The p-version of the FEM for slowly oscillating singularities in one dimension *

Norbert Heuer † Clemens Purer ‡

Abstract

We analyse the p-version of the finite element method on an interval for the approximation of slowly oscillating singularities. We prove an asymptotically exact a priori error estimate by studying Legendre expansions of these singularities. The dependence on the frequency is explicitly given and theoretical results are confirmed by numerical experiments.

Key words: p-version, finite element method, singularities, oscillations
AMS Subject Classification: 65N30, 41A10

1 Introduction and main result

In recent years there has been a substantial interest in studying oscillating singularities both theoretically and numerically. A wide area is signal processing where information is most often carried by irregular phenomena including singularities. Other areas are image recognition, electrocardiography, radar signals, and physics dealing with irregular structures [15]. Wavelet analysis and transform is now a standard tool for localising and characterising oscillating singularities, see, e.g., [14, 2, 11]. One standard form of studied singularities is

$$g(t) := t^\alpha \sin(\kappa t^\gamma), \quad t > 0, \alpha, \gamma, \kappa \in \mathbf{R}.$$

For non-integer α, the term t^α means that g has a reduced Sobolev regularity at $t = 0$ and the trigonometric part makes g oscillating ($\gamma \neq 0$, $\kappa \neq 0$). The oscillations increase when $t \to 0$ if $\gamma < 0$ and $\kappa \neq 0$. In this case singularities are called oscillating or strongly oscillating.

In this paper we study the polynomial approximation of slowly oscillating singularities with $\gamma = 1$. Note that the term 'slowly oscillating' does not mean that the frequency κ must be small, it only refers to the exponent γ. Other cases can be transformed to this one by mapping through a power function. Nevertheless, approximation theory details are not immediate, cf. [16]

*This work was done while C. Purer was an ERASMUS-Socrates exchange student at Brunel University.

†BICOM, Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex UB8 3PH, UK. email: `norbert.heuer@brunel.ac.uk`

‡Vienna University of Technology, Wiedner Hauptstrasse 8-10/105, 1040 Wien, Austria. email: `clemensp@gmx.at`

for logarithmic oscillations where, however, no exact expression for appearing constants is given. We prove an asymptotically exact a priori error estimate for the p-version of the finite element method (FEM) for the approximation of functions g with $\gamma = 1$. As expected, the rate of convergence depends in the known way on the singularity exponent α. More importantly, our results explicitly state the dependence on α and κ of the coefficient (of the principal error term).

There is well-established literature on the approximation theory of singularities to derive a priori error estimates for the FEM when dealing with boundary value problems and singular solutions. Standard variants of the FEM converge only slowly for such problems and there are different strategies to improve convergence. For instance, mesh refinement towards singularities [6] is appropriate, special elements can be used [3] or finite element spaces can be enriched by global functions which resemble the strongest singularities [18]. Particularly convincing is the hp-version with geometrically graded meshes which converges faster than algebraically [8].

When approximating oscillating functions by piecewise polynomials numerical dispersion poses a severe problem. In order to obtain acceptable results, and depending on the frequency, one has to use extremely fine meshes or high polynomial degrees, cf. [12, 13]. However, increasing polynomial degrees is much more efficient than mesh refinement to reduce the numerical dispersion error [1]. We also note that, for problems with singularities, the p-version of the FEM has twice the rate of convergence of the h-version with quasi-uniform meshes [7, 5]. Therefore, high order polynomials are attractive for the approximation of functions that possess both singular and oscillatory behaviour.

In this paper we consider the extreme case of the p-version where the mesh of the FEM is fixed and polynomial degrees are increased to improve the approximation. In fact, below we use only one element since a method with more elements exhibits the same asymptotic convergence (the element close to the singularity dominates the error). To study the best polynomial approximation error one generally uses expansions of approximants into orthogonal polynomials. For the FEM in one dimension this reduces straightforwardly to Legendre expansions of L_2-functions. In higher dimensions only recently the right tool, namely Jacobi polynomials in Jacobi-weighted Besov spaces, has been established. See [4] and [9] for the FEM in two and three dimensions, respectively, and [10] for the boundary element method in two dimensions. Here, for the FEM in one dimension, we study the Legendre expansion of blended singularities $t^\alpha u(t)$ where u is a smooth but possibly oscillating function. This general case is then made precise for slowly oscillating functions.

Our model problem is

$$-v'' = f \quad \text{on } I := (0,1)$$

subject to Dirichlet boundary conditions and with f given such that the solution is

$$v(t) = (t-\xi)_+^\alpha u(t) \tag{1.1}$$

where

$$(t-\xi)_+^\alpha = \begin{cases} (t-\xi)^\alpha & \text{if } t > \xi \\ 0 & \text{if } t \le \xi \end{cases},$$

$$\begin{array}{l} u(t) = \sum_{l=0}^{\infty} u_l (t-\xi)^l \quad \text{uniformly on } [0,1] \\ \text{with} \quad u_0 = u_1 = \cdots = u_{L-1} = 0, \ u_L \neq 0, \end{array} \tag{1.2}$$

and $\alpha > 1/2$ is a real parameter. We will need the following convention:

$$(t-\xi)_+^0 = \begin{cases} 1 & \text{if } t > \xi \\ 0 & \text{if } t \leq \xi \end{cases}$$

To introduce the p-version of the FEM we define the finite element space

$$V_p(I) := \{\phi \in \mathcal{P}_p(I);\ \phi(0) = 0,\ \phi(1) = v(1)\} \subset H^1(I).$$

Here, $\mathcal{P}_p(I)$ is the space of polynomials on I up to degree p. Below, $\mathcal{P}_p^0(I) \subset \mathcal{P}_p(I)$ denotes the subspace of polynomials that vanish at the endpoints of I.

The p-version FEM approximant $v_p \in V_p(I)$ is defined by

$$\int_I v_p'(t)\phi'(t)\, dt = \int_I f(t)\phi(t)\, dt \quad \forall \phi \in \mathcal{P}_p^0(I). \tag{1.3}$$

Gui and Babuška [7] proved that, for the case $u \equiv 1$, non-integer $\alpha > 1/2$ and singularity position $\xi = 0$, there holds

$$|v - v_p|_{H^1(I)} = \frac{\alpha \Gamma(\alpha)^2\, |\sin \pi\alpha|}{\pi \sqrt{2\alpha - 1}} \frac{1}{p^{2\alpha-1}} \left(1 + O\left(\frac{1}{p}\right)\right) \quad (p \to \infty)\,.$$

Here, $H^1(I)$ denotes the usual Sobolev space and $|v|_{H^1(I)} = \|v'\|_{L_2(I)}$. We study Legendre expansions of more general singularities (1.1). As a particular case, and in order to show the applicability of our expansions, we analyse the p-version of the FEM for slowly oscillating singularities of the form

$$v(t) = t_+^{\alpha}(A \cos \kappa t + B \sin \kappa t), \quad t \in (0,1), \tag{1.4}$$

i.e. there holds (1.1) with

$$\xi = 0, \quad u(t) = A \cos \kappa t + B \sin \kappa t.$$

Our main result is the following.

Theorem 1.1 *For the model problem with solution (1.4), non-integer $\alpha > \frac{1}{2}$, $\kappa > 0$, and A, $B \in \mathbf{R}$ not both vanishing there holds*

$$|v - v_p|_{H^1(I)} = |K(A, B, \kappa, \alpha)|\, p^{-\beta} \left(1 + O\left(\frac{1}{p}\right)\right) \quad (p \to \infty)$$

where

$$K(A, B, \kappa, \alpha) = \begin{cases} A\, \alpha\, \beta^{-1/2}\, \dfrac{\sin \pi\alpha}{\pi} \, [\Gamma(\alpha)]^2 & \text{if } A \neq 0 \\ B\, \kappa\, \beta^{-1/2}\, \dfrac{\sin \pi\alpha}{\pi} (\alpha+1)\, [\Gamma(\alpha+1)]^2 & \text{if } A = 0,\ B \neq 0 \end{cases},$$

$$\beta = \beta(A, B, \alpha) = \begin{cases} 2\alpha - 1 & \textit{if } A \neq 0 \\ 2\alpha + 1 & \textit{if } A = 0, \; B \neq 0 \end{cases}.$$

In case that α is an integer there holds for any $y \geq 0$

$$|v - v_p|_{H^1(I)} = O\left(\frac{1}{p^y}\right) \quad (p \to \infty).$$

Remark 1.1 *The rate of convergence β depends on the particular example, i.e. the choice of A and B. This is of course expected since the sine function has a linear root at $t = 0$ whereas the cosine function does not. Apart from this, the theorem describes the dependence of the constant coefficient in the asymptotic region on the frequency κ, see also the numerical results in Section 5.*

The rest of the paper is as follows. In Section 2 we study the Legendre expansion of blended singularities. The particular case of slowly oscillating singularities is analysed in Section 3. A detailed proof of the main result is given in Section 4. Numerical results that confirm the theoretical estimates are given in Section 5. Some technical lemmas are collected as an appendix.

