|^2_{H^{1/2}(T)} \le \|\tilde{u}_{I_1}\|^2_{\tilde{H}^{1/2}(T, I_2 \cup I_3)} \le C(1+\log p) \|\tilde{u}_{I_1}\|^2_{L^2(I_1)}. \tag{27}$$

Therefore we have the intermediate result

$$|\tilde{u}_W|^2_{H^{1/2}(T)} \le C(1+\log p) \left(\sum_{i=1}^3 \|\tilde{u}_{V_i}\|^2_{L^2(\partial T)} + \sum_{i=1}^3 \|\tilde{u}_{I_i}\|^2_{L^2(I_i)} \right). \tag{28}$$

Recalling that ϕ_0 is orthogonal on $(0,1)$ to any edge function and using the relation $(\phi_0, \phi_0^-)_{L^2(0,1)} = \frac{(-1)^{p+1}}{2(p+1)}\|\phi_0\|^2_{L^2(0,1)}$ (see [23]) one easily deduces that

$$\|\tilde{u}_W\|_{L^2(I_1)} = \|\tilde{u}_{V_1} + \tilde{u}_{V_2} + \tilde{u}_{I_1}\|_{L^2(I_1)} \quad \text{and} \quad \left(\|\tilde{u}_{V_1}\|^2_{L^2(I_1)} + \|\tilde{u}_{V_2}\|^2_{L^2(I_1)} + \|\tilde{u}_{I_1}\|^2_{L^2(I_1)}\right)^{1/2} \quad (29)$$

are equivalent norms. Analogous equivalencies hold on the edges I_2 and I_3.
It follows from (28) that

$$|\tilde{u}_W|^2_{H^{1/2}(T)} \leq C(1 + \log p)\|\tilde{u}_W\|^2_{L^2(\partial T)} = C(1 + \log p)\|u\|^2_{L^2(\partial T)}, \quad (30)$$

and using Lemma 2 we find

$$|\tilde{u}_W|^2_{H^{1/2}(T)} \leq C(1 + \log p)^2\|u\|^2_{H^{1/2}(T)}. \quad (31)$$

To bound $|\mathcal{F}|_{H^{1/2}(T)}$ we use (30):

$$|\mathcal{F}|^2_{H^{1/2}(T)} = |1 - \tilde{I}^W 1|^2_{H^{1/2}(T)} \leq 2|1|^2_{H^{1/2}(T)} + 2|\tilde{I}^W 1|^2_{H^{1/2}(T)} \leq C(1 + \log p)\|1\|^2_{L^2(\partial T)}. \quad (32)$$

Now the assertion of the lemma follows by combining (31) and (32) with Lemma 3. □

Lemma 6. *For a wire basket function u_W there holds*

$$\|u_W\|_{L^2(T)} \leq C\|u_W\|_{L^2(\partial T)}.$$

Proof. By the definition of u_W we get

$$\|u_W\|^2_{L^2(T)} \leq C\left(\sum_{i=1}^{3}\|\tilde{u}_{V_i}\|^2_{L^2(T)} + \sum_{i=1}^{3}\|\tilde{u}_{I_i}\|^2_{L^2(T)} + \|\mathcal{F}\|^2_{L^2(T)}\overline{u}^2_{W_T}\right).$$

By Lemma 1 there holds

$$\begin{aligned}
\|\tilde{u}_{I_1}\|^2_{L^2(T)} &= \int_0^1\int_0^{1-x}\left(\frac{x(1-x-y)}{y}\right)^2\left(\int_x^{x+y}\frac{\tilde{u}_{I_1}(t,0)}{t(1-t)}\,dt\right)^2 dy\,dx \\
&\leq 4\int_0^1\int_x^1 (\tilde{u}_{I_1}(t,0))^2\,dt\,dx \\
&\leq C\|\tilde{u}_{I_1}\|^2_{L^2(I_1)}.
\end{aligned}$$

Similarly, with $\tilde{u}_{V_1} = c_1(\psi_1 - \psi_3) = c_1(E_2^1\phi_0 - E^3(\psi_1|_{I_3} - \phi_0))$ we get

$$\|\tilde{u}_{V_1}\|^2_{L^2(T)} \leq C\,c_1^2\left(\|\psi_1\|^2_{L^2(T)} + \|\psi_3\|^2_{L^2(T)}\right) \leq C\|\tilde{u}_{V_1}\|^2_{L^2(I_1\cup I_3)}.$$

Analogous estimates hold for the other vertex and edge functions.

