Discontinuous Galerkin finite element methods for dynamic linear solid viscoelasticity problems

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Abstract

We consider the usual linear elastodynamics equations augmented with evolution equations for viscoelastic internal stresses. A fully discrete approximation is defined, based on a spatially symmetric or non-symmetric interior penalty discontinuous Galerkin finite element method, and a displacement-velocity centred difference time discretisation. An *a priori* error estimate is given but only the main ideas in the proof of the error estimate are reported here due to the large number of (mostly technical) estimates that are required. The full details are referenced to a technical report.

1 Introduction

This is the second in a series of papers, [12, 11], extending spatially discontinuous Galerkin methods to linear solid viscoelasticity problems.

We consider a model for the dynamic response of linear viscoelastic solids. This comprises the usual equations of elastodynamics, but augmented with evolution equations for the viscoelastic internal stresses. The spatial discretisation is effected by a discontinuous Galerkin finite element

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method (DG FEM), which can be taken as either a symmetric or non-symmetric scheme, and the time discretisation is a standard finite difference method of Crank-Nicolson type.

For the analogous quasistatic problem considered in [12] we represented the viscoelasticity through a hereditary integral. Here we have chosen the alternative representation through internal variables. The reasons for this are, firstly, to show that the error estimates can be extended to this case and, secondly, because some practitioners prefer to work with internal variables rather than history integrals (see e.g. [6, 5]). It is important to realise that, contrary to first impressions, the introduction of internal variables does not enlarge the discrete system by creating many 'new' unknown functions. Each internal variable is actually associated with a decaying exponential term in a Prony series Volterra kernel. If the internal variables were not used then an alternative variable for each term would have to be introduced in order to carry the 'history' implied by the Volterra integral. For both types of scheme only a basic matrix inversion is required for the primary unknown function, and then simple updates to either the history or internal variables can be carried out.

For background to viscoelasticity and the assumptions we make we refer back to [12], and for more general background to the application of DG methods we refer to [9, 8, 3, 14, 10] and, in particular, to the elastic problem studied in [10].

This article is arranged as follows. We finish this section with some notation and then in Section 2 describe the model problem and the spatial discretisation. A fully discrete scheme with an *a priori* error estimate is given in Section 3, and we conclude with Section 4.

The main result is the error estimate given in Theorem 3.10, but to get to a point where this can be proven requires a great many supporting technical estimates. We present these in a series of preliminary results, Lemmas 3.5 to 3.8. On the one hand the proofs of these are long and detailed and obscure the main point of the paper but, on the other hand, because we are including viscoelastic internal variables for the first time in this type of analysis, it seems that some attention ought to be paid to the fine detail. It is also the case that the error terms are so intertwined that we cannot simply extract and explain the 'new' terms arising from the viscoelasticity, and then refer to the literature for the more standard 'elasticity' terms.

To resolve this dilemma we have chosen not to include proofs of the supporting lemmas but, instead, to give some explanatory comments for each one and use notation that will enable the easy 'tracking' of each term to its final role in the error estimate. Full proofs can be found in the technical report [13]—they largely follow standard arguments associated with DG FEM and finite difference error estimation.

In the last part of this section we recall some standard notation.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open bounded domain with polygonal/polyhedral boundary and let $I = (0, T)$ be a finite time interval. The following notation is standard. For $\omega \subseteq \bar{\Omega}$,

$$(\boldsymbol{v}, \boldsymbol{w})_\omega := \int_\omega \boldsymbol{v} \cdot \boldsymbol{w} \, d\omega,$$

but we drop the subscript when $\omega = \Omega$, We use $\| \cdot \|_{p,\omega}$ to denote the $\boldsymbol{H}^p(\omega) := (H^p(\omega))^d$ norm and again abbreviate, $\| \cdot \|_m = \| \cdot \|_{m,\Omega}$, when $\omega = \Omega$. Since we are dealing with time dependent functions we take the usual approach of treating these as maps from time into a Banach space and set,

$$\|v\|_{L_p(0,t;X)} := \left(\int_0^t \|v(t)\|_X^p \, dt \right)^{1/p},$$

for $t \leqslant T$, $1 \leqslant p < \infty$ and with the obvious 'ess sup' modification for $p = \infty$. When $t = T$ we abbreviate: $\| \cdot \|_{L_2(L_2)} := \| \cdot \|_{L_2(0,T;L_2(\Omega))}$ and so on.

We need also to deal with scalar- and tensor-valued functions and, to ease notation, we make no distinction with the inner products and norms in these cases.

2 Model problem and spatial discretisation

The basic equations for the linear dynamic response of a solid are well known and are,

$$\rho \boldsymbol{u}_{tt}(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}; \boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, t) \quad \text{in} \quad \Omega \times I, \tag{1}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \bar{\boldsymbol{u}}(\boldsymbol{x}) \quad \text{in} \quad \bar{\Omega}, \tag{2}$$

$$\boldsymbol{u}_t(\boldsymbol{x}, 0) = \bar{\boldsymbol{z}}(\boldsymbol{x}) \quad \text{in} \quad \bar{\Omega}, \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \quad \text{on} \quad \Gamma_D \times \bar{I}, \tag{4}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}; \boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}, t), \quad \text{on} \quad \Gamma_N \times \bar{I}. \tag{5}$$

In these $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \varnothing$, and we assume that Γ_D is a closed set with positive surface measure. We do not explicitly display the $\boldsymbol{x}$ dependence in most of what follows. Also, to ease notation, we denote partial time differentiation with either a subscript, as above, or a dot. Thus $\dot{\boldsymbol{u}} = \boldsymbol{u}_t$, $\ddot{\boldsymbol{u}} = \boldsymbol{u}_{tt}$, and so on. We also assume that the boundary and initial data are compatible at $t = 0$.

In the theory of linear solid viscoelasticity the symmetric second-order stress tensor satisfies the constitutive relation,

$$\boldsymbol{\sigma}(\boldsymbol{u}(t)) = \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \sum_{i=1}^{N_\varphi} \gamma_i {}^*\boldsymbol{\sigma}_i(t),$$

where: $\varepsilon_{ij}(\boldsymbol{u}) := (u_{i,j} + u_{j,i})/2$; for $i = 1, \ldots, N_\varphi$,

$$^*\boldsymbol{\sigma}_i(t) = \int_0^t \gamma_i e^{-(t-s)/\tau_i} \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds; \tag{6}$$

and, the fourth order Hooke's tensor, $\boldsymbol{D}$, satisfies the symmetries,

$$D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij},$$

and is positive definite over symmetric second order tensors. Also,

$$\gamma_i = \left(\frac{\varphi_i}{\tau_i}\right)^{1/2},$$

where the φ_i and τ_i are positive constants, and we impose the normalisation,

$$\sum_{i=0}^{N_\varphi} \varphi_i = 1$$

with, additionally (see (35) later), $\varphi_0 > 0$ (note that we have a φ_0 but not a τ_0). Then it follows that,

$$\sum_{i=1}^{N_\varphi} \gamma_i^2 \tau_i = 1 - \varphi_0 > 0. \tag{7}$$

From (6), we see that each of the internal stress tensors, $^*\boldsymbol{\sigma}_i$, satisfies an initial value problem,

$$^*\dot{\boldsymbol{\sigma}}_i(t) + \frac{1}{\tau_i} {}^*\boldsymbol{\sigma}_i(t) = \gamma_i \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{8}$$

$$^*\boldsymbol{\sigma}_i(0) = \boldsymbol{0}, \tag{9}$$

and on the other hand, if we eliminate the viscous stresses, our basic equation becomes a second-order hyperbolic partial differential equation with a fading memory Volterra integral. For the well-posedness of these types of equations we refer to [2], and for numerical analysis we cite, for example, [15, 7, 16, 4, 1]. All of these deal with the Volterra form of the problem whereas, here, we include the viscoelasticity through the evolution equations for the internal variables, (8). We are not aware of literature containing error estimates for this approach.