2 Legendre expansion of blended singularities

We study the expansion into Legendre polynomials of blended singularities of the type

$$w(t) = (t - \xi)_+^\alpha u(t) \tag{2.1}$$

where

$$u(t) = \sum_{l=L}^{\infty} u_l (t - \xi)^l \qquad \text{uniformly on } [-1, 1], \qquad u_L \neq 0.$$

Throughout this section we assume that $\alpha > -\frac{1}{2}$. Let

$$a_n(u, \alpha, \xi) := \frac{2n+1}{2} \int_{-1}^{1} (t - \xi)_+^\alpha u(t) P_n(t) \, dt$$

and

$$A_n(\alpha, \xi) := \frac{2n+1}{2} \int_{-1}^{1} (t - \xi)_+^\alpha P_n(t) \, dt$$

be the nth Legendre coefficients of $(\cdot - \xi)_+^\alpha u(\cdot)$ and $(\cdot - \xi)_+^\alpha$, respectively. The coefficients A_n have been analysed by Gui and Babuška [7], and we will use those results to study the coefficients a_n. An immediate result is the following lemma.

Lemma 2.1 *There holds*

$$a_n(u(t), \alpha, \xi) = \sum_{l=L}^{\infty} u_l A_n(\alpha + l, \xi).$$

Proof. Making use of the uniform convergence of the power series of u on $[-1,1]$ we find

$$\begin{aligned} a_n(u(t),\alpha,\xi) &= \frac{2n+1}{2}\int_{-1}^{1}\sum_{l=L}^{\infty} u_l (t-\xi)_+^{\alpha+l} P_n(t)\,dt \\ &= \sum_{l=L}^{\infty} u_l \frac{2n+1}{2}\int_{-1}^{1}(t-\xi)_+^{\alpha+l}P_n(t)\,dt = \sum_{l=L}^{\infty} u_l A_n(\alpha+l,\xi). \end{aligned}$$

□

Let us recall relations for A_n from [7]. The next proposition gives exact expressions for the Legendre coefficients A_n and in Proposition 2.2 below their asymptotic behaviour in n is presented.

Proposition 2.1 [7] Thms 3/4
(i) For $\xi=-1$ there holds

$$\begin{aligned} A_0(\alpha,-1) &= \frac{1}{\alpha+1}2^{\alpha}, \\ A_n(\alpha,-1) &= \frac{\alpha\,(\alpha-1)\ldots(\alpha-n+1)}{(\alpha+1)\,(\alpha+2)\ldots(\alpha+n+1)}(2n+1)2^{\alpha} \qquad (n\geq 1). \end{aligned}$$

For non-integer α this writes like

$$A_n(\alpha,-1) = (-1)^{n-1}\frac{\left[\Gamma\left(1+\alpha\right)\right]^2\sin\pi\alpha}{\pi}\frac{\Gamma\left(n-\alpha\right)(2n+1)2^{\alpha}}{\Gamma\left(\alpha+n+2\right)}.$$

(ii) For $-1<\xi<1$ there holds

$$A_n(\alpha,\xi) = \frac{2n+1}{2}\frac{\Gamma\left(1+\alpha\right)(1-\xi)^{\alpha+1}\Gamma\left(n+1\right)}{\Gamma\left(\alpha+n+2\right)}P_n^{(\alpha+1,-\alpha-1)}\left(\xi\right)$$

where $P_n^{(\alpha+1,-\alpha-1)}$ is the Jacobi polynomial

$$P_n^{(\mu,\nu)}\left(t\right) = \frac{(-1)^n}{2^n n!(1-t)^{\mu}(1+t)^{\nu}}\frac{d^n}{dt^n}\left[(1-t)^{n+\mu}(1+t)^{n+\nu}\right]$$

with $\mu=\alpha+1$, $\nu=-\alpha-1$.
(iii) For $\xi<-1$ there holds

$$A_n(\alpha,\xi) = \frac{\alpha\,(\alpha-1)\ldots(\alpha-n+1)}{(2n-1)!!}(2r)^{n-\alpha}\Phi_{n,\alpha}\left(r^2\right)$$

where

$$r = \frac{1}{-\xi+\sqrt{\xi^2-1}} \quad (0<r<1)$$

and $\Phi_{n,\alpha}$ is the Gauss hypergeometric function

$$\Phi_{n,\alpha}(x) = 1 + \sum_{k=1}^{\infty} \left[\prod_{j=1}^{k} \left(\frac{n-\alpha+j-1}{n+j+\frac{1}{2}} \right) \right] (-1)^k \binom{\alpha+\frac{1}{2}}{k} x^k.$$

For non-integer α this renders like

$$A_n(\alpha,\xi) = (-1)^{n-1} \frac{\Gamma(1+\alpha)\sin\pi\alpha}{2^\alpha\sqrt{\pi}} \frac{\Gamma(n-\alpha)}{\Gamma\left(n+\frac{1}{2}\right)} r^{n-\alpha} \Phi_{n,\alpha}\left(r^2\right).$$

For properties of the Jacobi polynomials and the hypergeometric function see, e.g., [19, 17].

Proposition 2.2 [7] Thm 6
(i) For $\xi=-1$ and non-integer α there holds

$$A_n(\alpha,-1) = (-1)^{n-1} \frac{C_0(\alpha)}{n^{2\alpha+1}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

with

$$C_0(\alpha) = \frac{2^{\alpha+1}\left[\Gamma(1+\alpha)\right]^2 \sin\pi\alpha}{\pi}$$

(ii) For $-1<\xi<1$ there holds

$$A_n(\alpha,\xi) = \sqrt{\frac{2}{\pi}}\Gamma(1+\alpha)\left(\frac{\sin\theta}{n}\right)^{\alpha+\frac{1}{2}} \left\{\cos\left[(n+\frac{1}{2})\theta - (\alpha+\frac{3}{2})\frac{\pi}{2}\right] + O\left(\frac{1}{n}\right)\right\}$$

with $\theta=\arccos\xi$, and $O\left(\frac{1}{n}\right)$ holding uniformly for $|\xi|\le 1-\epsilon, \epsilon>0$.
(iii) For $\xi<-1$ and non-integer α there holds

$$A_n(\alpha,\xi) = (-1)^{n-1} \frac{C_1(\alpha)}{n^{\alpha+\frac{1}{2}}} \left[\left(1-r^2\right)^{\alpha+\frac{1}{2}} + O\left(\frac{1}{n^\sigma}\right)\right] \cdot r^{n-\alpha}$$

where

$$C_1(\alpha) = \frac{\Gamma(1+\alpha)\sin\pi\alpha}{2^\alpha\sqrt{\pi}}, \quad r = \frac{1}{|\xi|+\sqrt{\xi^2-1}}, \quad \sigma = \frac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}} > 0$$

and $O\left(\frac{1}{n^\sigma}\right)$ holds uniformly with respect to $r\in(0,1)$.

The following three theorems state explicit asymptotic expressions for the Legendre coefficients of blended singularities (2.1) (under appropriate conditions upon u via its Taylor coefficients), for the cases $\xi=-1$, $\xi<-1$ and $\xi\in(-1,1)$, respectively.

Theorem 2.1 Thms 10/11

(i) For non-integer α there holds

$$a_n(u(t),\alpha,-1) = (-1)^{n-1}(-2)^L u_L \left[\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\right]^2 \frac{C_0(\alpha)}{n^{2\alpha+2L+1}}\left(1+O\left(\frac{1}{n}\right)\right)$$

for $n\to\infty$ if

$$\frac{a_n(u(t),\alpha,-1)}{A_n(\alpha,-1)} = \sum_{l=L}^{\infty} u_l(-2)^l \left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}$$

converges uniformly in n. Here,

$$C_0(\alpha) = \frac{2^{\alpha+1}\left[\Gamma\left(1+\alpha\right)\right]^2 \sin\pi\alpha}{\pi}.$$

(ii) For integer α there holds

$$a_n(u(t),\alpha,-1) = O\left(\frac{1}{n^y}\right)$$

for all y such that $\sum_{l=1}^{\infty}|u_l|\,2^l l^y$ converges.