Proceeding as in the proof of Lemma 5 (see after (28)) we therefore get

$$\|\tilde{u}_W\|^2_{L^2(T)} \le C\|\tilde{u}_W\|^2_{L^2(\partial T)} = C\|u\|^2_{L^2(\partial T)}. \tag{33}$$

Analogously we find

$$\|\tilde{I}^W 1\|^2_{L^2(T)} \le C\,\|1\|^2_{L^2(\partial T)}$$

and therefore

$$\|\mathcal{F}\|^2_{L^2(T)} = \|1 - \tilde{I}^W 1\|^2_{L^2(T)} \le C\|1\|^2_{L^2(T)} + C\|\tilde{I}^W 1\|^2_{L^2(T)} \le C. \tag{34}$$

From Lemma 3 we know that

$$\overline{u}^2_{W_T} \le C\|u\|^2_{L^2(\partial T)} = C\|u_W\|^2_{L^2(\partial T)}.$$

Combining this with (33) and (34) finishes the proof. □

4 Proof of the main result

In this section we prove our main result, Theorem 3. In Theorem 4 and Theorem 5 we consider the bilinear forms corresponding to the additive Schwarz operator P_W and Theorem 6 deals with P_D.

Theorem 4. *There exists a positive constant C, independent of h and p, such that for any $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p$ there holds*

$$(1+\log p)^3 \sum_{i=1}^n \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} + \sum_{i=1}^n \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \le C\,(1+\log p)^4 |u|^2_{H^{1/2}(\Gamma)}. \tag{35}$$

Proof. Let Γ_i be an arbitrary triangle with boundary W_i and diameter h. Then, using a transformation to the reference triangle T and Lemma 2 we obtain

$$\begin{aligned}\|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} &\le C\,h\|v - \overline{v}_{W_i}\|^2_{L^2(\partial T)} \le Ch(1+\log p)|v|^2_{H^{1/2}(T)}\\ &\le C(1+\log p)|u|^2_{H^{1/2}(\Gamma_i)},\end{aligned} \tag{36}$$

see, e.g., [13]. Here v denotes the linearly transformed function u. Therefore,

$$(1+\log p)^3 \sum_{i=1}^n \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} \le C(1+\log p)^4 \sum_{i=1}^n |u|^2_{H^{1/2}(\Gamma_i)} \le C(1+\log p)^4 |u|^2_{H^{1/2}(\Gamma)}.$$

It remains to bound the norms of the interior components of u. On the reference triangle T we have $u_T = (u - u_W)|_T$. By Lemma 5 there holds

$$|u_W|^2_{H^{1/2}(T)} \le C(1+\log p)^2 \|u\|^2_{H^{1/2}(T)},$$

and Lemmas 6 and 2 yield

$$\|u_W\|^2_{L^2(T)} \le C\|u_W\|^2_{L^2(\partial T)} \le C(1+\log p)\|u\|^2_{H^{1/2}(T)}.$$

Therefore,

$$\|u_W\|^2_{H^{1/2}(T)} = \|u_W\|^2_{L^2(T)} + |u_W|^2_{H^{1/2}(T)} \le C(1+\log p)^2\|u\|^2_{H^{1/2}(T)}.$$

Using Lemma 4 and the triangle inequality we obtain

$$\begin{aligned}\|u-u_W\|^2_{\tilde H^{1/2}(T)} &\le C(1+\log p)^2\left(\|u\|^2_{H^{1/2}(T)} + \|u_W\|^2_{H^{1/2}(T)}\right)\\ &\le C(1+\log p)^4\|u\|^2_{H^{1/2}(T)}.\end{aligned}$$

Since the wire basket functions contain the constants we thus have for any $c \in \mathbb{R}$

$$\begin{aligned}\|u - I^W u\|^2_{\tilde H^{1/2}(T)} &= \|u + c - I^W(u+c)\|^2_{\tilde H^{1/2}(T)}\\ &\le C(1+\log p)^4\|u+c\|^2_{H^{1/2}(T)}.\end{aligned} \tag{37}$$