From our definitions we obtain the following regularity estimates.

Lemma 2.1 *For each $i = 1, \ldots, N_\varphi$ we have,*

$$\left\| \frac{\partial^{n} {}^{*}\boldsymbol{\sigma}_i}{\partial t^n} \right\|_{L_2(0,t;\boldsymbol{L}_2(\Omega))} \leqslant C \sum_{j=0}^{n-1} \left\| \frac{\partial^j \boldsymbol{u}}{\partial t^j} \right\|_{L_2(0,t;\boldsymbol{H}^1(\Omega))},$$

for $n = 1, 2, \ldots$.

Proof. Taking norms in (6) and using Hölder's inequality for convolutions gives, $\|{}^{*}\boldsymbol{\sigma}_i\|_{L_2(0,t;\boldsymbol{L}_2(\Omega))} \leqslant C\|\boldsymbol{u}\|_{L_2(0,t;\boldsymbol{H}^1(\Omega))}$. Now use successive differentiation on (8) and recursively apply the estimates obtained. □

We also have the following.

Lemma 2.2 *For each $i = 1, \ldots, N_\varphi$ we have,*

$$\|{}^{*}\boldsymbol{\sigma}_i(t)\|_r \leqslant C\|\boldsymbol{u}(t)\|_{r+1}$$

for $r \geqslant 0$.

The first step towards spatial discretisation is to establish some more notation. Let $\mathcal{E}_h = \{E_1, E_2, \ldots, E_{N_h}\}$ be a nondegenerate quasiuniform subdivision of Ω, where E_j is a triangle if $d = 2$, or a tetrahedron if $d = 3$. The nondegeneracy requirement is that there exists $\rho > 0$ such that if $h_j = \text{diam}(E_j)$, then E_j contains a ball of radius ρh_j in its interior. Let $h = \max\{h_j : 1 \leqslant j \leqslant N_h\}$, the quasiuniformity requirement is that there exists $\tau > 0$ such that $h/h_j \leqslant \tau$ for all $j \in \{1, \ldots, N_h\}$. We denote the set of interior edges (faces for $d = 3$) of $\mathcal{E}_h$ by Γ_h. With each edge (or face) e, we associate a unit normal vector $\boldsymbol{n}_e$. For a boundary edge (or face), $\boldsymbol{n}_e$ is taken to be the unit outward vector normal to $\partial\Omega$.

We now define the average and the jump operators. For each of the interior edges, e, suppose the neighbouring elements of e are E_e^1 and E_e^2 so that $e = \partial E_e^1 \cap \partial E_e^2$, and for a boundary edge suppose that E_e is the neighbouring element. We define the averaging operator $\{\cdot\}$ by,

$$\{\boldsymbol{w}\} := \begin{cases} \frac{1}{2}(\boldsymbol{w}|_{E_e^1})|_e + \frac{1}{2}(\boldsymbol{w}|_{E_e^2})|_e & \text{if } e \subset \Omega, \\ (\boldsymbol{w}|_{E_e})|_e & \text{if } e \subset \partial\Omega. \end{cases}$$

and the jump operator $[\cdot]$ by,

$$[\boldsymbol{w}] := \begin{cases} (\boldsymbol{w}|_{E_e^1})|_e - (\boldsymbol{w}|_{E_e^2})|_e & \text{if } e \subset \Omega, \\ (\boldsymbol{w}|_{E_e})|_e & \text{if } e \subset \partial\Omega. \end{cases}$$

The distinction between $[\cdot]$ and $-[\cdot]$ can be made because each edge e has a unit normal associated with it. The "direction" in which the jump takes place is unimportant.

These operators are well defined if $\boldsymbol{w}|_{E_e^i} \in (H^{\frac{1}{2}+\epsilon}(E_e^i))^d$ for $i = 1, 2$ and $\epsilon > 0$. Below, we use $|e|$ to denote the $(d-1)$-dimensional surface measure of the edge/face e. We also frequently use the estimate, $|e| \leqslant Ch^{d-1}$ which arises as a consequence of our assumptions.

Define the broken spaces for any integer $r \geqslant 0$,

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L_2(\Omega) : v|_E \in I\!P_r(E) \quad \forall E \in \mathcal{E}_h\}, \tag{10}$$

$$\mathbf{D}_r(\mathcal{E}_h) = \mathcal{D}_r(\mathcal{E}_h)^d, \tag{11}$$

$$\mathbf{L}_r(\mathcal{E}_h) = \mathcal{D}_r(\mathcal{E}_h)^{d\times d}. \tag{12}$$

For these finite element spaces we have the following interpolation-error estimates. If $\boldsymbol{v} \in \boldsymbol{H}^n(\mathcal{E}_h) \cap C(\bar{\Omega})^d$ and $\mu = \min\{r+1, n\}$ then there is an interpolant $\hat{\boldsymbol{v}} \in \mathbf{D}_r(\mathcal{E}_h) \cap C(\bar{\Omega})^d$ such that for each $E \in \mathcal{E}_h$,

$$\|\boldsymbol{v} - \hat{\boldsymbol{v}}\|_{m,E} \leqslant Ch_E^{\mu-m}\|\boldsymbol{v}\|_{n,E} \qquad \text{for } n \geqslant m \geqslant 0, \tag{13}$$

$$\|\boldsymbol{v} - \hat{\boldsymbol{v}}\|_{m,\gamma} \leqslant Ch_E^{\mu-m-1/2}\|\boldsymbol{v}\|_{n,E} \qquad \text{for } m = 0, 1 \text{ and } n \geqslant m, \tag{14}$$

where $\gamma \subseteq \partial E$.

For positive constants, δ and β, define the bilinear forms,

$$J_0^{\delta,\beta}(\boldsymbol{w}, \boldsymbol{v}) = \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{\delta}{|e|^\beta} \int_e [\boldsymbol{w}]\cdot[\boldsymbol{v}], \tag{15}$$

$$\begin{aligned} A(\boldsymbol{w}, \boldsymbol{v}) &= \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w})\colon \boldsymbol{\varepsilon}(\boldsymbol{v}) - \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w})\boldsymbol{n}_e\}\cdot[\boldsymbol{v}] \\ &\quad + \kappa \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n}_e\}\cdot[\boldsymbol{w}] + J_0^{\delta,\beta}(\boldsymbol{w}, \boldsymbol{v}). \end{aligned} \tag{16}$$

Here κ is a switch: we set $\kappa = 1$ to obtain the non-symmetric DG scheme, and $\kappa = -1$ to obtain the symmetric scheme.