Proof. First we consider the case that α is non-integer. We prove that

$$a_n(u(t),\alpha,-1) = \frac{A_n(\alpha,-1)}{n^{2L}}\left((-2)^L u_L\left[\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\right]^2 + O\left(\frac{1}{n}\right)\right). \tag{2.2}$$

The assertion then follows by using Proposition 2.2(i). To prove (2.2) we use Lemma 2.1 and Proposition 2.1(i) to calculate

$$\begin{aligned}\frac{a_n(u(t),\alpha,-1)}{A_n(\alpha,-1)} &= \sum_{l=L}^{\infty} u_l \frac{A_n(\alpha+l,-1)}{A_n(\alpha,-1)}\\ &= \sum_{l=L}^{\infty} u_l(-2)^l\left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}.\end{aligned}$$

This gives

$$\begin{aligned}&\frac{a_n(u(t),\alpha,-1)}{A_n(\alpha,-1)}n^{2L} - (-2)^L u_L\left[\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\right]^2\\ &= \sum_{l=L+1}^{\infty} u_l(-2)^l\left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}n^{2L} + O(\frac{1}{n})\end{aligned} \tag{2.3}$$

where we used that

$$\frac{\Gamma(n-\alpha-L)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+L)}n^{2L} = 1+O\left(\frac{1}{n}\right) \quad \text{for } n\to\infty$$

by Stirling's formula. By assumption the last series converges uniformly such that we can consider the limit $n \to \infty$ term-wise. Using again Stirling's formula we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{l=L+1}^{\infty} u_l(-2)^l \left[\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)} \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)} n^{2L+1} \\
&= \sum_{l=L+1}^{\infty} u_l(-2)^l \left[\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\right]^2 \lim_{n\to\infty} \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)} \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)} n^{2L+1} = 0
\end{aligned}$$

which, together with (2.3), proves (2.2). This yields part (i) of the theorem.

To prove part (ii) of the theorem we first bound the coefficients A_n. By Proposition 2.1 we have that

$$A_n(\alpha,-1) = \frac{\alpha\,(\alpha-1)\ldots(\alpha-n+1)}{(\alpha+1)\,(\alpha+2)\ldots(\alpha+n+1)}(2n+1)2^\alpha \qquad (n \geq 1).$$

Since $A_n(\alpha,-1) = 0$ for $n \geq \alpha+1$ or $\alpha = 0$ it suffices to consider the case $1 \leq n \leq \alpha$, $\alpha \geq 1$ and $y \geq 0$. We obtain

$$\begin{aligned}
\frac{\alpha\,(\alpha-1)\ldots(\alpha-n+1)}{(\alpha+1)\,(\alpha+2)\ldots(\alpha+n+1)} n^y &= \prod_{j=0}^{n-1} \frac{\alpha-j}{\alpha+j+1} \frac{n^y}{\alpha+n+1} \leq \prod_{j=0}^{n-1} 1 \cdot \frac{n^y}{\alpha+n+1} \\
&= \frac{n^y}{\alpha+n+1} \leq \frac{1}{\alpha+1}\alpha^y
\end{aligned}$$

such that, for arbitrary real y,

$$\begin{aligned}
|A_n(\alpha,-1)|\,n^y &= \left|\frac{\alpha\,(\alpha-1)\ldots(\alpha-n+1)}{(\alpha+1)\,(\alpha+2)\ldots(\alpha+n+1)}\right| (2n+1)2^\alpha n^y \\
&= \left|\frac{\alpha\,(\alpha-1)\ldots(\alpha-n+1)}{(\alpha+1)\,(\alpha+2)\ldots(\alpha+n+1)}\right| n^{y+1} \left|\frac{2n+1}{n}\right| 2^\alpha.
\end{aligned}$$

This yields

$$|A_n(\alpha,-1)|\,n^y \leq 3\frac{2^\alpha}{\alpha+1}\alpha^{y+1} \qquad (1 \leq n \leq \alpha).$$

Together with Lemma 2.1 this gives, after some calculations,

$$|a_n(u(t),\alpha,-1)n^y| \leq |u_0|\,3\frac{2^\alpha}{\alpha+1}\alpha^{y+1} + 3\,(1+\alpha)^{y+1}\,2^\alpha \sum_{l=1}^{\infty} |u_l|\,2^l l^y$$

if the last series converges, which is an assumption. This finishes the proof of the theorem. □

Theorem 2.2 Thms 17/18 *($\xi < -1$)*

(i) For non-integer α there holds

$$a_n(u(t),\alpha,\xi) = \frac{(-1)^{n-1+L} C_1(\alpha) r^{n-\alpha} u_L}{n^{\alpha+L+\frac{1}{2}}\,(2r)^L} \frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)} \left(1-r^2\right)^{\alpha+L+\frac{1}{2}} \left(1+O\left(\frac{1}{n^\sigma}\right)\right)$$

for $n \to \infty$ uniformly with respect to $r \in (0,1)$ if

$$\Phi_{n,\alpha}\left(r^2\right)\frac{a_n(u(t),\alpha,\xi)}{A_n(\alpha,\xi)} = \sum_{l=L}^{\infty} u_l \left(\frac{-1}{2r}\right)^l \frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\Phi_{n,\alpha+l}\left(r^2\right)$$

converges uniformly in n. Here

$$C_1(\alpha) = \frac{\Gamma(1+\alpha)\sin\pi\alpha}{2^\alpha\sqrt{\pi}}, \quad r = \frac{1}{|\xi| + \sqrt{\xi^2-1}} \quad \text{and} \quad \sigma = \frac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}}.$$

(ii) For integer α and all $y \in \mathbf{R}$ there holds

$$a_n(u(t),\alpha,\xi) = O\left(\frac{1}{n^y}\right) \quad (n \to \infty) \qquad \text{if}$$

$$a_n(u(t),\alpha,\xi) = \sum_{l=L}^{\infty} u_l \frac{(\alpha+l)(\alpha+l-1)\dots(\alpha+l-n+1)}{(2n-1)!!}(2r)^{n-\alpha-l}\Phi_{n,\alpha+l}\left(r^2\right)$$

converges uniformly in n.

Proof. The structure of the proof is analogous to the one of Theorem 2.1. Using Lemma 2.1 and Proposition 2.1(iii) one obtains

$$\frac{a_n(u(t),\alpha,\xi)}{A_n(\alpha,\xi)} = \sum_{l=L}^{\infty} u_l \left(\frac{-1}{2r}\right)^l \frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Phi_{n,\alpha+l}\left(r^2\right)}{\Phi_{n,\alpha}\left(r^2\right)}.$$

Therefore,

$$\begin{aligned}
&\left\{\frac{a_n(u(t),\alpha,\xi)}{A_n(\alpha,\xi)}n^L - u_L\left(\frac{-1}{2r}\right)^L\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\left(1-r^2\right)^L\right\} n^\sigma \\
&= \; u_L\left(\frac{-1}{2r}\right)^L\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\left\{\frac{\Gamma(n-\alpha-L)}{\Gamma(n-\alpha)}n^L\frac{\Phi_{n,\alpha+L}\left(r^2\right)}{\Phi_{n,\alpha}\left(r^2\right)} - \left(1-r^2\right)^L\right\} n^\sigma \\
&\quad + \sum_{l=L+1}^{\infty} u_l\left(\frac{-1}{2r}\right)^l\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}n^{L+\sigma}\frac{\Phi_{n,\alpha+l}\left(r^2\right)}{\Phi_{n,\alpha}\left(r^2\right)}.
\end{aligned} \tag{2.4}$$

There holds [7, p. 586]

$$\Phi_{n,\alpha}\left(r^2\right) = \left(1-r^2\right)^{\alpha+\frac{1}{2}} + O\left(\frac{1}{n^\sigma}\right) \quad (n\to\infty) \quad \text{with } \sigma = \frac{\alpha+\frac{1}{2}}{\alpha+\frac{3}{2}}$$

such that

$$\frac{\Phi_{n,\alpha+l}\left(r^2\right)}{\Phi_{n,\alpha}\left(r^2\right)} = \left(1-r^2\right)^l\left(1+O\left(\frac{1}{n^\sigma}\right)\right).$$

This yields the boundedness

$$\left\{\frac{\Gamma(n-\alpha-L)}{\Gamma(n-\alpha)}n^L\frac{\Phi_{n,\alpha+L}\left(r^2\right)}{\Phi_{n,\alpha}\left(r^2\right)}-\left(1-r^2\right)^L\right\}n^\sigma=O\left(1\right)\quad(n\to\infty).$$