By a quotient space argument we conclude that

$$\|u - I^W u\|^2_{\tilde H^{1/2}(T)} \le C(1+\log p)^4|u|^2_{H^{1/2}(T)}.$$

Since the norm in $\tilde H^{1/2}(T)$ scales like the semi-norm in $H^{1/2}(T)$ this proves

$$\|u_{\Gamma_i}\|^2_{\tilde H^{1/2}(\Gamma_i)} \le C(1+\log p)^4|u|^2_{H^{1/2}(\Gamma_i)}, \tag{38}$$

and summing over all elements this finishes the proof of this theorem. □

Theorem 5. *There exists a positive constant C, independent of h and p, such that for any $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p$ there holds*

$$\|u\|^2_{\tilde H^{1/2}(\Gamma)} \le C\left((1+\log p)^3\sum_{i=1}^n \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} + \sum_{i=1}^n \|u_{\Gamma_i}\|^2_{\tilde H^{1/2}(\Gamma_i)}\right).$$

Proof. We extend our basis functions in a discrete harmonic way into the interior of a tetrahedron, see Bică [6] for details. We consider a reference tetrahedron Ω_{ref} with T as one of its sides and maintain the notation for the basis functions on Ω_{ref}.
Let $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p$ be given. We remark that there holds (see von Petersdorff [24])

$$\|u\|^2_{\tilde H^{1/2}(\Gamma)} \le C\left(\|u_W\|^2_{\tilde H^{1/2}(\Gamma)} + \sum_{i=1}^n \|u - u_W\|^2_{\tilde H^{1/2}(\Gamma_i)}\right) \tag{39}$$

with $(u-u_W)|_{\Gamma_i} = u_{\Gamma_i}$. Therefore we only have to prove

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,(1+\log p)^3 \sum_{i=1}^{n} \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)}.$$

Consider a three dimensional domain Ω such that $\Gamma \subset \partial\Omega$. We decompose Ω into tetrahedra Ω_i such that the trace of this mesh is compatible with the mesh on Γ. For an arbitrary extension U_W of u_W with $U_W = 0$ on $\partial\Omega \setminus \Gamma$ there holds

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,|U_W|^2_{H^1(\Omega)} = C \sum_i |U_{W_i}|^2_{H^1(\Omega_i)}. \tag{40}$$

Now we consider the reference tetrahedron Ω_{ref} and the reference triangle $T \subset \partial\Omega_{\mathrm{ref}}$. We extend the wire basket component u_W defined on ∂T onto Ω_{ref} by using Theorem 2 and the discrete harmonic extension. Similarly as in the proof Theorem 2 (see [14] for details) there holds

$$\|u_W\|_{H^{1/2}(T)} \leq C\,(1+\log p)^{1/2} \|u_W\|_{L^2(\partial T)}.$$

In the same way we extend u_W onto the other sides of Ω_{ref}. Then we get a continuous function on $\partial\Omega_{\mathrm{ref}}$. For the discrete harmonic extension from the faces of the tetrahedron into the interior there holds

$$\|U_W\|_{H^1(\Omega_{\mathrm{ref}})} \leq C\,\|u_W\|_{H^{1/2}(\partial\Omega_{\mathrm{ref}})}. \tag{41}$$

This follows from the minimising property of the discrete harmonic extension and the extension theorem of Muñoz-Sola [18, Theorem 1]. Using Lemma 10 (see the appendix) it follows that

$$|U_W|^2_{H^1(\Omega_{\mathrm{ref}})} \leq C(1+\log p)^3 \|u_W\|^2_{L^2(\partial T)}.$$

All the extension operators used reproduce constant functions and therefore we get for any $c \in \mathbb{R}$

$$|U_W|^2_{H^1(\Omega_{\mathrm{ref}})} = |U_W + c|^2_{H^1(\Omega_{\mathrm{ref}})} \leq C\,(1+\log p)^3 \|u_W + c\|^2_{L^2(\partial T)}.$$

Transforming this result to an arbitrary element we get

$$|U_{W_i}|^2_{H^1(\Omega_i)} \leq C\,(1+\log p)^3 \|u_{W_i} + c\|^2_{L^2(\partial\Gamma_i)}.$$

Together with (40) this yields

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,(1+\log p)^3 \inf_{c\in\mathbb{R}} \sum_i \|u - c\|^2_{L^2(W_i)} = C\,(1+\log p)^3 \sum_i \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)}, \tag{42}$$

which was left to be proved. □

Theorem 6. *There exist positive constants C_0, C_1, independent of h and p, such that for any $u = u_W + \sum_{i=1}^n u_{\Gamma_i} \in S_h^p$ there holds*

$$C_0\,\|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} + \sum_{i=1}^{n} \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \leq C_1(1+\log p)^4 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