Defining $\boldsymbol{z}(t) := \boldsymbol{u}_t(t)$, we first note that if $\boldsymbol{z}(t)$, $\boldsymbol{u}(t) \in C(\bar{\Omega})^d$ for each t, then we have,

$$\begin{aligned} &(\rho\dot{\boldsymbol{z}}(t), \boldsymbol{v}) + A(\boldsymbol{u}(t), \boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{z}(t), \boldsymbol{v}) \\ &+ \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \gamma_i\{{}^*\boldsymbol{\sigma}_i(t)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}] - \sum_{i=1}^{N_\varphi}\sum_E \int_E \gamma_i{}^*\boldsymbol{\sigma}_i(t)\colon\boldsymbol{\varepsilon}(\boldsymbol{v}) \\ &= L(t;\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h), \end{aligned} \tag{17}$$

where

$$L(t;\boldsymbol{v}) := (\boldsymbol{f}(t), \boldsymbol{v}) + (\boldsymbol{g}(t), \boldsymbol{v})_{\Gamma_N},$$

and, for each $i = 1, \dots, N_\varphi$,

$$\sum_E ({}^*\dot{\boldsymbol{\sigma}}(t) + \frac{1}{\tau_i}{}^*\boldsymbol{\sigma}(t), \boldsymbol{w}_i)_E = \sum_E \gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{w}_i)_E$$
$$- \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\} \cdot [\boldsymbol{u}(t)] \qquad \forall \boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \tag{18}$$

and,

$$(\rho\boldsymbol{z}(t), \boldsymbol{v})_E = (\rho\dot{\boldsymbol{u}}(t), \boldsymbol{v})_E \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{19}$$

Equations (17) and (18) arise from elementwise integration by parts, see [12], and 'adding zero'.

We will also use the following norm and semi-norm,

$$\begin{aligned} \|\boldsymbol{v}\|_{\mathcal{A}} &:= \left(|\boldsymbol{v}|_{\mathcal{E}}^2 + J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})\right)^{\frac{1}{2}}, \\ |\boldsymbol{v}|_{\mathcal{E}} &:= \left(\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\right)^{\frac{1}{2}}. \end{aligned}$$

3 Fully discrete estimates

Let us define $k = T/N$ for some positive integer N and set $t_j = jk$. Setting,

$$L_j(\boldsymbol{v}) := \frac{1}{2}\Big(L(t_j;\boldsymbol{v}) + L(t_{j-1};\boldsymbol{v})\Big),$$

our fully discrete approximation to the problem described by (17), (18) and (19) is as follows: for each $j = 1, \dots, N$, find $\{\boldsymbol{z}_j^h, \boldsymbol{u}_j^h, \dots, {}^*\boldsymbol{\sigma}_{ij}^h, \dots\} \in \mathbf{D}_r(\mathcal{E}_h) \times \mathbf{D}_r(\mathcal{E}_h) \times \mathbf{L}_{r-1}(\mathcal{E}_h)^{N_\varphi}$ such that,

$$\left(\rho\frac{\boldsymbol{z}_j^h - \boldsymbol{z}_{j-1}^h}{k}, \boldsymbol{v}\right) + A\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}, \boldsymbol{v}\right) + J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \boldsymbol{v}\right)$$
$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \gamma_i\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}\boldsymbol{n}_e\right\}\cdot[\boldsymbol{v}]$$
$$-\sum_{i=1}^{N_\varphi}\sum_E\int_E \gamma_i\frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) = L_j(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h), \tag{20}$$

with, for each $i = 1, \ldots, N_\varphi$,

$$\sum_E \left(\frac{{}^*\boldsymbol{\sigma}^h_{ij} - {}^*\boldsymbol{\sigma}^h_{i,j-1}}{k} + \frac{1}{\tau_i}\frac{{}^*\boldsymbol{\sigma}^h_{ij} + {}^*\boldsymbol{\sigma}^h_{i,j-1}}{2}, \boldsymbol{w}_i \right)_E$$
$$= \sum_E \gamma_i \left(\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}^h_j + \boldsymbol{u}^h_{j-1}}{2} \right), \boldsymbol{w}_i \right)_E$$
$$- \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{D}\boldsymbol{w}_i \boldsymbol{n}_e\} \cdot \left[\frac{\boldsymbol{u}^h_j + \boldsymbol{u}^h_{j-1}}{2} \right] \quad \forall \boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \tag{21}$$

and,

$$\left(\rho \frac{\boldsymbol{z}^h_j + \boldsymbol{z}^h_{j-1}}{2}, \boldsymbol{v} \right)_E = \left(\rho \frac{\boldsymbol{u}^h_j - \boldsymbol{u}^h_{j-1}}{k}, \boldsymbol{v} \right)_E \quad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{22}$$

It follows from this last equation that,

$$\frac{\boldsymbol{z}^h_j + \boldsymbol{z}^h_{j-1}}{2} = \frac{\boldsymbol{u}^h_j - \boldsymbol{u}^h_{j-1}}{k}. \tag{23}$$

For the initial data we set ${}^*\boldsymbol{\sigma}^h_{i0} = \mathbf{0}$, for $i = 1, \ldots, N_\varphi$, and,

$$A(\boldsymbol{u}^h_0, \boldsymbol{v}) = A(\bar{\boldsymbol{u}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h), \tag{24}$$
$$(\rho \boldsymbol{z}^h_0, \boldsymbol{v}) = (\rho \bar{\boldsymbol{z}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{25}$$

From [13] we now quote a stability estimate.

Theorem 3.1 (discrete stability) *Assume that $\beta \geqslant (d-1)^{-1}$ along with $k \leqslant \hat{k}$ and $h \leqslant \hat{h}$. Then, for δ large enough, $\hat{k}$ and $\hat{h}$ small enough, and $m = 1, 2, \ldots, N$,*

$$\|\rho^{\frac{1}{2}} \boldsymbol{z}^h_m\|_0^2 + \|\boldsymbol{u}^h_m\|^2_{\mathcal{A}} + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\boldsymbol{D}^{-\frac{1}{2}} {}^*\boldsymbol{\sigma}^h_{im}\|_0^2$$
$$+ k \sum_{j=1}^{m} J_0^{\delta,\beta} \left(\frac{\boldsymbol{z}^h_j + \boldsymbol{z}^h_{j-1}}{2}, \frac{\boldsymbol{z}^h_j + \boldsymbol{z}^h_{j-1}}{2} \right)$$
$$+ k \sum_{i=1}^{N_\varphi} \sum_{j=1}^{m} \left\| \boldsymbol{D}^{-\frac{1}{2}} \left(\frac{{}^*\boldsymbol{\sigma}^h_{ij} - {}^*\boldsymbol{\sigma}^h_{i,j-1}}{k} \right) \right\|_0^2$$
$$\leqslant C\|\rho^{\frac{1}{2}} \bar{\boldsymbol{z}}\|_0^2 + C\|\bar{\boldsymbol{u}}\|_2^2 + Ch^{-1}\|\boldsymbol{g}(0)\|^2_{0,\Gamma_N} + Ch^{-1}\|\boldsymbol{g}(t_m)\|^2_{0,\Gamma_N}$$
$$+ Ch^{-1}k \sum_{j=1}^{m} \left\| \frac{\boldsymbol{g}(t_j) - \boldsymbol{g}(t_{j-1})}{k} \right\|^2_{0,\Gamma_N} + Ck \sum_{j=1}^{m} \left\| \frac{\boldsymbol{f}(t_j) + \boldsymbol{f}(t_{j-1})}{2} \right\|_0^2,$$

where C represents a generic positive constant.

