Numerical Approximation of a Time Dependent, Non-linear, Fractional Order Diffusion Equation*

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Abstract. In this article we analyze a fully discrete numerical approximation to a time dependent fractional order diffusion equation which contains a non-local, quadratic non-linearity. The analysis is performed for a general fractional order diffusion operator. The non-linear term studied is a product of the unknown function and a convolution operator of order 0. Convergence of the approximation and a priori error estimates are given. Numerical computations are included which confirm the theoretical predictions.

Key words. anomalous diffusion, non-linear parabolic equation, finite element approximation

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1 Introduction

In this paper we study the numerical approximation to time dependent, fractional order, diffusion equations containing a non-local quadratic non-linearity. Specifically, we consider equations of the form:

$$u_t \ + \ \mathcal{D}^{2\alpha} u \ - \ \nabla \cdot (u\, B(u)) \ = \ f(x)\,, \quad x \in \Omega\ ,\ t \in (0, T]\ , \tag{1.1}$$
$$u(x,t) \ = \ 0\,, \quad x \in \partial\Omega\ ,\ t \in (0,T]\ , \tag{1.2}$$
$$u(x,0) \ = \ u^0(x)\,, \quad x \in \Omega\ , \tag{1.3}$$

which arise from models in statistical mechanics. In such a setting u can be thought of as describing the density of particles filling up a domain $\Omega \subset \mathbb{R}^d$. In (1.1), $\mathcal{D}^{2\alpha}$ denotes a general fractional order diffusion operator of order 2α, $1/2 < \alpha \leq 1$. The term $\nabla \cdot (u\, B(u))$ models particle interactions.

For the classical diffusion case ($\alpha = 1$) the diffusion operator models a Brownian diffusion process. For fractional diffusion ($1/2 < \alpha < 1$) the $\mathcal{D}^{2\alpha}$ operator is commonly referred to as *anomalous dif-*

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fusion, where the underlying stochastic process is a Lévy α-stable flight. A key difference between fractional diffusion operators and the usual diffusion operator is that fractional diffusion operators are *non-local* operators. Equations containing fractional diffusion have also been investigated in modeling turbulent flow [6, 17], chaotic dynamics of classical conservative systems [18] and contaminant transport in groundwater flow [2, 12].

In this paper we do not investigate the existence of u satisfying (1.1)-(1.3) but, assuming a sufficiently regular solution u exists, the existence and convergence properties of its approximation u_h. (For a discussion on the existence of u see [4].) The results presented in this paper extend the work developed in [7], [8] (see also [14]) for a steady-state, linear fractional advection dispersion equation.

Different definitions for fractional diffusion operators have been used. Two such definitions are given in Section 2.2. Our analysis does not rely on the particular form of the fractional diffusion operator, only that it satisfies properties of *continuity* and *coercivity* (see (2.2),(2.3)). Several examples of non-local operators $B(\cdot)$ are given in Section 2.3. Our analysis does not assume a particular form for $B(\cdot)$, only that it is linear, and an *operator of order* 0 (see (2.4)).

A finite element approximation scheme is described and shown to be computable in Section 3. A priori error estimates for the approximation are presented in Section 4. Hölder type inequalities for Sobolev spaces, used in the analysis, are derived in Section 2.4. Finally, in Section 5 we present some numerical experiments which support the theoretical estimates.

2 Mathematical Preliminaries

2.1 Mathematical Notation

In this section we summarize the mathematical notation used, and state our assumptions regarding properties satisfied by the fractional diffusion operator $\mathcal{D}^{2\alpha}$ and the operator $B(\cdot)$.

The following notation is used. The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$, and the $L^p(\Omega)$ norm by $\|\cdot\|_{L^p}$, with the special cases of $L^2(\Omega)$ and $L^\infty(\Omega)$ norms being written as $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. For $k \in \mathbb{N}$, we denote the norm associated with the Sobolev space $W^{k,p}(\Omega)$ by $\|\cdot\|_{W^{k,p}}$, with the special case $W^{k,2}(\Omega)$ being written as $H^k(\Omega)$ with norm $\|\cdot\|_k$ and seminorm $|\cdot|_k$. For the definition of fractional order Sobolev spaces $W^{s,p}(\Omega)$, $s \in \mathbb{R}^+\backslash\mathbb{N}$, we use the real method of interpolation between two Banach spaces [3, 15].