Proof. The first inequality has already been proved by (39).
It remains to prove the second inequality. The bound

$$\sum_{i=1}^{n} \|u_{\Gamma_i}\|^2_{\tilde{H}^{1/2}(\Gamma_i)} \leq C\,(1+\log p)^4 \sum_{i=1}^{n} |u|^2_{H^{1/2}(\Gamma_i)} \leq C(1+\log p)^4 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

is an immediate consequence of Theorem 4.
From inequality (42) we know that there holds

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C(1+\log p)^3 \sum_{i=1}^{n} \inf_{c_i\in\mathbb{R}} \|u_W - c_i\|^2_{L^2(\partial\Gamma_i)}$$

and by (36) we get

$$\inf_{c\in\mathbb{R}} \|u-c\|^2_{L^2(\partial\Gamma_i)} \leq C(1+\log p)|u|^2_{H^{1/2}(\Gamma_i)}.$$

Since $u = u_W$ on $\partial\Gamma_i$ the latter two estimates imply

$$\|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C\,(1+\log p)^4 \sum_{i=1}^{n} |u|^2_{H^{1/2}(\Gamma_i)} \leq C\,(1+\log p)^4 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

This finishes the proof of the theorem. □

Proof of Theorem 3. The assertions are direct consequences of the previous theorems by noting that there holds

$$a(u,u) = \langle Du, u\rangle_\Gamma \cong \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \cong \|u\|^2_{\tilde{H}^{1/2}(\Gamma_i)},$$

for any $u \in \tilde{H}^{1/2}(\Gamma)$ with support on $\bar{\Gamma}_i \subset \Gamma$, see (2). □

5 Numerical results

In this section we present some numerical experiments to confirm our theoretical results about the behaviour of the condition number of the preconditioned boundary element matrix.

First we comment on the implementation of the preconditioner. When ordering the basis functions of the boundary element space appropriately the preconditioning matrix has a block diagonal form

$$S := \begin{pmatrix} S_W & 0 & 0 & 0 \\ 0 & S_{\Gamma_1} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & S_{\Gamma_n} \end{pmatrix}.$$

Here, S_W is the discretisation of the bilinear form b_0 involving the wire basket functions and S_{Γ_i} discretises the energy bilinear form involving interior functions defined on Γ_i, cf. (15), (16), (17). For the calculation of the bilinear form $\hat{a}_W$ we remark that the mean value $\overline{u}_{W_i} = \frac{\int_{W_i} u\,ds}{\int_{W_i} 1\,ds}$ of u on W_i minimises $\|u - c_i\|_{L^2(W_i)}$ with respect to c_i and therefore

$$\min_{c_i \in \mathbb{R}} \|u - c_i\|^2_{L^2(W_i)} = \|u - \overline{u}_{W_i}\|^2_{L^2(W_i)} = \int_{W_i} \left(u(t) - \frac{1}{|W_i|} \int_{W_i} u\,ds \right)^2 dt$$
$$= \int_{W_i} u^2\,dt - \frac{1}{|W_i|} \left(\int_{W_i} 1 \cdot u\,ds \right)^2,$$

see also [23]. The first term of the right-hand side above is calculated by using the mass matrix $M^{(i)}$ for the wire basket functions on W_i. Furthermore, there holds $|W_i| = \int_{W_i} 1 \cdot 1\,ds = \vec{z}^{(i)T} M^{(i)} \vec{z}^{(i)}$. Here, $\vec{z}^{(i)}$ contains the coefficients for the constant function 1 on Γ_i. With this notation we can also write

$$\int_{W_i} 1 \cdot u\,ds = \vec{z}^{(i)T} M^{(i)} \vec{u} = \left(M^{(i)} \vec{z}^{(i)} \right)^T \vec{u},$$

where $\vec{u}$ contains the coefficients of u for the basis in use, and

$$\left(\int_{W_i} 1 \cdot u\,ds \right)^2 = \left(\left(M^{(i)} \vec{z}^{(i)} \right)^T \vec{u} \right)^2 = \vec{u}^T \left(M^{(i)} \vec{z}^{(i)} \right) \left(M^{(i)} \vec{z}^{(i)} \right)^T \vec{u}.$$

Thus, locally on one element we obtain

$$S_{W_i} = (1 + \log p)^3 \left(M^{(i)} - \frac{(M^{(i)} \vec{z}^{(i)}) \cdot (M^{(i)} \vec{z}^{(i)})^T}{\vec{z}^{(i)T} M^{(i)} \vec{z}^{(i)}} \right).$$

To calculate the preconditioning block S_W we sum over all the elements.