Also, since the following series converges uniformly in n by assumption, we find that

$$\lim_{n\to\infty}\sum_{l=L+1}^{\infty}u_l\left(\frac{-1}{2r}\right)^l\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}n^{L+\sigma}\frac{\Phi_{n,\alpha+l}\left(r^2\right)}{\Phi_{n,\alpha}\left(r^2\right)}=0$$

where we used that $\sigma<1$. Referring back to (2.4) we conclude that

$$a_n(u(t),\alpha,\xi)=\frac{A_n(\alpha,\xi)}{n^L}\left(u_L\left(\frac{-1}{2r}\right)^L\frac{\Gamma(\alpha+1+L)}{\Gamma(\alpha+1)}\left(1-r^2\right)^L+O\left(\frac{1}{n^\sigma}\right)\right).$$

The assertion for non-integer α then follows by using Proposition 2.2(iii). For integer α we proceed as follows. Proposition 2.1(iii) says that

$$A_n(\alpha+l,\xi)=\frac{(\alpha+l)\,(\alpha+l-1)\ldots(\alpha+l-n+1)}{(2n-1)!!}(2r)^{n-\alpha-l}\Phi_{n,\alpha+l}\left(r^2\right).$$

Therefore, for integer α, $A_n(\alpha+l,\xi)$ vanishes for $n\geq\alpha+l+1$. In particular, $A_n(\alpha+l,\xi)=O\left(\frac{1}{n^y}\right)$ for any real y and integer l. By Lemma 2.1 this means that

$$\lim_{n\to\infty}n^y a_n(u(t),\alpha,\xi)=\lim_{n\to\infty}n^y\sum_{l=L}^{\infty}u_l A_n(\alpha+l,\xi)=0$$

if

$$\begin{aligned}a_n(u(t),\alpha,\xi)&=\sum_{l=L}^{\infty}u_l A_n(\alpha+l,\xi)\\&=\sum_{l=L}^{\infty}u_l\frac{(\alpha+l)\,(\alpha+l-1)\ldots(\alpha+l-n+1)}{(2n-1)!!}(2r)^{n-\alpha-l}\Phi_{n,\alpha+l}\left(r^2\right)\end{aligned}$$

converges uniformly in n. This is the case by assumption and the proof of the theorem is finished. □

Theorem 2.3 Thm 20 *($\xi\in(-1,1)$)*
For $\alpha>-\frac{1}{2}$ there holds

$$\begin{aligned}a_n(u(t),\alpha,\xi)&=u_L\sqrt{\frac{2}{\pi}}\Gamma\left(1+\alpha+L\right)\left(\frac{\sin\theta}{n}\right)^{\alpha+L+\frac{1}{2}}\\&\quad\cdot\left\{\cos\left[(n+\frac{1}{2})\theta-(\alpha+L+\frac{3}{2})\frac{\pi}{2}\right]+O\left(\frac{1}{n}\right)\right\}+O\left(\frac{1}{n^{\alpha+L+\frac{3}{2}}}\right)\end{aligned}$$

for $n \to \infty$ uniformly in $|\xi| \le 1-\epsilon$ ($\epsilon > 0$, $\theta := \arccos\xi$) if

$$a_n(u(t),\alpha,\xi) = \frac{2n+1}{2}\Gamma(n+1)\sum_{l=0}^{\infty} u_l \frac{\Gamma(1+\alpha+l)(1-\xi)^{\alpha+l+1}}{\Gamma(\alpha+l+n+2)} P_n^{(\alpha+l+1,-\alpha-l-1)}(\xi)$$

converges uniformly in n.

Proof. Due to Lemma 2.1 and Proposition 2.1(ii) there holds

$$\begin{aligned}\Big(a_n(u(t),\alpha,\xi) - u_L A_n(\alpha+L,\xi)\Big)n^{\alpha+L+\frac{3}{2}} &= \sum_{l=L+1}^{\infty} u_l A_n(\alpha+l,\xi)n^{\alpha+L+\frac{3}{2}} \\ = \frac{2n+1}{2}\Gamma(n+1)\sum_{l=L+1}^{\infty} &u_l \frac{\Gamma(1+\alpha+l)(1-\xi)^{\alpha+l+1}}{\Gamma(\alpha+l+n+2)} P_n^{(\alpha+l+1,-\alpha-l-1)}(\xi)\, n^{\alpha+L+\frac{3}{2}}.\end{aligned}$$

By the uniform convergence of the last series, and making use of the asymptotic behaviour of $A_n(\alpha+l,\xi)$ by Proposition 2.2(ii), one finds that

$$\lim_{n\to\infty}\Big(a_n(u(t),\alpha,\xi) - u_L A_n(\alpha+L,\xi)\Big)n^{\alpha+L+\frac{3}{2}} = \lim_{n\to\infty} u_{L+1}A_n(\alpha+L+1,\xi)n^{\alpha+L+\frac{3}{2}} = C < \infty.$$

The assertion then follows by again using Proposition 2.2(ii). □

3 Legendre expansion of slowly oscillating singularities

In this section we analyse the asymptotic behaviour of Legendre coefficients of slowly oscillating singularities with singularity position at $\xi = -1$. This is the particularly interesting case where, in the finite element method, the singularity is met by a mesh point. The p-version then converges twice as fast as the h-version with quasi-uniform meshes (if the singularity is strong enough). Throughout this section we assume that

$$\kappa \ge 0.$$

Theorem 3.1 Thm 21/23

(i) For $u(t) = \cos\kappa(1+t)$ and non-integer α there holds

$$a_n(u(t),\alpha,-1) = (-1)^{n-1}\frac{C_0(\alpha)}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right) \quad (n\to\infty)$$

with

$$C_0(\alpha) = \frac{2^{\alpha+1}\left[\Gamma(1+\alpha)\right]^2\sin\pi\alpha}{\pi}.$$

(ii) For $u(t) = \cos\kappa(1+t)$, integer α and any $y \ge 0$ there holds

$$a_n(u(t),\alpha,-1) = O\left(\frac{1}{n^y}\right) \quad (n\to\infty).$$

Proof. Referring to the representation (1.2) of u we have $u_l = 0$ for odd l, $u_l = (-1)^{l/2}\kappa^l/l!$ for even l and, in particular, $L = 0$ and $u_L = 1$. To prove (i) we note that by Theorem 2.1(i) there holds

$$a_n(u(t),\alpha,-1) = (-1)^{n-1}\frac{C_0(\alpha)}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right) \quad (n\to\infty)$$

if

$$\frac{a_n(u(t),\alpha,-1)}{A_n(\alpha,-1)} = \sum_{l=0}^{\infty} u_l(-2)^l\left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}$$

converges uniformly in n. Since (using Lemma A.4)

$$\begin{aligned}
&\sum_{l=0}^{\infty}\left|u_l(-2)^l\left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}\right| \\
\leq\ & |u_0| + \frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}\frac{\alpha+1}{\Gamma(\alpha+1)}\max(1,\Gamma(\alpha+2))\sum_{l=1}^{\infty}|u_l|\,4^l l^{\alpha+2}
\end{aligned}$$

this is the case if the latter series converges. We finish the proof by showing that

$$\sum_{l=0}^{\infty}|u_l|\,4^l l^{\alpha+2} \leq \frac{(\lfloor\alpha\rfloor+3)^{\lfloor\alpha\rfloor+3}}{(\lfloor\alpha\rfloor+3)!}(4\kappa)^{\lfloor\alpha\rfloor+3}\cosh(4\kappa) + 4^{\lfloor\alpha\rfloor+3}\left(\lfloor\alpha\rfloor+3\right)^{\alpha+2}\cosh\kappa. \tag{3.1}$$

Using Lemma A.5 with $N = \lfloor\alpha\rfloor+3$ we obtain

$$\begin{aligned}
\sum_{l=\lfloor\alpha\rfloor+3}^{\infty}|u_l|\,4^l l^{\alpha+2} &\leq \sum_{l=\lfloor\alpha\rfloor+3}^{\infty}|u_l|\,4^l l^{\lfloor\alpha\rfloor+3} \leq \frac{(\lfloor\alpha\rfloor+3)^{\lfloor\alpha\rfloor+3}}{(\lfloor\alpha\rfloor+3)!}\sum_{l=\lfloor\alpha\rfloor+3}^{\infty}|u_l|\,4^l\frac{l!}{(l-\lfloor\alpha\rfloor-3)!} \\
&= \frac{(\lfloor\alpha\rfloor+3)^{\lfloor\alpha\rfloor+3}}{(\lfloor\alpha\rfloor+3)!}(4\kappa)^{\lfloor\alpha\rfloor+3}\sum_{\substack{l=\lfloor\alpha\rfloor+3\\ l\text{ even}}}^{\infty}\frac{(4\kappa)^{l-\lfloor\alpha\rfloor-3}}{(l-\lfloor\alpha\rfloor-3)!} \\
&\leq \frac{(\lfloor\alpha\rfloor+3)^{\lfloor\alpha\rfloor+3}}{(\lfloor\alpha\rfloor+3)!}(4\kappa)^{\lfloor\alpha\rfloor+3}\cosh(4\kappa).
\end{aligned}$$