Note that the factor 'h^{-1}' is not observed in practical computations; its presence seems to be due only to a weakness in the proof. However, recalling that uniqueness implies existence for linear finite dimensional problems, Theorem 3.1 allows us to assert the existence and uniqueness of the discrete solution.

Theorem 3.2 (well-posedness) *Under the conditions of Theorem 3.1, the discrete solution exists and is unique.*

Our next goal is a fully discrete error estimate and the first step toward this is to derive an error equation. For this we set,

$$\begin{aligned} \boldsymbol{\chi}_j &:= \boldsymbol{u}_j^h - \breve{\boldsymbol{u}}(t_j), & \boldsymbol{\psi}_j &:= \boldsymbol{z}_j^h - \breve{\boldsymbol{z}}(t_j), & \boldsymbol{\eta}_{ij} &:= {}^*\boldsymbol{\sigma}_{ij}^h - {}^*\breve{\boldsymbol{\sigma}}_i(t_j), \\ \boldsymbol{\xi}_j &:= \boldsymbol{u}(t_j) - \breve{\boldsymbol{u}}(t_j), & \boldsymbol{\phi}_j &:= \boldsymbol{z}(t_j) - \breve{\boldsymbol{z}}(t_j), & \boldsymbol{\theta}_{ij} &:= {}^*\boldsymbol{\sigma}_i(t_j) - {}^*\breve{\boldsymbol{\sigma}}_i(t_j), \end{aligned}$$

where $\{\breve{\boldsymbol{u}}(t), {}^*\breve{\boldsymbol{\sigma}}_1(t), \ldots\} \subset \mathbf{D}_r(\mathcal{E}_h)$ for each t and with $\boldsymbol{z} := \boldsymbol{u}_t$ and $\breve{\boldsymbol{z}} := \breve{\boldsymbol{u}}_t$.

We choose $\breve{\boldsymbol{u}}(t) \in \mathbf{D}_r(\mathcal{E}_h)$ as the continuous interpolant of $\boldsymbol{u}(t)$ and ${}^*\breve{\boldsymbol{\sigma}}_i$ as the $\boldsymbol{L}_2(\Omega)$ projection of ${}^*\boldsymbol{\sigma}_i$ into $\mathbf{L}_{r-1}(\mathcal{E}_h)$. We then have, $\breve{\boldsymbol{z}}(t) = \breve{\boldsymbol{u}}_t(t) \in \mathbf{D}_r(\mathcal{E}_h)$ and if $\boldsymbol{u}(t_j),\ \boldsymbol{u}_t(t_j) \in C(\bar{\Omega})^d$ it follows that,

$$[\boldsymbol{u}(t_j)] = \mathbf{0}, \qquad [\breve{\boldsymbol{u}}(t_j)] = \mathbf{0}, \qquad [\boldsymbol{\xi}_j] = \mathbf{0}, \tag{26}$$

and

$$[\boldsymbol{u}_t(t_j)] = \mathbf{0}, \qquad [\breve{\boldsymbol{u}}_t(t_j)] = \mathbf{0}, \qquad [\boldsymbol{\phi}_j] = \mathbf{0}. \tag{27}$$

Moreover,

$$(\boldsymbol{\theta}_i, \boldsymbol{w}_i) = (\dot{\boldsymbol{\theta}}_i, \boldsymbol{w}_i) = (\ddot{\boldsymbol{\theta}}_i, \boldsymbol{w}_i) = \cdots = 0 \qquad \forall \boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \tag{28}$$

and from standard arguments we also have,

$$\left\| \frac{\partial^n \breve{\boldsymbol{\sigma}}_i}{\partial t^n} \right\|_0 \leqslant \left\| \frac{\partial^n \boldsymbol{\sigma}}{\partial t^n} \right\|_0.$$

Furthermore, from (13) we have,

$$\|\boldsymbol{\theta}_{ij}\|_0 = \|{}^*\boldsymbol{\sigma}_i(t_j) - {}^*\breve{\boldsymbol{\sigma}}_i(t_j)\|_0 \leqslant Ch^r \|{}^*\boldsymbol{\sigma}_i(t_j)\|_r.$$

Now, motivated by the terms that arise below, define,

$$\Delta_j \boldsymbol{v} := \frac{\boldsymbol{v}_t(t_j) + \boldsymbol{v}_t(t_{j-1})}{2} - \frac{\boldsymbol{v}(t_j) - \boldsymbol{v}(t_{j-1})}{k}.$$

Then by standard estimates for the trapezoidal quadrature rule and the Cauchy-Schwarz inequality we have the following result.

Lemma 3.3 *We have,*

$$\Delta_j \boldsymbol{v} = \frac{1}{2k} \int_{t_{j-1}}^{t_j} \boldsymbol{v}_{ttt}(t)(t_j - t)(t - t_{j-1}).$$

Moreover, if $\boldsymbol{v}_{ttt} \in L_2((t_{j-1}, t_j); \boldsymbol{L}_2(\Omega))$*, then,*

$$\|\Delta_j \boldsymbol{v}\|_0^2 \leqslant \frac{k^3}{4} \int_{t_{j-1}}^{t_j} \|\boldsymbol{v}_{ttt}(t)\|_0^2.$$

We also need the following estimate which is proven by using Taylor's series with integral remainder.

Lemma 3.4 *We have,*

$$\Delta_j \boldsymbol{v} - \Delta_{j+1} \boldsymbol{v} = k \left(\frac{\boldsymbol{v}(t_{j+1}) - 2\boldsymbol{v}(t_j) + \boldsymbol{v}(t_{j-1})}{k^2} - \frac{\boldsymbol{v}_t(t_{j+1}) - \boldsymbol{v}_t(t_{j-1})}{2k} \right).$$

Moreover, if, a.e. in Ω*, we have* $\boldsymbol{v}_{tttt} \in L_2(t_{j-1}, t_{j+1})$*, then,*

$$|\Delta_j \boldsymbol{v} - \Delta_{j+1} \boldsymbol{v}|^2 \leqslant C k^5 \int_{t_{j-1}}^{t_{j+1}} |\boldsymbol{v}_{tttt}(t)|^2.$$

These two lemmas will result in the error estimate being of the optimal, $O(k^2)$, order in the time discretisation. We are now in a position to derive an error equation.