For our model problem we choose the domain $\Gamma = (-1/2, 1/2)^2 \times \{0\}$ and use uniform triangular meshes. We do not specify any right-hand side function f in (1) since we only report on the spectral behaviour of the stiffness matrix. For smooth right-hand side functions f the hp-version with quasi-uniform meshes converges like $\mathcal{O}(h^{1/2} p^{-1})$ in the energy norm, see [4, 5].

In Figure 5 the condition numbers of the Galerkin matrix with preconditioner are plotted. Here we consider only the wire basket preconditioner, i.e. the additive Schwarz operator P_W based upon the bilinear form $\hat{a}_W$ defined by (16). In the plot we also give the curve of $(1 + \log p)^4$, and the numerical results behave similarly in the given range of p ($p = 1, \ldots, 6$). Exact numbers are given in Table 1 together with the condition numbers of the un-preconditioned stiffness matrix. We also give the iteration numbers of the conjugate gradient method needed for fixed precision. As expected the iteration numbers increase only moderately when one of the preconditioners is used and grow substantially without preconditioner.
In Figure 6 we present condition numbers of the preconditioned matrix P_W for the h-version and different polynomial degrees. The results confirm the asymptotic independence of the condition numbers on h.

p	DOF	cond(w/o)	niter	cond(pre, L^2)	niter	cond(pre, $\tilde{H}^{1/2}$)	niter
1	1	0.100E+01	1	0.100E+01	1	0.100E+01	1
2	9	0.781E+01	6	0.561E+01	7	0.100E+01	1
3	25	0.734E+02	29	0.443E+02	18	0.547E+01	11
4	49	0.739E+03	100	0.745E+02	34	0.796E+01	17
5	81	0.102E+05	315	0.112E+03	44	0.103E+02	22
6	121	0.189E+06	993	0.150E+03	56	0.124E+02	27

Table 1: Condition numbers and iteration numbers for the p-version without and with preconditioning (using the L^2- and the energy bilinear form).

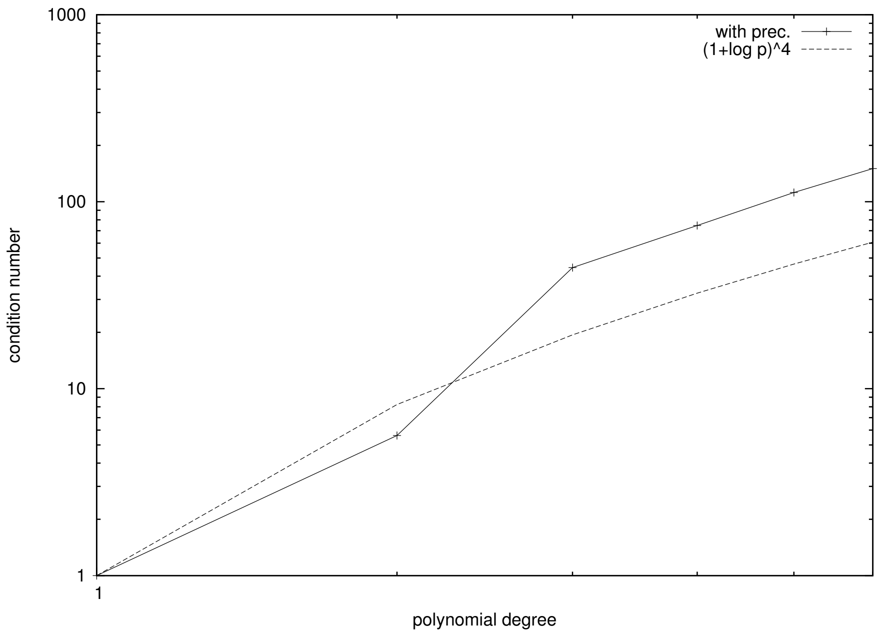

Figure 5: Condition number of the preconditioned Galerkin matrix, p-version.