When $v(\mathbf{x},t)$ is defined on the entire time interval $(0,T)$, we define

$$\|v\|_{\infty,k} \;:=\; \sup_{0<t<T} \|v(\cdot,t)\|_k \;, \quad \|v\|_{0,k} \;:=\; \left(\int_0^T \|v(\cdot,t)\|_k^2 \, dt\right)^{1/2} \;, \quad \|v\|(t) \;:=\; \|v(\cdot,t)\| \;.$$

For convenience we let X denote the space

$$X := H_0^\alpha(\Omega) := \texttt{ closure of } C_0^\infty(\Omega) \texttt{ in } H^\alpha(\Omega)\,. \tag{2.1}$$

We use $H^{-\alpha}(\Omega)$ to denote the dual space of $H_0^\alpha(\Omega)$, with norm denoted $\|\cdot\|_{-\alpha}$.

Throughout the paper we use C to denote a *generic constant* whose actual value may change from line to line.

We make the following general assumptions regarding the diffusion operator. There exist constants $C_c, C_t > 0$ such that for $v,\, w \in X$

$$\langle \mathcal{D}^{2\alpha} v, w \rangle \ \leq \ C_t \, \|v\|_\alpha \, \|w\|_\alpha \, , \quad \text{(continuity on } X \times X) \, , \tag{2.2}$$

$$\langle \mathcal{D}^{2\alpha} v, v \rangle \ \geq \ C_c \, \|v\|_\alpha^2 \, , \quad \text{(coercivity on } X) \, , \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $H^{-\alpha}(\Omega)$ and $H_0^\alpha(\Omega)$.

For the non-local operator $B(\cdot)$ we assume:
(i) $B(\cdot)$ is linear,
(ii) $B(\cdot)$ is an operator of order 0, i.e., for $\beta \geq 0$, $u \in H^\beta(\Omega)$,

$$\|B(u)\|_\beta \ \leq \ C_B \|u\|_\beta \ . \tag{2.4}$$

2.2 Examples of Fractional Diffusion Operators satisfying (2.2),(2.3)

1. Fractional Laplacian Operator
In the context of pseudo-differential operators, [16], a fractional diffusion operator may be defined in terms of the negative Laplacian operator, $-\Delta$, [4].

We have that for ω the Fourier transform variable,

$$\mathcal{F}(-\Delta u(x)) \ = \ |\omega|^2 \, \hat{u}(\omega) \ .$$

The Fractional Laplacian operator is then defined via the inverse Fourier transform as

$$(-\Delta)^{\gamma/2} \, u(x) \ := \ \mathcal{F}^{-1} \left(|\omega|^\gamma \, \hat{u}(\omega) \right) \ . \tag{2.5}$$

Associated with (2.5) a fractional differential operator of order 2α may be formally defined as

$$\mathcal{D}^{2\alpha} u(x) \ := \ (-\Delta)^\alpha \, u(x) \ . \tag{2.6}$$

For $\mathcal{D}^{2\alpha}$ defined by (2.6), we have for $v, w \in H_0^\alpha(\Omega)$, $\alpha > 1/2$,

$$\begin{aligned} \langle \mathcal{D}^{2\alpha} v, w \rangle \ &= \ \left((-\Delta)^{\alpha/2} \, v \, , \, (-\Delta)^{\alpha/2} \, w \right) \\ &\leq \ C_1 \, \|v\|_\alpha \, \|w\|_\alpha \, . \end{aligned}$$

Also,

$$\begin{aligned} \langle \mathcal{D}^{2\alpha} v, v \rangle \ &= \ \left((-\Delta)^{\alpha/2} \, v \, , \, (-\Delta)^{\alpha/2} \, v \right) \\ &\geq \ C_2 \, \|v\|_\alpha^2 \, . \end{aligned}$$

2. Weighted Directional, Fractional Diffusion Operator
In [8, 13] the following fractional diffusion operator was introduced and analyzed.

$$D_M^{2\alpha} u(x) \ := \ - \int_{\|\nu\|=1} D_\nu^{2\alpha} u(x) \, M(d\nu) \, , \tag{2.7}$$

where $M(d\nu)$ denotes a general probability measure on the unit sphere in $\mathbb{R}^{\acute{d}}$; for $\sigma > 0$,

$$D_\nu^{-\sigma} v(x) \;:=\; \frac{1}{\Gamma(\sigma)} \int_0^\infty \xi^{\sigma-1} v(x \,-\, \xi\nu)\, d\xi \quad ,$$

and for $n-1 < \beta \le n$, $\sigma = n - \beta$,

$$D_\nu^{\beta} v(x) \;:=\; (\nu \cdot \nabla)^n\, D_\nu^{-\sigma} v(x) \quad .$$

Properties (2.2),(2.3) were established in [8, 13] for $D_M^{2\alpha}$.