Averaging (17), (18) and (19) between t_j and t_{j-1}, and subtracting the result from the fully discrete scheme given by (20), (21) and (22) then gives three error equations,

$$\begin{aligned}
&\left(\rho \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k}, \boldsymbol{v} \right) + A \left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2}, \boldsymbol{v} \right) + J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \boldsymbol{v} \right) \\
&\quad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} \boldsymbol{n}_e \right\} \cdot [\boldsymbol{v}] \\
&\quad - \sum_{i=1}^{N_\varphi} \sum_{E} \int_E \gamma_i \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \\
&= (\rho \Delta_j \boldsymbol{z}, \boldsymbol{v}) + \left(\rho \frac{\boldsymbol{\phi}_j - \boldsymbol{\phi}_{j-1}}{k}, \boldsymbol{v} \right) + A \left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}, \boldsymbol{v} \right) \\
&\quad + J_0^{\delta,\beta} \left(\frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2}, \boldsymbol{v} \right) + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \boldsymbol{n}_e \right\} \cdot [\boldsymbol{v}] \\
&\quad - \sum_{i=1}^{N_\varphi} \sum_{E} \int_E \gamma_i \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h),
\end{aligned} \tag{29}$$

with,

$$
\begin{aligned}
&\sum_{i=1}^{N_\varphi}\sum_E\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}+\frac{1}{\tau_i}\frac{\boldsymbol{\eta}_{ij}+\boldsymbol{\eta}_{i,j-1}}{2},\boldsymbol{w}_i\right)_E\\
&\quad-\sum_{i=1}^{N_\varphi}\sum_E\gamma_i\left(\boldsymbol{D\varepsilon}\left(\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right),\boldsymbol{w}_i\right)_E\\
&\quad+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot\left[\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right]\\
&=\sum_{i=1}^{N_\varphi}\sum_E(\Delta_j{}^*\boldsymbol{\sigma}_i,\boldsymbol{w}_i)_E+\sum_{i=1}^{N_\varphi}\sum_E\left(\frac{\boldsymbol{\theta}_{ij}-\boldsymbol{\theta}_{i,j-1}}{k}+\frac{1}{\tau_i}\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2},\boldsymbol{w}_i\right)_E\\
&\quad+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot\left[\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right]\\
&-\sum_{i=1}^{N_\varphi}\sum_E\gamma_i\left(\boldsymbol{D\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right),\boldsymbol{w}_i\right)_E\qquad\forall\{\boldsymbol{w}_i\}\in\{\mathbf{L}_{r-1}(\mathcal{E}_h)\},
\end{aligned}\tag{30}
$$

and,

$$
\begin{aligned}
&\left(\rho\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\boldsymbol{v}\right)_E-\left(\rho\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k},\boldsymbol{v}\right)_E=-(\rho\Delta_j\boldsymbol{u},\boldsymbol{v})_E\\
&+\left(\rho\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2},\boldsymbol{v}\right)_E-\left(\rho\frac{\boldsymbol{\xi}_j-\boldsymbol{\xi}_{j-1}}{k},\boldsymbol{v}\right)_E\qquad\forall\boldsymbol{v}\in\mathbf{D}_r(\mathcal{E}_h).
\end{aligned}\tag{31}
$$

Now, choosing $\boldsymbol{v}=(\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1})/k$ in (29), $\boldsymbol{w}_i=\boldsymbol{D}^{-1}(\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1})/k$ in (30) and $\boldsymbol{v}=(\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1})/k$ in (31), adding the first two resulting equations together and noting from the third that,

$$
\begin{aligned}
\left(\rho\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)&=\left(\rho\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right)\\
-\left(\rho\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right)&+\left(\rho\frac{\boldsymbol{\xi}_j-\boldsymbol{\xi}_{j-1}}{k},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right)\\
&+\left(\rho\Delta_j\boldsymbol{u},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right),
\end{aligned}
$$

we multiply by $2k$ and sum over $j = 1, \dots, m$ to obtain,

$$
\begin{aligned}
&\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + 2k\sum_{j=1}^{m} J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}\right) \\
&+ 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2 \\
&\qquad = \|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + \|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{i0}\|_0^2 \\
&+ 2k\sum_{j=1}^{m} G_j\left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k}\right) + 2k\sum_{j=1}^{m} H_j\left(\boldsymbol{D}^{-1}\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k}\right) \\
&\qquad + T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \qquad (32)
\end{aligned}
$$

where,

$$
\begin{aligned}
T_1 &:= -2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left(\left\{\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right]\right.\\
&\qquad\left.+\left\{\frac{\boldsymbol{\eta}_{ij}+\boldsymbol{\eta}_{i,j-1}}{2}\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right]\right),\\
T_2 &:= 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\sum_{E}\gamma_i\int_E\left(\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right):\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right.\\
&\qquad\left.+\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right):\frac{\boldsymbol{\eta}_{ij}+\boldsymbol{\eta}_{i,j-1}}{2}\right),\\
T_3 &:= 2k\sum_{j=1}^{m}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\left(\left\{\boldsymbol{D\varepsilon}\left(\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right)\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right]\right.\\
&\qquad\left.-\kappa\left\{\boldsymbol{D\varepsilon}\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right]\right),\\
T_4 &:= -2k\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\Delta_j\breve{\boldsymbol{u}},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right),\\
T_5 &:= -2k\sum_{j=1}^{m}\left(\rho\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k},\Delta_j\breve{\boldsymbol{u}}\right),\\
T_6 &:= 2k\sum_{j=1}^{m}\left(\rho\Delta_j\boldsymbol{z},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)\\
&\qquad+2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\sum_{E}\left(\Delta_j{}^*\boldsymbol{\sigma}_i,\boldsymbol{D}^{-1}\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)_E,
\end{aligned}
$$

along with,

$$
\begin{aligned}
G_j(\boldsymbol{v}) &:= J_0^{\delta,\beta}\left(\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2},\boldsymbol{v}\right)+\left(\rho\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k},\boldsymbol{v}\right)\\
&\qquad+A\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2},\boldsymbol{v}\right)-\sum_{i=1}^{N_\varphi}\sum_{E}\int_E\gamma_i\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}:\boldsymbol{\varepsilon}(\boldsymbol{v})\\
&\qquad+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left\{\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}\boldsymbol{n}_e\right\}\cdot[\boldsymbol{v}],
\end{aligned}
$$

and,

$$\begin{aligned} H_j(\boldsymbol{v}) &:= \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{D}\boldsymbol{v}\boldsymbol{n}_e\} \cdot \left[\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}\right] \\ &\quad - \sum_{i=1}^{N_\varphi} \sum_{E} \gamma_i \left(\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}\right), \boldsymbol{v}\right)_E \\ &\quad + \sum_{i=1}^{N_\varphi} \sum_{E} \left(\frac{\boldsymbol{\theta}_{ij} - \boldsymbol{\theta}_{i,j-1}}{k} + \frac{1}{\tau_i}\frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2}, \boldsymbol{v}\right)_E . \end{aligned}$$

To get these we noted first that,

$$\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} - \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} = -\Delta_j \breve{\boldsymbol{u}}, \tag{33}$$

because $(\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h)/2 = (\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h)/k$, and secondly that,

$$\frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2} - \frac{\boldsymbol{\xi}_j - \boldsymbol{\xi}_{j-1}}{k} = \Delta_j \boldsymbol{u} - \Delta_j \breve{\boldsymbol{u}}.$$

The goal now is to derive an *a priori* error estimate for this scheme and, as will become apparent, the starting point for this will be (34). This inequality will be formed by estimating the terms on the right of (32), and these estimates are now given in the next series of lemmas. We note that (34) estimates the error at a generic 'current' time level t_m.

As already explained in the introduction, the proofs of these lemmas are not given here (but can be found in [13]) so as not to obscure the main result. Instead we attempt to describe what each of these preliminary results actually represents in terms of its contribution to (34) and, thereby, hope we can convey the main points of the proof without over-burdening the reader with technicalities. In the results that follow much use is made of Young's inequality in the form,

$$ab \leqslant \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon} \qquad \forall a, b \in \mathbb{R} \text{ and } \forall \epsilon > 0.$$

We use many differently labelled ϵ's in the estimates that follow and these re-appear in (34). This notation makes it possible for the reader to track each of the following estimates through to its final destination in the main proof and, we hope, will lend a little more meaning to the comments following each lemma.