A Additional technical results

For the convenience of the reader we repeat some of the results and proofs of Bică [6] who deals with wire basket preconditioners for the p-version of the finite element method on tetrahedral meshes. In fact, at several places he uses an unknown factor $N(p)$ stemming from an unproved extension theorem. Here we use the extension theorem from [14] to fill this gap.
We denote by Ω_{ref} the reference tetrahedron

$$\Omega_{ref} := \{(x, y, z);\ 0 \leq x, y, z \leq 1,\ x + y + z \leq 1\}$$

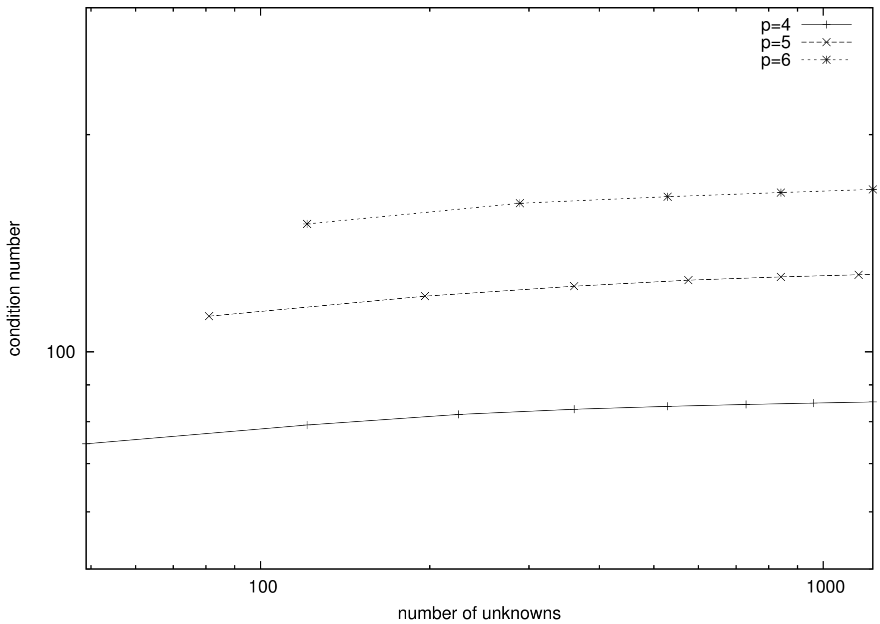

Figure 6: Condition number of the preconditioned Galerkin matrix, h-version.

and define for integer p

$$P^p(\Omega_{\mathrm{ref}}) := \mathrm{span}\{x^i y^j z^k;\ 0 \le i+j+k \le p\}.$$

Before dealing with results from [6] we collect three technical lemmas needed below.

Lemma 7. *[25] Let $u \in P^p(0,1)$. Then*

$$\max_{[0,1]} \left| \frac{d}{dx} u(x) \right| \le 2p^2 \max_{[0,1]} |u(x)|.$$

Lemma 8. *[2, Theorem 6.2] Let $u \in P^p(0,1)$. Then,*

$$\|u\|^2_{L^\infty(0,1)} \le C(1+\log p)\|u\|^2_{H^{1/2}(0,1)}.$$

Lemma 9. *[23, Lemma 5.3] Let I be any line segment in the closure of the reference tetrahedron $\overline{\Omega}_{\mathrm{ref}}$ and let $u \in P^p(\Omega_{\mathrm{ref}})$. Then,*

$$\|u\|^2_{L^2(I)} \le C(1+\log p)\|u\|^2_{H^1(\Omega_{ref})}.$$

If $\overline{u}_W$ is the average of u over the wire basket W, then,

$$\|u - \overline{u}_W\|^2_{L^2(I)} \le C(1+\log p)|u|^2_{H^1(\Omega_{ref}.)}$$

Now let us turn to the theory of finite elements. The wire basket decomposition used in [6] is the three-dimensional analogue of our decomposition obtained by discrete harmonic extensions of basis functions onto the reference tetrahedron. The corresponding wire basket interpolation operators in three dimensions will be again denoted by I^W and $\tilde{I}^W$. Here, in this section, W denotes the wire basket of the reference tetrahedron. As in the two-dimensional setting we define $\mathcal{F} := 1 - \tilde{I}^W 1$ on the boundary of the reference element. But now $\mathcal{F}$ has four components, each associated with one of the faces and vanishing on the other faces, $\mathcal{F} = \sum_{k=1}^4 \mathcal{F}_k$. Note that by definition of the basis functions $\mathcal{F}_k$ is discrete harmonic.
The next lemma is needed for the proof of Theorem 5 and its proof is based on the two lemmas that follow.