2.3 Examples of Operators B(·) satisfying (2.4)

Typically $B(\cdot)$ is of the form

$$B(u)(x) \;=\; \int b(x,y)\, u(y)\, dy \;. \tag{2.8}$$

For the ordinary diffusion equation the following operators have been considered. The choice

$$b(x,y) \;=\; c\,(x-y)\,|x-y|^{-\acute{d}} \tag{2.9}$$

has been used in a model for Brownian diffusion of charge carriers interacting via Coulomb forces. For $c > 0$ (2.8) has been used to model the mutual gravitational attraction of particles in a cloud [4]. For $\acute{d} = 2$, and

$$b(x,y) \;=\; (x_2 - y_2\,,\, -(x_1 - y_1))\,|x-y|^{-2} \tag{2.10}$$

the ordinary diffusion equation becomes the vorticity equation for the Navier-Stokes equations.

A general *potential kernel* for $B(\cdot)$ has the form

$$b(x,y) \;=\; c\,(x-y)\,|x-y|^{-\acute{d}+\beta-1}\;, \qquad \text{for } 0 < \beta \le \acute{d} - 1. \tag{2.11}$$

To determine the order of the operators $B(\cdot)$ defined above we have from [9]:
1. For $P(x_1, \ldots, x_{\acute{d}})$ a polynomial in $\acute{d}$ variables

$$\mathcal{F}(P(x_1, \ldots, x_{\acute{d}}))(\omega) \;=\; (2\pi)^{\acute{d}}\, P(-i\partial/\partial\omega_1, \ldots, -i\partial/\partial\omega_{\acute{d}})\,\delta(\omega) \;. \tag{2.12}$$

2. For $r = |x|$, $\rho = |\omega|$, $m \in \{0, 1, 2, \ldots\}$, c_{-1}, c_0 constants dependent on $\acute{d} + 2m$

$$\mathcal{F}(r^{-\lambda}) \;=\; 2^{\acute{d}-\lambda}\pi^{\acute{d}/2}\,\Gamma((\acute{d}-\lambda)/2)\,\rho^{\lambda-\acute{d}}\,/\,\Gamma(\lambda/2)\;, \qquad \lambda \ne \acute{d} + 2m \tag{2.13}$$

$$\mathcal{F}(r^{-\acute{d}-2m}) \;=\; \frac{1}{2}\,\Gamma(\acute{d}/2)\,\pi^{-\acute{d}/2}\,\left(c_{-1}\rho^{2m}\ln\rho \,+\, c_0\rho^{2m}\right)\;. \tag{2.14}$$

For the k^{th} component of $x/|x|^\lambda \;=\; x_k/|x|^\lambda$, $\lambda \ne \acute{d} + 2m$, combining (2.12) and (2.13), we have using the convolution property $\star$ of Fourier transforms

$$\begin{aligned}
\mathcal{F}(x_k/|x|^\lambda) \;&=\; (2\pi)^{\acute{d}}\,(-i\,\partial/\partial\omega_k)\,\delta(\omega)\,\star\,2^{\acute{d}-\lambda}\pi^{\acute{d}/2}\,\Gamma((\acute{d}-\lambda)/2)\,\rho^{\lambda-\acute{d}}\,/\,\Gamma(\lambda/2) \\
&=\; C\int (-i\,\partial/\partial\sigma_k)\;\delta(\sigma)\,|\omega-\sigma|^{\lambda-\acute{d}}\,d\sigma \\
&=\; C\,i\int \delta(\sigma)\,(\sigma_k - \omega_k)|\omega-\sigma|^{\lambda-\acute{d}-2}\,d\sigma \\
&=\; C\,i\,\omega_k\,|\omega|^{\lambda-\acute{d}-2}\;.
\end{aligned} \tag{2.15}$$