The lemmas that now follow are all subject to the assumptions made later in Theorem 3.10.

Lemma 3.5 *We have,*

$$|T_4+T_5+T_6| \leqslant 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\epsilon_6'}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2$$
$$+\frac{k}{\epsilon_6}\sum_{j=0}^{m}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2+\epsilon_5\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2+\|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2+k\sum_{j=1}^{m-1}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2$$
$$+Ck^3\int_0^{t_1}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2+\frac{Ck^3}{\epsilon_5}\int_{t_{m-1}}^{t_m}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2+Ck^4\int_0^{t_m}\|\breve{\boldsymbol{u}}_{tttt}(t)\|_0^2$$
$$+Ck^4\int_0^{t_m}\left((\epsilon_6+1)\|\boldsymbol{z}_{ttt}(t)\|_0^2+\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2+\frac{1}{\epsilon_6'}\sum_{i=1}^{N_\varphi}\|{}^*\ddot{\boldsymbol{\sigma}}_i(t)\|_0^2\right),$$

for all ϵ_5, ϵ_6, $\epsilon_6'>0$.

The first and third terms on the right in this result contain ‘current’ errors (at t_m) and, as such, appear on the left of (34). The second and fifth terms contribute to a ‘Gronwall summation’ on the right of (34) while the ‘initial’ error in the fourth term stays on the right and contributes a term of order $O(h^r)$ to the final estimate. The remaining terms in the the last two lines on the right contribute the temporal $O(k^2)$ to the final estimate.

The terms T_1, T_2 and T_3 are handled, with some modifications, in much the same way as in the proof of Theorem 3.1 (given in [13]). The result follows.

Lemma 3.6 *We have*

$$|T_1+T_2+T_3| \leqslant 2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2$$
$$+2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\hat{\epsilon}_i}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2$$
$$+\frac{2Ck}{\check{\epsilon}}\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2$$
$$+\sum_{i=1}^{N_\varphi}\frac{\bar{\epsilon}}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2+\frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2+\frac{2C^2h^{(d-1)\beta-1}}{\epsilon''}J_0^{1,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m)$$
$$+Ck\sum_{j=0}^{m}\frac{1}{\epsilon'}\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2+\frac{\epsilon''}{2}\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2+Ck\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2h^{(d-1)\beta-1}}{\hat{\epsilon}_i}J_0^{1,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j)$$
$$+2h^{(d-1)\beta-1}k(2\check{\epsilon}(1-\varphi_0)+\epsilon')\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right),$$

for all $\hat{\epsilon}_1, \ldots, \hat{\epsilon}_{N_\varphi}, \check{\epsilon}, \bar{\epsilon}, \epsilon', \epsilon'' > 0$.

In this estimate the first term on the right will be subject (see Theorem 3.10) to the restriction that $(d-1)\beta \geqslant 1$ and, as such, it remains on the right in (34) and contributes an initial error of $O(h^r)$ to the final estimate. The seventh term also remains on the right in (34) and provides a summation against which Gronwall's inequality can be invoked. The remaining terms on the right in the lemma above contain 'current' errors at t_m and are thus moved to the left of (34).

The 'H_j' terms in (32) are a specific consequence of the presence of the viscoelastic internal variables, and are dealt with as follows.

Lemma 3.7 *Assuming that* $h \leqslant \hat{h}$, $\beta \geqslant (d-1)^{-1}$, $|e| \leqslant Ch^{d-1}$ *and* $\|\boldsymbol{v}\|_{0,e} \leqslant Ch^{-1/2}\|\boldsymbol{v}\|_{0,E}$ *if* e *is an edge of* E, *we have,*

$$\left| 2k \sum_{j=1}^{m} H_j \left(\boldsymbol{D}^{-1} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right|$$
$$\leqslant 2k \sum_{j=1}^{m} \left(\sum_{i=1}^{N_\varphi} \epsilon'_H \left\| \boldsymbol{D}^{-1/2} \left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right) \right\|_0^2 + \frac{C}{\epsilon'_H} \left| \frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2} \right|_{\mathcal{E}}^2 \right),$$

for all $\epsilon'_H > 0$.

The first term in the sum on the right is moved to the left in (34) while the second term remains on the right and contributes a spatial error term of the order $O(h^r)$.

The 'G_j' terms in (32) require a more involved treatment. The result is as follows.

Lemma 3.8 *We have,*

$$\left| 2k\sum_{j=1}^{m} G_j\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)\right| \leqslant 2k\sum_{j=1}^{m}\frac{1}{2}\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0^2$$
$$+ 2k\sum_{j=1}^{m}\frac{1+\epsilon''_G}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2 + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi} C\epsilon'_G\left\|\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}\right\|_0^2$$
$$+ \frac{k}{\epsilon'_G}\sum_{j=1}^{m} J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right) + 2k\sum_{j=0}^{m}\frac{1}{2\epsilon''_G}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2$$
$$+ Ch^{2r}\left((1+\epsilon'''_G)\|\boldsymbol{u}\|^2_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))} + \|\boldsymbol{u}_t\|^2_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}\right)$$
$$+\frac{1}{2}\|\boldsymbol{\chi}_0\|^2_{\mathcal{A}} + \frac{1}{2\epsilon'''_G}\|\boldsymbol{\chi}_m\|^2_{\mathcal{A}} + \frac{k}{2}\sum_{j=1}^{m-1}\|\boldsymbol{\chi}_j\|^2_{\mathcal{A}},$$

for all positive ϵ'_G, ϵ''_G and ϵ'''_G.

The first, third, sixth and seventh terms on the right in this estimate produce a quantity of the order $O(h^r + k^2)$ in the final estimate while the fourth and eighth terms contain current errors at t_m and are moved to the left in (34). The fifth and ninth terms provide Gronwall summations and so remain on the right in (34), and the second term is dealt with as follows.

Lemma 3.9 *We have,*

$$2k\sum_{j=1}^{m}\frac{1+\epsilon''_G}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2$$
$$\leqslant Ch^{2r}\|\boldsymbol{u}_t\|^2_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))} + C(1+\epsilon''_G)k^4\|\boldsymbol{u}_{tttt}\|^2_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}.$$

Now we can use these preliminary estimates to derive the error estimate.

Theorem 3.10 (fully discrete 'energy' error estimate) *Assume that we have $h \leqslant \hat{h}$, $k \leqslant \hat{k}$, $\beta \geqslant (d-1)^{-1}$, $\bar{\boldsymbol{u}} \in \boldsymbol{H}^{r+1}(\Omega)$, $\bar{\boldsymbol{z}} \in \boldsymbol{H}^r(\Omega)$ and $\boldsymbol{u} \in H^4(\boldsymbol{L}_2)\cap H^2(\boldsymbol{H}^1)\cap W^1_\infty(\boldsymbol{H}^{r+1})\cap C^1(C(\bar{\Omega})^d)$. Then, for $\hat{h}$ and $\hat{k}$ small enough, and δ large enough,*

$$\|\rho^{1/2}(\boldsymbol{u}_t(t_m)-\boldsymbol{z}^h_m)\|_0 + \|\boldsymbol{u}(t_m)-\boldsymbol{u}^h_m\|_{\mathcal{A}}$$
$$+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}({}^*\boldsymbol{\sigma}_i(t_m)-{}^*\boldsymbol{\sigma}^h_{im})\|_0 \leqslant C(h^r+k^2),$$

where C is a 'Gronwall' constant independent of h and k.