Lemma 10. *(Compare [6, Lemma 4.16]) Setting $U_W := I^W u$ there holds*

$$|U_W|^2_{H^1(\Omega_{\mathrm{ref}})} \leq C\,(1+\log p)^3 \|u\|^2_{L^2(W)} \quad \forall\, u \in P^p(\Omega_{\mathrm{ref}})$$

Proof. Let F_k, $k = 1, \ldots, 4$, denote the faces of Ω_{ref}. Since $I^W u = \tilde{I}^W u + \sum_{k=1}^4 \overline{u}_{\partial F_k} \mathcal{F}_k$ we get

$$|I^W u|^2_{H^1(\Omega_{\mathrm{ref}})} \leq 5 \left(|\tilde{I}^W u|^2_{H^1(\Omega_{\mathrm{ref}})} + \sum_{k=1}^{4} \overline{u}^2_{\partial F_k} |\mathcal{F}_k|^2_{H^1(\Omega_{\mathrm{ref}})} \right).$$

By Lemma 11 there holds

$$|\tilde{I}^W u|^2_{H^1(\Omega_{\mathrm{ref}})} \leq C\,(1+\log p) \|u\|^2_{L^2(W)}.$$

Using the discrete harmonicity of $\mathcal{F}$ and combining Lemma 4 with (32) and (34) we obtain

$$|\mathcal{F}_k|^2_{H^1(\Omega_{\mathrm{ref}})} \leq C\,(1+\log p)^3.$$

Together with

$$\overline{u}^2_{\partial F_k} = \left(\frac{\int_{\partial F_k} u}{\int_{\partial F_k} 1} \right)^2 \leq C \|u\|^2_{L^2(W)}$$

we get

$$\begin{aligned} |I^W u|^2_{H^1(\Omega_{\mathrm{ref}})} &\leq C \left((1+\log p)\|u\|^2_{L^2(W)} + \sum_{k=1}^{4} \overline{u}^2_{\partial F_k} (1+\log p)^3 \right) \\ &\leq C\,(1+\log p)^3 \|u\|^2_{L^2(W)}. \end{aligned}$$

□

Lemma 11. *(Compare [6, Lemma 4.13])*

$$|\tilde{I}^W u|^2_{H^1(\Omega_{\mathrm{ref}})} \leq C\,(1+\log p) \|u\|^2_{L^2(W)} \quad \forall\, u \in P^p(\Omega_{\mathrm{ref}})$$

Proof. Using the estimates for the vertex and edge functions in Lemma 12, and noting that $L^2(W)$-inner products between different wire basket components are negligible (see (29)), we get

$$
\begin{aligned}
|\tilde{I}^W u|^2_{H^1(\Omega_{\mathrm{ref}})} &\le C\left(\sum_{i=1}^{4}|\tilde{u}_{V_i}|^2_{H^1(\Omega_{\mathrm{ref}})}+\sum_{j=1}^{6}|\tilde{u}_{E_j}|^2_{H^1(\Omega_{\mathrm{ref}})}\right)\\
&\le C\,(1+\log p)\left(\sum_{i=1}^{4}\|\tilde{u}_{V_i}\|^2_{L^2(W)}+\sum_{j=1}^{6}\|\tilde{u}_{E_j}\|^2_{L^2(W)}\right)\\
&\le C\,(1+\log p)\|\tilde{I}^W u\|^2_{L^2(W)} = C\,(1+\log p)\|u\|^2_{L^2(W)}.
\end{aligned}
$$

□

Lemma 12. *(Compare [6, Lemmas 4.11, 4.12]) Let Φ_V and Φ_E be a vertex function and an edge function, respectively. There holds*

$$\|\Phi_V\|_{H^1(\Omega_{\mathrm{ref}})} \le C\,(1+\log p)^{1/2}\|\Phi_V\|_{L^2(W)}$$

and

$$\|\Phi_E\|_{H^1(\Omega_{\mathrm{ref}})} \le C(1+\log p)^{1/2}\|\Phi_E\|_{L^2(W)}.$$

Proof. This follows by using the property of discrete harmonic extensions, cf. (41), and Theorem 2. □

References

[1] M. Ainsworth and B. Q. Guo, *An additive Schwarz preconditioner for the p-version boundary element approximation of the hypersingular operator in three dimensions*, Numer. Math., 85 (2000), pp. 343–366.

[2] I. Babuška, A. Craig, J. Mandel, and J. Pitkäranta, *Efficient preconditioning for the p-version finite element method in two dimensions*, SIAM J. Numer. Anal., 28 (1991), pp. 624–661.