The zero extension of $f \in H_0^\gamma(\Omega)$, $\tilde{f}$, satisfies $\tilde{f} \in H^\gamma(\mathbb{R}^{\acute{d}})$. Thus, $f \in H_0^\gamma(\Omega)$ implies $|\omega|^j \mathcal{F}(\tilde{f}) \in L^2(\mathbb{R}^{\acute{d}})$, for $0 \le j \le \gamma$.
For the k^{th} component of $B(u)(x)$ defined by (2.8),(2.11) with $\beta \neq 1$, we have, as $B(\cdot)$ is a convolution operator,

$$\begin{aligned}
\int_{\mathbb{R}^{\acute{d}}} |\omega|^{2j} \, |\mathcal{F}(B(u)_k)|^2 \, d\omega &= \int_{\mathbb{R}^{\acute{d}}} |\omega|^{2j} \left| C\, \omega_k \, |\omega|^{-\beta-1} \hat{u}(\omega) \right|^2 \, d\omega \\
&\le C \int_{\mathbb{R}^{\acute{d}}} |\omega|^{2(j-\beta)} \, |\hat{u}|^2 \, (\omega) \, d\omega \; .
\end{aligned}$$

Thus, if $u \in H_0^\gamma(\Omega)$ then $(B(u))_k \in H^{\beta+\gamma}(\Omega)$ for $k = 1, \ldots \acute{d}$. Hence $B(u) \in H^{\beta+\gamma}(\Omega)$. Consequently, $B(\cdot)$ is an operator of order $-\beta$. (Also, then an operator of order 0). For $\beta = 1$, using (2.14), $B(\cdot)$ is an operator of order -1.

2.4 Hölder type inequalities for Sobolev Spaces

In this section we present a number of estimates which are useful in handling the non-linear term in the error analysis.

Lemma 1 *Let $\Omega \subset \mathbb{R}^{\acute{d}}$ be bounded, $\partial\Omega \in C^1$. Then for u and v such that the given norms are finite we have*

$$\|uv\| \le C \begin{cases} \|u\|_s \, \|v\|_{\acute{d}/2-s} \, , \; 0 < s < \acute{d}/2 \\ \|u\|_\infty \, \|v\| \\ \|u\|_s \, \|v\| \, , \; s > \acute{d}/2 \end{cases} . \tag{2.16}$$

Proof: For $z \in W^{j,p}(\Omega) \cap W^{m,r}(\Omega)$ we have the following embedding properties for Sobolev spaces ([1] Pg. 218). For $1 < r \le p < \infty$,

$$\|z\|_{W^{j,p}} \le C \, \|z\|_{W^{m,r}} \tag{2.17}$$

$$\text{where } \frac{1}{p} = \frac{j}{\acute{d}} + \frac{1}{r} - \frac{m}{\acute{d}} \, , \quad \text{and} \quad \begin{cases} j \ge 0, \;\text{if } r < p \, , & \text{or} \\ j > 0, \;\; j \text{ not an integer} \; , & \text{or} \\ j \ge 0, \;\; 1 < r \le 2 \, . \end{cases}$$

Note the above inequality (2.17) holds for $m \in \mathbb{R}$, $m > 0$. Using Hölder's inequality, with $p, \tilde{p} > 1$, satisfying $1/p + 1/\tilde{p} = 1$, and the embedding theorem

$$\begin{aligned}
\|uv\| &\le \|u\|_{L^{2p}} \, \|v\|_{L^{2\tilde{p}}} \\
&= \|u\|_{W^{0,2p}} \, \|v\|_{W^{0,2\tilde{p}}} \\
&\le C\|u\|_{W^{\acute{d}(p-1)/(2p),2}} \, \|v\|_{W^{\acute{d}/(2p),2}} \\
&= C\|u\|_{\acute{d}(p-1)/(2p)} \, \|v\|_{\acute{d}/(2p)} \, .
\end{aligned} \tag{2.18}$$

The first inequality in (2.16) follows from (2.18) with the choice $s = \acute{d}(p-1)/(2p)$. The second and third inequalities are straightforward to establish.

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Remark: The boundary regularity assumption on Ω in Lemma 1, $\partial\Omega \in C^1$, can be relaxed. The Sobolev embedding theorems on bounded domains require sufficiently regularity of the domain to enable functions defined in Ω to be appropriately extended to $\mathbb{R}^{\acute{d}}$. In particular, for the analysis presented in Sections 3 and 4 it suffices for Ω to be a Lipschitz domain.

The following results are Hölder type inequalities for Sobolev spaces.