The last inequality is due to

$$\sum_{\substack{l=\lfloor\alpha\rfloor+3\\ l\text{ even}}}^{\infty}\frac{(4\kappa)^{l-\lfloor\alpha\rfloor-3}}{(l-\lfloor\alpha\rfloor-3)!} = \left\{\begin{array}{ll}\cosh(2\kappa), & \lfloor\alpha\rfloor \text{ odd}\\ \sinh(2\kappa), & \lfloor\alpha\rfloor \text{ even}\end{array}\right\} \leq \cosh(2\kappa).$$

Noting that

$$\sum_{l=0}^{\lfloor\alpha\rfloor+2}|u_l|\,4^l l^{\alpha+2} \leq 4^{\lfloor\alpha\rfloor+2}\left(\lfloor\alpha\rfloor+2\right)^{\alpha+2}\sum_{l=0}^{\lfloor\alpha\rfloor+2}|u_l| \leq 4^{\lfloor\alpha\rfloor+2}\left(\lfloor\alpha\rfloor+2\right)^{\alpha+2}\cosh\kappa$$

this yields (3.1) and part (i) of the theorem is proved. To prove part (ii) we need to show that $\sum_{l=0}^{\infty}|u_l|\,2^l l^y$ converges for any $y \geq 0$, cf. Theorem 2.1(ii). But this is true since, similarly as before, one can estimate

$$\sum_{l=0}^{\infty}|u_l|\,2^l l^y \leq \frac{(\lfloor y\rfloor+1)^{\lfloor y\rfloor+1}}{(\lfloor y\rfloor+1)!}(2\kappa)^{\lfloor y\rfloor+1}\cosh(2\kappa)+\lfloor y\rfloor^y\,2^{\lfloor y\rfloor}\cosh\kappa.$$

□

Theorem 3.2 Thms 25/27

(i) For $u(t)=\sin\kappa(1+t)$ and non-integer α there holds

$$a_n(u(t),\alpha,-1)=2\kappa(\alpha+1)^2C_0(\alpha)\frac{(-1)^n}{n^{2\alpha+3}}\left(1+O\left(\frac{1}{n}\right)\right)\qquad(n\to\infty)$$

with

$$C_0(\alpha)=\frac{2^{\alpha+1}\left[\Gamma\left(1+\alpha\right)\right]^2\sin\pi\alpha}{\pi}.$$

(ii) For $u(t)=\sin\kappa(1+t)$, integer α and any $y\geq 0$ there holds

$$a_n(u(t),\alpha,-1)=O\left(\frac{1}{n^y}\right)\qquad(n\to\infty).$$

Proof. The proof is analogous to the one of Theorem 3.1. We are in the situation of (1.2) with $u_l=0$ for even l, $u_l=(-1)^{\frac{l-1}{2}}\kappa^l/l!$ for odd l and, in particular, $L=1$ and $u_L=\kappa$. To prove (i) we note that by Theorem 2.1(i) there holds

$$a_n(u(t),\alpha,-1)=(-1)^{n-1}(-2\kappa)\left[\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}\right]^2\frac{C_0(\alpha)}{n^{2\alpha+2+1}}\left(1+O\left(\frac{1}{n}\right)\right)\qquad(n\to\infty)$$

if

$$\frac{a_n(u(t),\alpha,-1)}{A_n(\alpha,-1)}=\sum_{l=0}^{\infty}u_l(-2)^l\left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}$$

converges uniformly in n. By Lemma A.4 we find

$$\begin{aligned}&\sum_{l=0}^{\infty}\left|u_l(-2)^l\left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}\right|\\ \leq\;&\frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}\frac{\alpha+1}{\Gamma(\alpha+1)}\max(1,\Gamma(\alpha+2))\sum_{l=1}^{\infty}|u_l|\,4^l l^{\alpha+2}\end{aligned}$$

if the latter series converges. In that case the uniform convergence needed above is shown. Similarly as for the cosine function one can estimate the latter series like

$$\sum_{l=0}^{\infty}|u_l|\,4^l l^{\alpha+2}\leq\frac{(\lfloor\alpha\rfloor+3)^{\lfloor\alpha\rfloor+3}}{(\lfloor\alpha\rfloor+3)!}(4\kappa)^{\lfloor\alpha\rfloor+3}\cosh(4\kappa)+4^{\lfloor\alpha\rfloor+3}\left(\lfloor\alpha\rfloor+3\right)^{\alpha+2}\sinh\kappa$$

and part (i) of the theorem is proved. To prove part (ii) we have to show that $\sum_{l=0}^{\infty} |u_l| \, 2^l l^y$ converges for any $y \geq 0$, cf. Theorem 2.1(ii). This is true since one can estimate

$$\sum_{l=0}^{\infty} |u_l| \, 2^l l^y \leq \frac{(\lfloor y \rfloor + 1)^{\lfloor y \rfloor + 1}}{(\lfloor y \rfloor + 1)!} (2\kappa)^{\lfloor y \rfloor + 1} \cosh(2\kappa) + \lfloor y \rfloor^y \, 2^{\lfloor y \rfloor} \sinh \kappa.$$

This finishes the proof of the theorem. □

4 Optimal a priori error estimate for the p-version of the FEM for slowly oscillating singularities

We now come back to the error analysis of the FEM for oscillating singularities. At the end of this section a proof of Theorem 1.1 is given.

For the finite element solution

$$v_p \in V_p(I) := \{\phi \in \mathcal{P}_p(I); \; \phi(0) = 0, \, \phi(1) = v(1)\}$$

defined by (1.3) one finds that there holds

$$|v - v_p|_{H^1(I)} = \min_{\phi \in V_p(I)} |v - \phi|_{H^1(I)} = \min_{\phi \in \mathcal{P}_{p-1}(I)} \|v' - \phi\|_{L_2(I)}.$$

Introducing the linear transformation $T: \; t \mapsto (t+1)/2$ and noting that

$$\|v' - \phi\|_{L_2(I)} = \frac{1}{\sqrt{2}} \|v'(T(\cdot)) - \phi(T(\cdot))\|_{L_2(-1,1)}$$

we have the following characterisation of the finite element error:

$$|v - v_p|_{H^1(I)} = \frac{1}{\sqrt{2}} \min_{\phi \in \mathcal{P}_{p-1}(-1,1)} \|v'(T(\cdot)) - \phi(\cdot)\|_{L_2(-1,1)}.$$

Therefore, expanding $v'(T(\cdot))$ into Legendre polynomials

$$v'(T(\cdot)) = \sum_{n=0}^{\infty} b_n l_n,$$

we find

$$|v - v_p|_{H^1(I)} = \frac{1}{\sqrt{2}} \left(\sum_{n=p}^{\infty} b_n^2 \frac{2}{2n+1} \right)^{1/2}. \tag{4.1}$$

This term can be estimated by using the knowledge about the Legendre coefficients of oscillating singularities from Section 3. We have

$$\begin{aligned} v'(T(t)) &= \alpha \left(\frac{1+t}{2} \right)^{\alpha - 1} \left(A \cos \kappa \frac{1+t}{2} + B \sin \kappa \frac{1+t}{2} \right) \\ &\quad + \left(\frac{1+t}{2} \right)^{\alpha} \left(-A\kappa \sin \kappa \frac{1+t}{2} + B\kappa \cos \kappa \frac{1+t}{2} \right) \end{aligned}$$

such that, by the linearity of the Legendre expansion,

$$\begin{aligned} b_n &= \alpha\left(\frac{1}{2}\right)^{\alpha-1}\left(A\,a_n(\cos\kappa\frac{1+t}{2},\alpha-1,-1)+B\,a_n(\sin\kappa\frac{1+t}{2},\alpha-1,-1)\right) \\ &\quad +\kappa\left(\frac{1}{2}\right)^{\alpha}\left(B\,a_n(\cos\kappa\frac{1+t}{2},\alpha,-1)-A\,a_n(\sin\kappa\frac{1+t}{2},\alpha,-1)\right). \end{aligned} \tag{4.2}$$

We note that the first term on the right-hand side is not present in the case $\alpha = 0$.