Proof. We start with (32) and invoke Lemmas 3.5, 3.6, 3.7, 3.8 and 3.9 to get,

$$\begin{aligned}
&\left(1-\epsilon_5-\frac{k}{\epsilon''_G}-\frac{k}{\epsilon_6}\right)\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 \\
&+\left(1-\frac{\epsilon''}{2}-\frac{1}{2\epsilon'''_G}-\frac{Ck}{\epsilon'}\right)\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 \\
&-\left(\frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2+\hat{h}^{(d-1)\beta-1}\left(\frac{2C^2}{\epsilon''}+Ck\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2}{\hat{\epsilon}_i}\right)J_0^{1,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m)\right) \\
&+2k\left(1-\frac{1}{2\epsilon'_G}-\hat{h}^{(d-1)\beta-1}(2\check{\epsilon}(1-\varphi_0)+\epsilon')\right) \\
&\times\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right) \\
&+2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left(1-\frac{\hat{\epsilon}_i}{2}-\epsilon'_H-\frac{\epsilon'_6}{2}\right)\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 \\
&+\left(1-\bar{\epsilon}-\frac{2Ck}{\check{\epsilon}}\right)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2 \\
&\leqslant \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4+\mathcal{T}_5,
\end{aligned} \tag{34}$$

where,

$$
\begin{aligned}
\mathcal{T}_1 &:= 2\|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + (\tfrac{3}{2} + 2C\hat{h}^{\frac{(d-1)\beta-1}{2}})\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{i0}\|_0^2,\\
\mathcal{T}_2 &:= Ck^4\int_0^{t_m}\|\breve{\boldsymbol{u}}_{tttt}(t)\|_0^2 + 2k\sum_{j=1}^{m}\frac{1}{2}\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0^2,\\
&\quad + Ck^3\int_0^{t_1}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{Ck^3}{\epsilon_5}\int_{t_{m-1}}^{t_m}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2\\
&\quad + Ck^4\int_0^{t_m}\left((1+\epsilon_6)\|\boldsymbol{z}_{ttt}(t)\|_0^2 + \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon_6'}\sum_{i=1}^{N_\varphi}\|{}^*\ddot{\boldsymbol{\sigma}}_i(t)\|_0^2\right)\\
\mathcal{T}_3 &:= 2k\sum_{j=1}^{m}\frac{1+\epsilon_G''}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2,\\
\mathcal{T}_4 &:= 2k\sum_{j=1}^{m}\frac{C}{\epsilon_H'}\left|\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right|_{\mathcal{E}}^2 + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}C\epsilon_G'\left\|\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}\right\|_0^2\\
&\quad + Ch^{2r}\big((1+\epsilon_G''')\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2 + \|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2\big),\\
\mathcal{T}_5 &:= \frac{2Ck}{\check{\epsilon}}\sum_{j=0}^{m-1}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2 + Ck\sum_{j=0}^{m-1}\left(\frac{1}{2}+\frac{1}{\epsilon'}\right)\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2\\
&\quad + Ck\sum_{j=0}^{m-1}\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2\hat{h}^{(d-1)\beta-1}}{\delta\hat{\epsilon}_i}J_0^{\delta,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j)\\
&\quad + k\sum_{j=0}^{m-1}\left(1+\frac{1}{\epsilon_G''}+\frac{1}{\epsilon_6}\right)\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2.
\end{aligned}
$$

We now choose,

$$\epsilon'' = \frac{\varphi_0/4}{1-\varphi_0/2}, \qquad \epsilon' = \varphi_0, \qquad \hat{\epsilon}_i = 1, \qquad \check{\epsilon} = \frac{1}{4}, \qquad \epsilon_G''' = \frac{2-\varphi_0}{\varphi_0},$$

$$\bar{\epsilon} = 1 - \frac{\varphi_0}{2}, \qquad \epsilon_5 = \frac{1}{2}, \qquad \epsilon_G'' = \epsilon_6 = \frac{2}{C} \quad \text{for some } C > 0,$$

$$\epsilon_G' = \frac{1}{1-\varphi_0}, \qquad \epsilon_H' = \frac{1}{6}, \qquad \epsilon_6' = \frac{1}{3},$$

and insist that,

$$\delta \geqslant \frac{4C^2(2-\varphi_0)^2\hat{h}^{(d-1)\beta-1}}{(2-2\varphi_0)\varphi_0}. \tag{35}$$

These lead to,

$$\left(\frac{1}{2} - C\hat{k}\right)\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \left(\frac{\varphi_0}{8-4\varphi_0} - C\hat{k}\right)\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2$$
$$+ k(1+\varphi_0)(1-\hat{h}^{(d-1)\beta-1})\sum_{j=1}^{m} J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)$$
$$+ \frac{k}{3}\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2$$
$$+ \left(\frac{\varphi_0}{2} - C\hat{k}\right)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2$$
$$\leqslant \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5.$$

Now, for $\mathcal{T}_2$, using Lemmas 2.1, 3.4 and stability properties of the interpolant,

$$|\mathcal{T}_2| \leqslant Ck^4\Big(\|\boldsymbol{u}_{ttt}\|_{L_\infty(0,t_m;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{u}\|_{H^2(0,t_m;\boldsymbol{H}^1(\Omega))} + \|\boldsymbol{u}_{ttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))} + \|\boldsymbol{u}_{tttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}\Big)^2.$$

For $\mathcal{T}_3$, using Lemma 3.9,

$$|\mathcal{T}_3| \leqslant Ch^{2r}\|\boldsymbol{u}_t\|^2_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))} + Ck^4\|\boldsymbol{u}_{tttt}\|^2_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}.$$

For $\mathcal{T}_4$,

$$|\mathcal{T}_4| \leqslant Ch^{2r}\Big(\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))} + \|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))} + \max_{1\leqslant i\leqslant N_\varphi}\|{}^*\boldsymbol{\sigma}_i\|_{L_\infty(0,t_m;\boldsymbol{H}^r(\Omega))}\Big)^2,$$

and use Lemma 2.2, and, for $\mathcal{T}_5$,

$$|\mathcal{T}_5| \leqslant Ck\sum_{j=0}^{m-1}\left(\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 + \|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2\right).$$

Now select $\hat{h}$ and $\hat{k}$ small enough, use the initial conditions ${}^*\boldsymbol{\sigma}_i(0) = \mathbf{0}$ and apply the discrete Gronwall lemma to get,

$$\|\rho^{1/2}\boldsymbol{\psi}_m\|_0 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0$$
$$\leqslant C(h^r + k^2) + C\|\rho^{1/2}\boldsymbol{\psi}_0\|_0 + C\|\boldsymbol{\chi}_0\|_{\mathcal{A}}.$$