Theorem 1 *Let $\Omega \subset \mathbb{R}^{\acute{d}}$ be bounded, $\partial\Omega \in C^1$. Then for $0 \le \alpha, \beta \le 1$, $\tilde{\epsilon} > 0$, $p > 1$, $0 < s < 1/2$, u and v such that the given norms are finite we have*

$$\|uv\|_{\alpha} \;\le\; C \begin{cases} \|u\|_1 \, \|v\|_{\alpha+\tilde{\epsilon}}, & \acute{d} = 2, \\ \|u\|_{3/2\,-\,s} \, \|v\|_{\alpha+s+\tilde{\epsilon}}, & \acute{d} = 3, \quad 0 < s \le 1/2, \end{cases} \tag{2.19}$$

$$\|uv\|_{\alpha\beta} \;\le\; C\|u\|_{\beta\,+\,\acute{d}(p-1)(1-\beta)/2p} \, \|v\|_{\acute{d}(1-\alpha+\alpha p)/(2p)\,+\,\tilde{\epsilon}} \;, \; \text{for} \begin{cases} \acute{d} = 2, & 1 < p \\ \acute{d} = 3, & 1 < p \le 3. \end{cases} \tag{2.20}$$

Proof: We have that

$$\|uv\|_1 \le \|uv\| \,+\, |uv|_1 \;, \tag{2.21}$$

and

$$|uv|_1 \le \|u\,\nabla v\| \,+\, \|\nabla u\, v\| \;.$$

Proceeding as in the proof of Lemma 1, with $q, \tilde{q} > 1$, $1/q + 1/\tilde{q} = 1$,

$$\begin{aligned} \|u\,\nabla v\| \;&\le\; \|u\|_{L^{2q}} \, \|\nabla v\|_{L^{2\tilde{q}}} \;\le\; \|u\|_{W^{0,2q}} \, \|v\|_{W^{1,2\tilde{q}}} \\ &\le\; C\|u\|_{\acute{d}(q-1)/(2q)} \, \|v\|_{(\acute{d}+2q)/(2q)} \,. \end{aligned} \tag{2.22}$$

Also, for $\epsilon > 0$

$$\|u\,\nabla v\| \;\le\; C\|u\|_{\acute{d}/2\,+\,\epsilon} \, \|v\|_1 \,. \tag{2.23}$$

Similarly, with $r > 1$

$$\|\nabla u\, v\| \;\le\; C\|u\|_{(\acute{d}+2r)/(2r)} \, \|v\|_{\acute{d}(r-1)/(2r)} \quad \text{and} \quad \|\nabla u\, v\| \;\le\; C\|u\|_1 \, \|v\|_{\acute{d}/2\,+\,\epsilon}. \tag{2.24}$$

Combining (2.21),(2.22),(2.24),(2.18), for $s > 1$, we have

$$\|uv\|_1 \;\le\; C\left(\|u\|_{\acute{d}(s-1)/(2s)} \, \|v\|_{\acute{d}/(2s)} \,+\, \|u\|_{\acute{d}(q-1)/(2q)} \, \|v\|_{(\acute{d}+2q)/(2q)} \,+\, \|u\|_{(\acute{d}+2r)/(2r)} \, \|v\|_{\acute{d}(r-1)/(2r)} \right). \tag{2.25}$$

From (2.22),(2.24)(b), and (2.25) it follows that

$$\|uv\|_1 \;\le\; C \; \|u\|_1 \, \|v\|_{\acute{d}(1+\epsilon)/2} \,, \; \acute{d} = 2, 3 \,. \tag{2.26}$$

Also, equating the norms for u in the last two terms of (2.25) we have, for s appropriately chosen,

$$\|uv\|_1 \;\le\; C \; \|u\|_{\acute{d}(q-1)/(2q)} \, \|v\|_{(\acute{d}+2q)/(2q)} \,, \; \text{for } q > 3 \,, \; \acute{d} = 3 \,. \tag{2.27}$$

Next we interpolate between spaces to obtain the stated estimates.

For u fixed, let operators T_0, T_1 be dependent on v defined by:

$$T_0 \;=\; T_1 \;=\; uv \; .$$

Using (2.26) we consider T_1 as a bounded linear operator between $H^{\acute{d}(1+\epsilon)/2}$ and H^1, with norm $\leq C\|u\|_1$. Also, using (2.18) we consider T_0 as a bounded linear operator between $H^{\acute{d}/(2p)}$ and L^2, with norm $\leq C\|u\|_{\acute{d}(p-1)/(2p)}$. By interpolation [3, 15] we obtain