Lemma 4.1 *For non-integer α there holds*

$$b_n = M(A,B,\kappa,\alpha)\,n^{-\beta}\left(1+O\left(\frac{1}{n}\right)\right) \qquad (n\to\infty)$$

where

$$M(A,B,\kappa,\alpha) = \begin{cases} 2A\alpha\frac{\sin\pi\alpha}{\pi}(-1)^n\,[\Gamma(\alpha)]^2 & \text{if } A\neq 0 \\ 2B\kappa\frac{\sin\pi\alpha}{\pi}(-1)^{n-1}(\alpha+1)\,[\Gamma(\alpha+1)]^2 & \text{if } A=0,\ B\neq 0 \end{cases}$$

and

$$\beta = \beta(A,B,\alpha) = \begin{cases} 2\alpha-1 & \text{if } A\neq 0 \\ 2\alpha+1 & \text{if } A=0, B\neq 0 \end{cases}.$$

Proof. Applying part (i) of Theorems 3.1 and 3.2 we find that there holds

$$\begin{aligned} a_n\left(\cos\kappa\frac{1+t}{2},\alpha-1,-1\right) &= (-1)^{n-1}\frac{C_0(\alpha-1)}{n^{2\alpha-1}}\left(1+O\left(\frac{1}{n}\right)\right), \\ a_n\left(\sin\kappa\frac{1+t}{2},\alpha-1,-1\right) &= \kappa\alpha^2(-1)^n\frac{C_0(\alpha-1)}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right), \\ a_n\left(\cos\kappa\frac{1+t}{2},\alpha,-1\right) &= (-1)^{n-1}\frac{C_0(\alpha)}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right), \\ a_n\left(\sin\kappa\frac{1+t}{2},\alpha,-1\right) &= \kappa(\alpha+1)^2(-1)^n\frac{C_0(\alpha)}{n^{2\alpha+3}}\left(1+O\left(\frac{1}{n}\right)\right). \end{aligned}$$

Remember that we assume that A and B are not both vanishing. In the case that $A\neq 0$ then one obtains

$$\begin{aligned} b_n &= \alpha\left(\frac{1}{2}\right)^{\alpha-1}\left[A(-1)^{n-1}\frac{C_0(\alpha-1)}{n^{2\alpha-1}}\left(1+O\left(\frac{1}{n}\right)\right)+O\left(\frac{1}{n^{2\alpha+1}}\right)\right]+O\left(\frac{1}{n^{2\alpha+1}}\right) \\ &= A\alpha\left(\frac{1}{2}\right)^{\alpha-1}(-1)^{n-1}\frac{C_0(\alpha-1)}{n^{2\alpha-1}}\left(1+O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{4.3}$$

If $A=0$ and $B\neq 0$ then we obtain

$$\begin{aligned} b_n &= \alpha\left(\frac{1}{2}\right)^{\alpha-1}B\kappa\alpha^2(-1)^n\frac{C_0(\alpha-1)}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right) \\ &\quad +\kappa\left(\frac{1}{2}\right)^{\alpha}B(-1)^{n-1}\frac{C_0(\alpha)}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right) \\ &= B\kappa(-1)^n\left(\frac{1}{2}\right)^{\alpha}\left[2\alpha^3C_0(\alpha-1)-C_0(\alpha)\right]\frac{1}{n^{2\alpha+1}}\left(1+O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{4.4}$$

Plugging into (4.3) the term

$$C_0(\alpha - 1) = -\frac{2^\alpha \left[\Gamma(\alpha)\right]^2 \sin \pi\alpha}{\pi}$$

and noting that, in (4.4),

$$\begin{aligned} & B\kappa(-1)^n \left(\frac{1}{2}\right)^\alpha \left[2\alpha^3 C_0(\alpha - 1) - C_0(\alpha)\right] \\ = \; & 2B\kappa(-1)^{n-1}\frac{\sin \pi\alpha}{\pi}(\alpha+1)\left[\Gamma(\alpha+1)\right]^2 \end{aligned}$$

this finishes the proof in the case of non-integer α. □

Proof of Theorem 1.1.

We use the representation (4.1) of the error and apply, for non-integer α, Lemma 4.1 to conclude that there holds

$$\begin{aligned} |v - v_p|_{H^1(I)} &= \frac{1}{\sqrt{2}}\left\{\sum_{n=p}^{\infty} b_n^2 \frac{2}{2n+1}\right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\left\{\sum_{n=p}^{\infty} M(A,B,\kappa,\alpha)^2 \frac{1}{n^{2\beta}}\left(1 + O\left(\frac{1}{n}\right)\right)\frac{1}{n}\left(1 + O\left(\frac{1}{n}\right)\right)\right\}^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\left|M(A,B,\kappa,\alpha)\right|\sqrt{\frac{1}{2\beta}\frac{1}{p^\beta}}\left(1 + O\left(\frac{1}{p}\right)\right). \end{aligned}$$

For integer α we use part (ii) of Theorems 3.1 and 3.2. Then (4.2) yields $b_n = O(1/n^y)$ for any $y \geq 0$ and application of (4.1) finishes the proof of the theorem. □

5 Numerical results

We present numerical results for the model situation of Theorem 1.1. Throughout we choose $\alpha = 1.1$ and use a double-logarithmic scale for the plots. In Figure 1 we show the errors in $H^1(I)$-semi-norm for different frequencies κ for the case $A = B = 1$, i.e., the model solution is $v(t) = t^{1.1}(\cos(\kappa t) + \sin(\kappa t))$ ($t \in I = (0,1)$). As expected, the pre-asymptotic behaviour of the errors strongly depend on κ, but the asymptotic error for large polynomial degrees is independent of κ as predicted by Theorem 1.1 since $A \neq 0$. In fact, in this case the asymptotic rate of convergence is $\beta = 2\alpha - 1 = 1.2$ and all the curves approach $K(A,B,\kappa,\alpha)p^{-\beta}$ with

$$K(A,B,\kappa,\alpha) = A\alpha\beta^{-1/2}\frac{\sin(\pi\alpha)}{\pi}\Gamma(\alpha)^2 \asymp 0.0894$$

independent of κ. In contrast, Figure 2 shows the results for $A = 0$ and $B = 1$. Here, the asymptotic rate of convergence is $\beta = 2\alpha + 1 = 3.2$ with coefficient

$$K(A, B, \kappa, \alpha) = B\kappa\beta^{-1/2}\frac{\sin \pi\alpha}{\pi}(\alpha+1)\Gamma(\alpha+1)^2 \asymp 0.1265\,\kappa$$

being linearly dependent on κ. The numerical results for $\kappa \in \{1, 10, 25\}$ are given and the errors approach, as predicted, the asymptotic error terms $0.1265\,\kappa\,p^{-3.2}$.

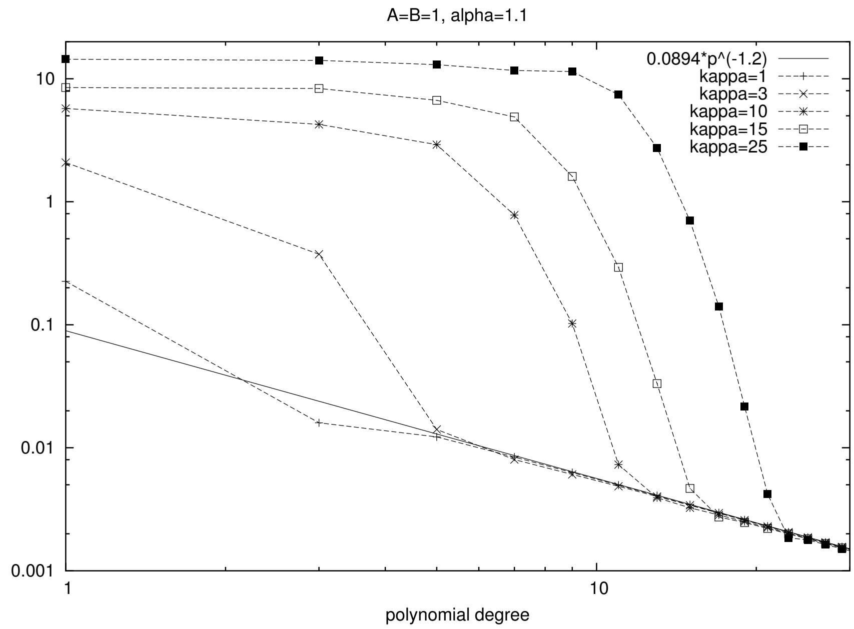

Figure 1: Approximation error for $v(t) = t^{1.1}(\cos(\kappa t) + \sin(\kappa t))$ for different frequencies κ.