[3] I. Babuška and M. Suri, *The h-p version of the finite element method with quasi-uniform meshes*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 199–238.

[4] A. Bespalov and N. Heuer, *The p-version of the boundary element method for hypersingular operators on piecewise plane open surfaces*, Numer. Math., 100 (2005), pp. 185–209.

[5] ——, *The hp-version of the boundary element method with quasi-uniform meshes in three dimensions*, Tech. Rep. 06/1, BICOM, Brunel University, Uxbridge, UK, 2006.

[6] I. Bică, *Iterative Substructuring Algorithms for the p-version Finite Element Method for Elliptic Problems*, PhD thesis, Courant Institute of Mathematical Sciences, New York University, 1997.

[7] ——, *Nonoverlapping domain decomposition algorithms for the p-version finite element method for elliptic problems*, in Domain decomposition methods, 10 (Boulder, CO, 1997), Amer. Math. Soc., Providence, RI, 1998, pp. 231–237.

[8] W. Cao and B. Q. Guo, *Preconditioning for the p-version boundary element method in three dimensions with triangular elements*, J. Korean Math. Soc., 41 (2004), pp. 345–368.

[9] P. Grisvard, *Caractérisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal., 25 (1967), pp. 40–63.

[10] B. Guo and W. Cao, *An additive Schwarz method for the h-p version of the finite element method in three dimensions*, SIAM J. Numer. Anal., 35 (1998), pp. 632–654.

[11] N. Heuer, *An iterative substructuring method for the p-version of the boundary element method for hypersingular integral equations in three dimensions*, Numer. Math., 79 (1998), pp. 371–396. Erratum vol. 87, no. 4, pp. 793–794, 2001.

[12] ——, *Additive Schwarz methods for indefinite hypersingular integral equations in $\mathbb{R}^3$ – the p-version*, Appl. Anal., 72 (1999), pp. 411–437.

[13] N. Heuer, *Additive Schwarz method for the p-version of the boundary element method for the single layer potential operator on a plane screen*, Numer. Math., 88 (2001), pp. 485–511.

[14] N. Heuer and F. Leydecker, *An extension theorem for polynomials on triangles*, Tech. Rep. 06/4, BICOM, Brunel University, Uxbridge, UK, 2006.

[15] N. Heuer and E. P. Stephan, *An additive Schwarz method for the h-p version of the boundary element method for hypersingular integral equations in $\mathbb{R}^3$*, IMA J. Numer. Anal., 21 (2001), pp. 265–283.

[16] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972.

[17] Y. Maday, *Relèvements de traces polynomiales et interpolations hilbertiennes entre espaces de polynômes*, C. R. Acad. Sci. Paris Sér. I Math., 309 (1989), pp. 463–468.

[18] R. Muñoz-Sola, *Polynomial liftings on a tetrahedron and applications to the h-p version of the finite element method in three dimensions*, SIAM J. Numer. Anal., 34 (1997), pp. 282–314.

[19] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris, 1967.

[20] L. F. Pavarino, *Additive Schwarz methods for the p-version finite element method*, Numer. Math., 66 (1994), pp. 493–515.

[21] ——, *Schwarz methods with local refinement for the p-version finite element method*, Numer. Math., 69 (1994), pp. 185–211.

[22] L. F. Pavarino and T. Warburton, *Overlapping Schwarz methods for unstructured spectral elements*, J. Comput. Phys., 160 (2000), pp. 298–317.

[23] L. F. Pavarino and O. B. Widlund, *A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions*, SIAM J. Numer. Anal., 33 (1996), pp. 1303–1335.

[24] T. Petersdorff, *Randwertprobleme der Elastizitätstheorie für Polyeder – Singularitäten und Approximation mit Randelementmethoden*, PhD thesis, Technische Hochschule Darmstadt, Germany, 1989.

[25] T. J. Rivlin, *Chebyshev polynomials*, John Wiley & Sons Inc., New York, second ed., 1990.

[26] J. Schöberl, J. M. Melenk, C. G. A. Pechstein, and S. C. Zagelmayr, *Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements*, ICAM-Report No. 2005-11, Institute of Computational Mathematics, University of Linz, Austria, 2005.

[27] E. P. Stephan, *Boundary integral equations for screen problems in* $\mathbf{R}^3$, Integral Equations Operator Theory, 10 (1987), pp. 236–257.

[28] X. Zhang, *Multilevel Schwarz methods*, Numer. Math., 63 (1992), pp. 521–539.