$$\begin{aligned}
\|uv\|_\alpha &= \|uv\|_{[L^2,H^1]_{\alpha,2}} \\
&\leq \|T_1\|^\alpha \|T_0\|^{1-\alpha} \|v\|_{[H^{\acute{d}/(2p)},H^{\acute{d}(1+\epsilon)/2}]_{\alpha,2}} \\
&\leq C\|u\|_1^\alpha \, \|u\|^{1-\alpha}_{\acute{d}(p-1)/(2p)} \, \|v\|_{\acute{d}(1-\alpha+\alpha p)/(2p)\,+\,\tilde{\epsilon}} \\
&\leq C\|u\|_1 \, \|v\|_{\acute{d}(1-\alpha+\alpha p)/(2p)\,+\,\tilde{\epsilon}} \;, \qquad (2.28)
\end{aligned}$$

for $1 < p \leq 3$ in $\mathbb{R}^3$ (i.e. $\acute{d} = 3$), and no restriction in $\mathbb{R}^2$ (i.e. $\acute{d} = 2$). Note that $(\acute{d}(1-\alpha+\alpha p)/(2p))$ is a decreasing function of p. Minimizing with respect to p we obtain (2.19)(a) and (2.19)(b) for $s = 1/2$.

Next we interpolate with v held fixed. Consider T_0, T_1 defined above to be functions of u. Using (2.28) consider T_1 as a bounded linear operator between H^1 and H^α, with norm $\leq C\|v\|_{\acute{d}(1-\alpha+\alpha p)/(2p)\,+\,\tilde{\epsilon}}$. Using (2.18) we consider T_0 as a bounded linear operator between $H^{\acute{d}(p-1)/(2p)}$ and L^2, with norm $\leq C\|v\|_{\acute{d}/(2p)}$. By interpolation we obtain

$$\begin{aligned}
\|uv\|_{\alpha\beta} &= \|uv\|_{[L^2,H^\alpha]_{\beta,2}} \\
&\leq \|T_1\|^\beta \|T_0\|^{1-\beta} \|u\|_{[H^{\acute{d}(p-1)/(2p)},H^1]_{\beta,2}} \\
&\leq C\|v\|^\beta_{\acute{d}(1-\alpha+\alpha p)/(2p)\,+\,\tilde{\epsilon}} \, \|v\|^{1-\beta}_{\acute{d}/(2p)} \, \|u\|_{\beta\,+\,\acute{d}(p-1)(1-\beta)/(2p)} \\
&\leq C\|u\|_{\beta\,+\,\acute{d}(p-1)(1-\beta)/(2p)} \, \|v\|_{\acute{d}(1-\alpha+\alpha p)/(2p)\,+\,\tilde{\epsilon}} \, . \qquad (2.29)
\end{aligned}$$

For the case (2.19)(b), in view of (2.18), for the choice $q = p$, for $p > 3$, from (2.27)

$$\|uv\|_1 \;\leq\; C\|u\|_{\acute{d}(p-1)/(2p)} \, \|v\|_{\acute{d}/(2p)\,+\,1} \, . \qquad (2.30)$$

Interpolating, as above, with u fixed we obtain

$$\begin{aligned}
\|uv\|_\alpha &\leq C\|u\|_{\acute{d}(p-1)/(2p)} \, \|v\|_{\alpha(\acute{d}/(2p)\,+\,1)\,+\,(1-\alpha)(\acute{d}/2p)} \\
&= C\|u\|_{\acute{d}(p-1)/(2p)} \, \|v\|_{\acute{d}/(2p)\,+\,\alpha} \, . \qquad (2.31)
\end{aligned}$$

Letting $s = \acute{d}/(2p)$ in (2.31) the stated result follows.

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Also used below is the following lemma.

Lemma 2 *For $\Omega \subset \mathbb{R}^{\acute{d}}$, $\alpha > \acute{d}/4$, $v,\, w \,\in\, X$, $\epsilon > 0$, there exists $C > 0$ such that*

$$(v\,B(w)\,,\,\nabla v) \leq C\frac{(q\,\epsilon)^{-p/q}}{p}\|\nabla\cdot B(w)\|^p\|v\|^2 \,+\, \epsilon\,\|v\|^2_\alpha\,, \qquad (2.32)$$

where $p = 4\alpha/(4\alpha - \acute{d})$, $q = 4\alpha/\acute{d}$.