To illustrate how the h-version and the hp-version with geometrically graded meshes compare with the p-version we present Figure 3. Here, we consider the same example as in the second case (Figure 2) for $\kappa = 10$. The errors in $H^1(I)$-semi-norm are plotted versus the dimension N of the finite element space. The h-version uses uniform meshes and polynomial degree $p = 1$ and converges like $O(h)$ (h denoting the mesh size). The hp-version uses a non-uniform mesh that is geometrically graded towards the singularity (determined by the grading parameter σ, see [8]) and where polynomial degrees increase linearly from element to element away from the singularity. This method converges like $c\,e^{-b\sqrt{N}}$ with unspecified positive numbers c and b. There are no results about how the errors depend on the wave number for these cases. But as expected from the behaviour of numerical dispersion [1] the p-version enters much earlier its asymptotic range. Also, the less graded hp-version (with $\sigma = 0.5$) appears to be more appropriate in the pre-asymptotic range for oscillating problems than the more graded version ($\sigma = 0.17$). Note that for plain singularities (without oscillations) the case $\sigma = 0.17$ is known to be optimal, see [8].

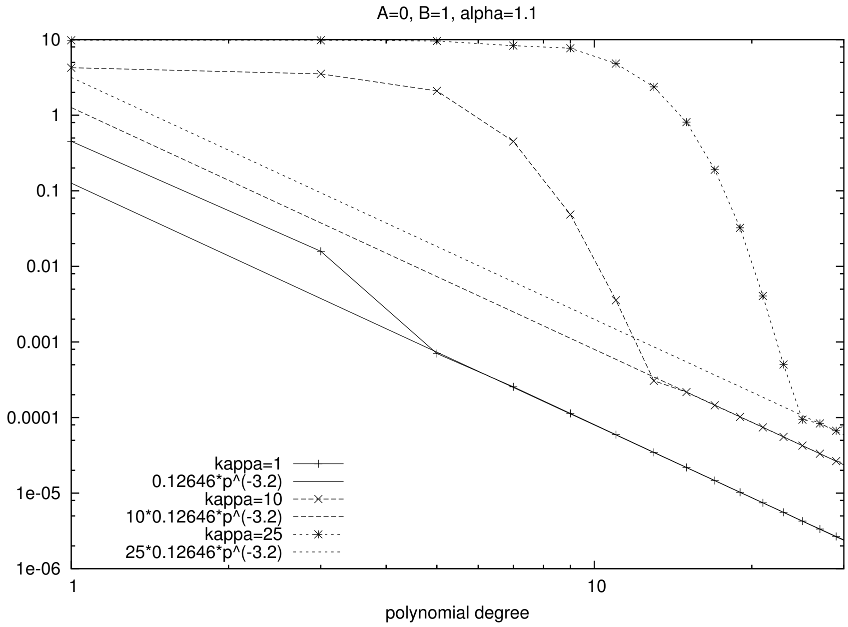

Figure 2: Approximation error for $v(t) = t^{1.1}\sin(\kappa t)$ for different frequencies κ.

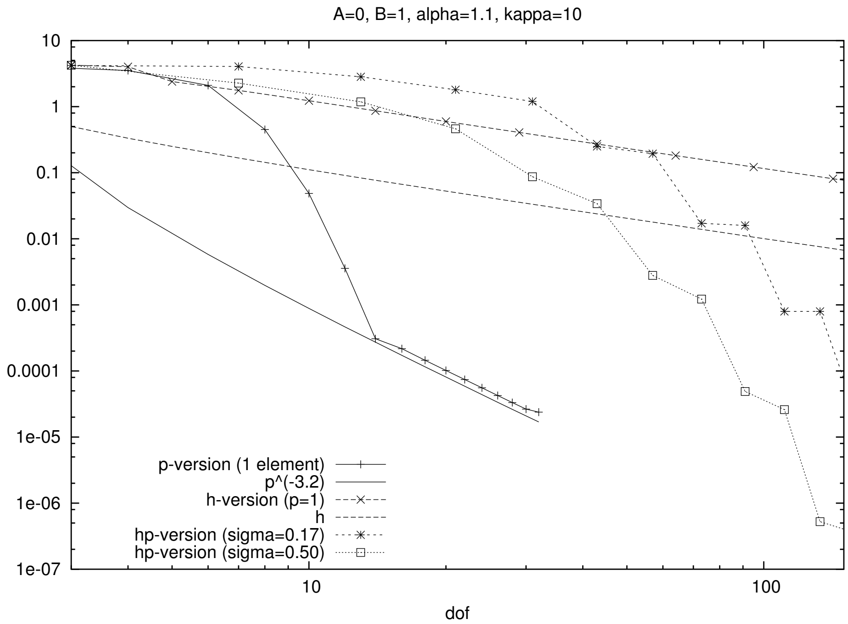

Figure 3: Approximation error for $v(t) = t^{1.1}\sin(10\,t)$ for h-, p- and hp-versions.

A Appendix

Lemma A.1 *For $l \geq 1$ there holds*

$$\left|\frac{\Gamma(\alpha+1+l)\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\right| \leq \frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}2^l l^2 \left|\frac{\Gamma(\alpha+1+l)}{\Gamma(l)}\right| \quad \textit{for all integer } n.$$

Proof. For fixed $l \geq 1$ let

$$c_n := \left|\frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\right| = \frac{\pi}{|\sin\pi\alpha|}\frac{1}{|\Gamma(n-\alpha)\Gamma(\alpha+l-n+1)|}.$$

For the equality we used the property of the Gamma function

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin\pi a}.$$

One finds that there holds $c_n \leq c_{n+1}$ for $n \leq \left\lfloor \alpha + \frac{l}{2} \right\rfloor$ and $c_n \geq c_{n+1}$ for $n \geq \left\lfloor \alpha + \frac{l}{2} \right\rfloor + 1$, that is

$$c_n \leq c_{\lfloor \alpha + l/2 \rfloor + 1}.$$

Therefore,

$$\begin{aligned}
\left|\frac{\Gamma(\alpha+1+l)\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\right| &\leq \Gamma(\alpha+1+l)c_{\lfloor \alpha+l/2\rfloor+1} \\
&= \frac{\pi}{|\sin\pi\alpha|}\frac{\Gamma(\alpha+1+l)}{|\Gamma(\lfloor\alpha+l/2\rfloor+1-\alpha)\Gamma(\alpha+l-\lfloor\alpha+l/2\rfloor-1+1)|} \\
&= \frac{\pi}{|\sin\pi\alpha|}\left|\frac{(\left\lfloor\alpha+\frac{l}{2}\right\rfloor+1-\alpha)(\left\lfloor\alpha+\frac{l}{2}\right\rfloor+2-\alpha)}{\Gamma(\left\lfloor\alpha+\frac{l}{2}\right\rfloor+3-\alpha)}\right| \\
&\quad \cdot \left|\frac{\Gamma(\alpha+1+l)(\alpha+l-\left\lfloor\alpha+\frac{l}{2}\right\rfloor)(\alpha+l-\left\lfloor\alpha+\frac{l}{2}\right\rfloor+1)}{\Gamma(\alpha+l-\left\lfloor\alpha+\frac{l}{2}\right\rfloor+2)}\right| \\
&\leq \frac{\pi}{|\sin\pi\alpha|}\left|\frac{\Gamma(\alpha+1+l)(\frac{l}{2}+1)(\frac{l}{2}+2)(\frac{l}{2}+1)(\frac{l}{2}+2)}{\Gamma(\frac{l}{2}+2)\Gamma(\frac{l}{2}+2)}\right| \\
&\leq \frac{\pi}{|\sin\pi\alpha|}\frac{(l+2)^2(l+4)^2}{16}\left|\frac{\Gamma(\alpha+1+l)}{\Gamma(\frac{l}{2}+2)\Gamma(\frac{l}{2}+\frac{3}{2})}\right| \\
&= \frac{\pi}{|\sin\pi\alpha|}\frac{(l+2)^2(l+4)^2}{16}\frac{2^{l+2}}{\sqrt{\pi}}\left|\frac{\Gamma(\alpha+1+l)}{\Gamma(l+3)}\right| \qquad (A.1) \\
&= \frac{\sqrt{\pi}}{|\sin\pi\alpha|}\frac{l+2}{l+1}\frac{(l+4)^2}{4}2^l\left|\frac{\Gamma(\alpha+1+l)}{\Gamma(l)}\right| \\
&\leq \frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}2^l l^2\left|\frac{\Gamma(\alpha+1+l)}{\Gamma(l)}\right|.
\end{aligned}$$