Now, by (25) and (13), we have,

$$\begin{aligned}
\|\rho^{\frac{1}{2}}\boldsymbol{\psi}_0\|_0 &= \|\rho^{\frac{1}{2}}(\boldsymbol{z}_0^h - \breve{\boldsymbol{z}}(0))\|_0, \\
&\leqslant \|\rho^{\frac{1}{2}}(\boldsymbol{z}_0^h - \bar{\boldsymbol{z}})\|_0 + \|\rho^{\frac{1}{2}}(\breve{\boldsymbol{z}}(0) - \bar{\boldsymbol{z}})\|_0, \\
&\leqslant 2\|\rho^{\frac{1}{2}}(\breve{\boldsymbol{z}}(0) - \bar{\boldsymbol{z}})\|_0 \leqslant Ch^r\|\bar{\boldsymbol{z}}\|_r.
\end{aligned}$$

Also, using standard results for the elliptic projection (see e.g. [12]) we have,

$$\begin{aligned}
\|\boldsymbol{\chi}_0\|_{\mathcal{A}} &= \|\boldsymbol{u}_0^h - \breve{\boldsymbol{u}}(0)\|_{\mathcal{A}}, \\
&\leqslant \|\bar{\boldsymbol{u}} - \boldsymbol{u}_0^h\|_{\mathcal{A}} + \|\bar{\boldsymbol{u}} - \breve{\boldsymbol{u}}(0)\|_{\mathcal{A}}, \\
&\leqslant \|\bar{\boldsymbol{u}} - \boldsymbol{u}_0^h\|_{\mathcal{A}} + |\bar{\boldsymbol{u}} - \breve{\boldsymbol{u}}(0)|_{\mathcal{E}} \leqslant Ch^r\|\bar{\boldsymbol{u}}\|_{r+1}.
\end{aligned}$$

Using the triangle inequality we now obtain,

$$\begin{aligned}
&\|\rho^{1/2}(\boldsymbol{u}_t(t_m) - \boldsymbol{z}_m^h)\|_0 + \|\boldsymbol{u}(t_m) - \boldsymbol{u}_m^h\|_{\mathcal{A}} \\
&\quad + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}({}^*\boldsymbol{\sigma}_i(t_m) - {}^*\boldsymbol{\sigma}_{im}^h)\|_0 \\
&\qquad \leqslant \|\rho^{1/2}\boldsymbol{\xi}_t(t_m)\|_0 + \|\boldsymbol{\xi}(t_m)\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\theta}_i(t_m)\|_0 \\
&\qquad\quad + \|\rho^{1/2}\boldsymbol{\psi}_m\|_0 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0, \\
&\qquad \leqslant Ch^r\|\boldsymbol{u}_t(t_m)\|_r + Ch^r\|\boldsymbol{u}(t_m)\|_{r+1} \\
&\qquad\quad + Ch^r \max_{1\leqslant i\leqslant N_\varphi}\|{}^*\boldsymbol{\sigma}_i(t_m)\|_r + C(h^r + k^2).
\end{aligned}$$

To get this we noted that, $\|\boldsymbol{\xi}(t_j)\|_{\mathcal{A}} = |\boldsymbol{\xi}(t_j)|_{\mathcal{E}} \leqslant Ch^r\|\boldsymbol{u}(t_j)\|_{r+1}$ for $j = 0, 1, \ldots, m$. Finally, using Lemma 2.2 completes the proof. □

4 Conclusion

In this article we have extended the application of the DG FEM to dynamic linear solid viscoelasticity problems. This builds upon the algorithm and estimates in [12] in that we have now included the inertia term. It also varies the approach in [12] in that here we have chosen to represent the viscoelastic history through evolution equations for internal stress tensors, rather than augment the momentum equation with a Volterra (hereditary) integral.

The technical report that accompanies this article, [13], also contains semidiscrete energy and L_2 error estimates. They are not given here because the latter seems to require rather restrictive assumptions.

This article with [12] represents the extension of DG FEM to elliptic and (second-order) hyperbolic problems with viscoelastic memory. The analogous parabolic problem has been studied in [11]. Since code development for these type of problems is non-trivial, we do not present numerical results here. Instead, numerics for all three problems will be presented elsewhere at a later date when all the numerical issues have been identified.

References

[1] J. Brilla. Error analysis for Laplace transform—finite element solution of hyperbolic equations. *Numer. Math.*, 41:55—62, 1983.

[2] C. M. Dafermos. An abstract Volterra equation with applications to linear viscoelasticity. *J. Diff. Eqns.*, 7:554—569, 1970.

[3] V. Girault, B. Riviere, and M. Wheeler. A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier-Stokes problems. *Mathematics of Computation*, TICAM Report 02-08 (2002), to appear.

[4] Gregory M. Hulbert and Thomas J. R. Hughes. Space-time finite element methods for second-order hyperbolic equations. *Comp. Meth. Appl. Mech. Eng.*, 84:327—348, 1990.

[5] A. R. Johnson. Modeling viscoelastic materials using internal variables. *The Shock and Vibration Digest*, 31:91—100, 1999.

[6] A. R. Johnson, A. Tessler, and M. Dambach. Dynamics of thick viscoelastic beams. *Journal of Engineering Materials and Technology*, 119:273—278, 1997.

[7] A. K. Pani, V. Thomée, and L. B. Wahlbin. Numerical methods for hyperbolic and parabolic integro-differential equations. *J. Integral Equations Appl.*, 4:533—584, 1992.

[8] B. Rivière, M. F. Wheeler, and V. Girault. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. *SIAM J. Numer. Anal.*, 39:902–931, 2001.

[9] B. Rivière and M.F. Wheeler. A discontinuous Galerkin method applied to nonlinear parabolic equations. In B. Cockburn, G.E. Karniadakis, and C.-W. Shu, editors, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, volume 11 of *Lecture Notes in Computational Science and Engineering*, pages 231–244. Springer, 1999.

[10] B. Riviere and M.F. Wheeler. Discontinuous finite element methods for acoustic and elastic wave problems. *Contemporary Mathematics*, 329:271–282, 2003.

[11] Béatrice Rivière and Simon Shaw. Discontinuous Galerkin finite element approximation of nonlinear non-Fickian diffusion in viscoelastic polymers. Submitted to SIAM J. Numer. Anal. See report 05/6 at `www.brunel.ac.uk/bicom`, 2005.

[12] Béatrice Rivière, Simon Shaw, Mary F. Wheeler, and J.R. Whiteman. Discontinuous Galerkin finite element methods for linear elasticity and quasistatic linear viscoelasticity. *Numer. Math.*, 95:347—376, 2003.

[13] Béatrice Rivière, Simon Shaw, and J.R. Whiteman. Discontinuous Galerkin finite element methods for dynamic linear solid viscoelasticity problems. Technical report, BICOM, Brunel University, 2004. 04/2, `www.brunel.ac.uk/bicom`.

[14] Béatrice Rivière, Mary F. Wheeler, and Vivette Girault. Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I. *Computational Geosciences*, 3:337—360, 1999.

[15] Simon Shaw, M. K. Warby, and J. R. Whiteman. An error bound via the Ritz-Volterra projection for a fully discrete approximation to a hyperbolic integrodifferential equation. Technical report, 94/3, BICOM, Brunel University, Uxbridge, U.K., 1994. (see `www.brunel.ac.uk/bicom`).

[16] E. G. Yanik and G. Fairweather. Finite element methods for parabolic and hyperbolic partial integro-differential equations. *Nonlinear Analysis, Theory, Methods & Applications*, 12:785—809, 1988.