Proof: We begin by rewriting the inner product as

$$\begin{aligned}(v\,B(w)\,,\,\nabla v) &= \frac{1}{2}\left(B(w)\,,\,\nabla v^2\right) \;=\; -\frac{1}{2}\left(\nabla\cdot B(w)\,,\,v^2\right) \\ &\leq \frac{1}{2}\,\|v\|_{L^4}^2\,\|\nabla\cdot B(w)\|\,.\end{aligned}$$

For $\Omega\subset\mathbb{R}^{\acute{d}}$, $H^{\acute{d}/4}(\Omega)$ is continuously imbedded in $L^4(\Omega)$ (cf. (2.17)), and as an interpolation space

$$H^{\acute{d}/4}(\Omega) \;=\; \left[L^2\,,\,H^{\alpha}\right]_{\frac{\acute{d}}{4\alpha},2}\,.$$

Hence,

$$\begin{aligned}\|v\|_{L^4}^2 &\leq C\,\|v\|_{H^{\acute{d}/4}}^2 \;\leq\; C\left(\|v\|^{1-\acute{d}/4\alpha}\,\|v\|_{\alpha}^{\acute{d}/4\alpha}\right)^2 \\ &\leq C\|v\|^{2-\acute{d}/2\alpha}\,\|v\|_{\alpha}^{\acute{d}/2\alpha}\,.\end{aligned}$$

Thus,

$$\|v\|_{L^4}^2\,\|\nabla\cdot B(w)\| \leq C\,\|v\|^{2-\acute{d}/2\alpha}\,\|\nabla\cdot B(w)\|\,\|v\|_{\alpha}^{\acute{d}/2\alpha}\,. \tag{2.33}$$

Applying Young's inequality: $ab\leq|a|^p/p+|b|^q/q$, for $1/p+1/q=1$, with the choice $p=4\alpha/(4\alpha-\acute{d})$, $q=4\alpha/\acute{d}$, the stated results (2.32) follows.

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3 Finite Element Approximation

In this section we formulate a fully discrete finite element method for (1.1)-(1.3). The estimates presented in Theorem 1 and Lemma 2 depend upon the spatial dimension. In order to avoid the added notational complexity of simultaneously deriving estimates for $\mathbb{R}^2$ and $\mathbb{R}^3$, we focus on the case $\Omega\subset\mathbb{R}^2$. Estimates for $\Omega\subset\mathbb{R}^3$ can analogously be derived.

We begin by describing the finite element approximation framework and listing the approximating properties and inverse estimates used in the analysis.

Let $\Omega\subset\mathbb{R}^2$ be a polygonal domain and let T_h be a triangulation of Ω made of triangles. Thus, the computational domain is defined by

$$\Omega=\bigcup K;\quad K\in T_h.$$

We assume that there exist constants c_1,c_2 such that

$$c_1 h\leq h_K\leq c_2\rho_K\,,$$

where h_K is the diameter of triangle K, ρ_K is the diameter of the greatest ball included in K, and $h=\max_{K\in T_h}h_K$. For $k\in\mathbb{N}$, let $P_k(A)$ denote the space of polynomials on A of degree no greater than k. Then we define the finite element space X_h as follows.

$$X_h:=\left\{v\in X\cap C(\bar{\Omega}):v|_K\in P_k(K),\ \forall K\in T_h\right\}. \tag{3.1}$$

We summarize several properties of finite element spaces and Sobolev spaces which we will use in our subsequent analysis. For $w \in H^{k+1}(\Omega)$ we have (see [10]) that there exists $\mathcal{W} \in X_h$ such that

$$\|w - \mathcal{W}\| \; + \; h\|\nabla(w - \mathcal{W})\| \; \leq \; C_I h^{k+1} \|w\|_{k+1} \; , \tag{3.2}$$

Lemma 3 *[5] Let $\{T_h\}$, $0 < h \leq 1$, denote a quasi-uniform family of subdivisions of a polyhedral domain $\Omega \subset \mathbb{R}^{\acute{d}}$. Let $(\hat{K}, P, N)$ be a reference finite element such that $P \subset W^{l,p}(\hat{K}) \cap W^{m,q}(\hat{K})$ is a finite-dimensional space of functions on $\hat{K}$, N is a basis for P', where $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $0 \leq m \leq l$. For $K \in T_h$, let (K, P_K, N_K) be the affine equivalent element, and $V_h = \{v : v$ is measurable and $v|_K \in P_K, \forall K \in T_h\}$. Then there exists $C = C(l,p,q)$ such that*

$$\left[\sum_{K \in T_h} \|v\|^p_{W^{l,p}(K)} \right]^{1/p} \; \leq \; C \, h^{m-l+\min(0\,,\,\frac{\acute{d}}{p} - \frac{\acute{d}}{q})} \left[\sum_{K \in T_h} \|v\|^q_{W^{m,q}(K)} \right]^{1/q} \; , \tag{3.3}$$

for all $v \in V_h$.