Here we used properties of the Gamma function, in particular (A.1) holds by

$$\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}} \Gamma(a) \Gamma\left(a + \frac{1}{2}\right) \quad (a > 0).$$

$\square$

Lemma A.2 *For $l \geq 1$ and $\alpha > -1$ there holds*

$$\frac{\Gamma(\alpha+1+l)}{\Gamma(l)} \frac{1}{l^{\alpha+1}} \leq \max(1, \Gamma(\alpha+2)).$$

Proof. Let us denote

$$h_l := \left| \frac{\Gamma(\alpha+1+l)}{\Gamma(l)} \right| \frac{1}{l^{\alpha+1}} = \frac{\Gamma(\alpha+1+l)}{\Gamma(l)} \frac{1}{l^{\alpha+1}} \geq 0 \quad (l \geq 1).$$

By Stirling's formula there holds $\lim_{l \to \infty} h_l = 1$. One finds that h_{l+1}/h_l is increasing for $\alpha \geq 0$ and decreasing for $-1 < \alpha < 0$ which leads us to $h_l \leq h_1 = \Gamma(\alpha+2)$ for $\alpha \geq 0$ and $h_l \leq 1$ for $-1 < \alpha < 0$. As a consequence $h_l \leq \max(1, \Gamma(\alpha+2))$ for general $\alpha > -1$. $\square$

Lemma A.3 *For $l \geq 1$ and $\alpha > -1$ there holds*

$$\Gamma(\alpha+1+l) \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+n+2+l)} \leq \frac{\Gamma(\alpha+2)}{l}.$$

Proof. Let us denote

$$e_n := \left| \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+n+2+l)} \right| = \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+n+2+l)} \geq 0 \quad (l \geq 1).$$

One finds that $\frac{e_n}{e_{n+1}} \geq 1$ for $l \geq 0$. Therefore,

$$e_n \leq e_0 = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+2+l)}$$

which leads to the conclusion. $\square$

Lemma A.4 *For $l \geq 1$ there holds*

$$\left| (-2)^l \left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)} \right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)} \frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)} \right| \leq E(\alpha) 4^l l^{\alpha+2}$$

where

$$E(\alpha) = \frac{75}{8} \frac{\sqrt{\pi}}{|\sin \pi\alpha|} \frac{\alpha+1}{\Gamma(\alpha+1)} \max(1, \Gamma(\alpha+2)).$$

Proof. We combine Lemmas A.1, A.2 and A.3 to conclude

$$\begin{aligned}
&\left|(-2)^l \left[\frac{\Gamma(\alpha+1+l)}{\Gamma(\alpha+1)}\right]^2 \frac{\Gamma(n-\alpha-l)}{\Gamma(n-\alpha)}\frac{\Gamma(n+\alpha+2)}{\Gamma(n+\alpha+2+l)}\right| \\
&\leq \quad \frac{2^l}{[\Gamma(\alpha+1)]^2}\frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}2^l l^2 \left|\frac{\Gamma(\alpha+1+l)}{\Gamma(l)}\right| \frac{\Gamma(\alpha+2)}{l} \\
&\leq \quad \frac{2^l}{[\Gamma(\alpha+1)]^2}\frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}2^l l^2 l^{\alpha+1}\max(1,\Gamma(\alpha+2))\frac{\Gamma(\alpha+2)}{l} \\
&= \quad \frac{75}{8}\frac{\sqrt{\pi}}{|\sin\pi\alpha|}\frac{\alpha+1}{\Gamma(\alpha+1)}\max(1,\Gamma(\alpha+2))4^l l^{\alpha+2}.
\end{aligned}$$

□

Lemma A.5 *For positive integers l, N there holds*

$$\frac{l^N\Gamma(l+1-N)}{\Gamma(l+1)} \leq \frac{N^N}{N!} \quad (l \geq N).$$

Proof. Let us denote

$$q_l := \frac{l^N\Gamma(l+1-N)}{\Gamma(l+1)} \geq 0 \quad (l \geq N).$$

There holds

$$\frac{q_{x+1}}{q_x} = \frac{x+1-N}{x}\left(\frac{x+1}{x}\right)^{N-1} \qquad (x \geq N)$$

such that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{q_{x+1}}{q_x}\right) &= \frac{N-1}{x^2}\left(\frac{x+1}{x}\right)^{N-1} + \frac{x+1-N}{x}(N-1)\left(\frac{x+1}{x}\right)^{N-2}\left(\frac{-1}{x^2}\right) \\
&= \frac{N-1}{x^2}\left(\frac{x+1}{x}\right)^{N-2}\frac{N}{x} \geq 0 \quad (x \geq N > 0).
\end{aligned}$$

Therefore q_{x+1}/q_x is a non-decreasing function in x for $x \geq N$. Moreover, $q_{x+1}/q_x \to 1$ for $x \to \infty$ such that $q_{x+1}/q_x \leq 1$ for $x \geq N$. Therefore q_n is non-increasing for $n \geq N$ and we obtain $q_n \leq q_N$. The lemma is proved by noting that

$$q_N = \frac{N^N\Gamma(1)}{\Gamma(N+1)}.$$

□

References

[1] M. Ainsworth, *Discrete dispersion relation for hp-version finite element approximation at high wave number*, SIAM J. Numer. Anal., 42 (2004), pp. 553–575.

[2] A. Arneodo, E. Bacry, S. Jaffard, and J. F. Muzy, *Oscillating singularities on Cantor sets: a grand-canonical multifractal formalism*, J. Statist. Phys., 87 (1997), pp. 179–209.

[3] I. Babuška, B. Andersson, B. Guo, J. M. Melenk, and H. S. Oh, *Finite element method for solving problems with singular solutions*, J. Comput. Appl. Math., 74 (1996), pp. 51–70.

[4] I. Babuška and B. Q. Guo, *Optimal estimates for lower and upper bounds of approximation errors in the p-version of the finite element method in two dimensions*, Numer. Math., 85 (2000), pp. 219–255.

[5] I. Babuška and M. Suri, *The h-p version of the finite element method with quasiuniform meshes*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 199–238.

[6] I. Babuška, R. B. Kellogg, and J. Pitkäranta, *Direct and inverse error estimates for finite elements with mesh refinement*, Numer. Math., 33 (1979), pp. 447–471.

[7] W. Gui and I. Babuška, *The h, p and h-p versions of the finite element method in 1 dimension, Part I: The error analysis of the p-version*, Numer. Math., 49 (1986), pp. 577–612.

[8] ——, *The h, p and h-p versions of the finite element method in 1 dimension, Part II: The error analysis of the h and h-p versions*, Numer. Math., 49 (1986), pp. 613–657.

[9] B. Q. Guo, *Approximation theory for the p-version of the finite element method in three dimensions. Part 1: approximabilities of singular functions in the framework of the Jacobi-weighted Besov and Sobolev spaces*, SIAM J. Numer. Anal., 44 (2006), pp. 246–269.

[10] B. Q. Guo and N. Heuer, *The optimal rate of convergence of the p-version of the boundary element method in two dimensions*, Numer. Math., 98 (2004), pp. 499–538.

[11] L. Hai-Feng, G. Xing, D. Zheng-Hua, and Y. Zun-Hong, *A new method to estimate the oscillating singularity exponents in locally self-similar functions*, Phys. Lett. A, 310 (2003), pp. 30–39.

[12] F. Ihlenburg and I. Babuška, *Finite element solution of the Helmholtz equation with high wave number, Part I: The h-version of the FEM*, Comput. Math. Appl., 30 (1995), pp. 9–37.

[13] ——, *Finite element solution of the Helmholtz equation with high wave number, Part II: The h-p version of the FEM*, SIAM J. Numer. Anal., 24 (1997), pp. 315–358.

[14] S. Mallat and W. L. Hwang, *Singularity detection and processing with wavelets*, IEEE Trans. Inform. Theory, 38 (1992), pp. 617–643.

[15] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.

[16] H.-S. Oh, H. Kim, and S.-J. Lee, *The numerical methods for oscillating singularities in elliptic boundary value problems*, J. Comput. Phys., 170 (2001), pp. 742–763.

[17] I. Ryshik and I. Gradstein, *Tables of Series Products and Integrals, Vol. 1*, Harri Deutsch, Thun und Frankfurt/Main, 1981.

[18] G. Strang and G. J. Fix, *An analysis of the finite-element method*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[19] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, 1967.