■

Let Δt denote the step size for t so that $t_n = n\Delta t$, $n = 0, 1, 2, \ldots, N$. For notational convenience, we denote $v^n := v(\cdot, t_n)$, and

$$d_t f^n \; := \; \frac{f(t_n) - f(t_{n-1})}{\Delta t} \, . \tag{3.4}$$

The following norms are also used in the analysis:

$$\begin{aligned} |||v|||_{\infty,k} \;\; &:= \;\; \max_{1 \leq n \leq N} \|v^n\|_k \\ |||v|||_{0,k} \;\; &:= \;\; \left[\sum_{n=1}^{N} \|v^n\|^2_k \Delta t \right]^{1/2} . \end{aligned}$$

Approximating System

For $n = 1, 2, \ldots, N$, find $u_h^n \in X_h$ such that

$$(d_t u_h^n, v) \; + \; \langle \mathcal{D}^{2\alpha} u_h^n \, , \, v \rangle \; + \; (u_h^n B(u_h^{n-1}) \, , \, \nabla v) \; = \; (f^n, v) \; , \quad \forall \, v \in X_h \, . \tag{3.5}$$

For notational convenience we define $A(w; u, v)$ as

$$A(w; u, v) := \langle \mathcal{D}^{2\alpha} u \, , \, v \rangle \; + \; (uB(w) \, , \, \nabla v) \, . \tag{3.6}$$

Then, the linear system of equations (3.5) can be written equivalently as

$$(d_t u_h^n, v) \; + \; A(u_h^{n-1}; u_h^n, v) \; = \; (f^n, v) \; , \quad \forall \, v \in X_h \, . \tag{3.7}$$

To ensure computability of the algorithm, we begin by showing that (3.5) is uniquely solvable for u_h^n at each time step n. We use the following induction hypothesis which simply states that the computed iterates u_h^n are bounded independent of h and n.

$$\textbf{(IH1)} \qquad \|u_h^j\|_1 \leq \mathcal{K} \, , \quad j = 0, \ldots, n-1. \tag{3.8}$$

Lemma 4 *Assume that **(IH1)**, i.e.* $\|u_h^j\|_1 \le \mathcal{K}$ *for* $j = 0, 1, \ldots, n-1$*. For a sufficiently small step size* Δt*, there exists a unique solution* $u_h^n \in X_h$ *satisfying (3.5).*

Proof: As (3.5) represents a finite system of linear equations, positivity of $(u_h^n, u_h^n)/\Delta t + A(u_h^{n-1}; u_h^n, u_h^n)$ is a sufficient condition for the existence and uniqueness of u_h^n.

We have, using (2.3) and (2.32),

$$
\begin{aligned}
(u_h^n, u_h^n)/\Delta t \;+\; A(u_h^{n-1}; u_h^n, u_h^n) \;&=\; \frac{1}{\Delta t}\|u_h^n\|^2 \,+\, \langle \mathcal{D}^{2\alpha} u_h^n \,,\, u_h^n \rangle \,+\, (u_h^n B(u_h^{n-1}) \,,\, \nabla u_h^n) \\
&\ge\; \frac{1}{\Delta t}\|u_h^n\|^2 \,+\, C_c \|u_h^n\|_\alpha^2 \,-\, C_1 \epsilon_2^{-C_2} \|\nabla \cdot B(u_h^{n-1})\|^{C_3} \|u_h^n\|^2 \,-\, \epsilon_2 \|u_h^n\|_\alpha^2 \\
&=\; \left(\frac{1}{\Delta t} \,-\, C_1 \epsilon_2^{-C_2} \|\nabla \cdot B(u_h^{n-1})\|^{C_3} \right) \|u_h^n\|^2 \,+\, (C_c \,-\, \epsilon_2)\, \|u_h^n\|_\alpha^2 \quad (3.9) \\
&\ge\; \left(\frac{1}{\Delta t} \,-\, \tilde{C}_1 \epsilon_2^{-C_2} \|B(u_h^{n-1})\|_1^{C_3} \right) \|u_h^n\|^2 \,+\, (C_c \,-\, \epsilon_2)\, \|u_h^n\|_\alpha^2 \quad (3.10) \\
&\ge\; \left(\frac{1}{\Delta t} \,-\, \tilde{C}_1 \epsilon_2^{-C_2} C_B^{C_3} \|u_h^{n-1}\|_1^{C_3} \right) \|u_h^n\|^2 \,+\, (C_c \,-\, \epsilon_2)\, \|u_h^n\|_\alpha^2 \quad (3.11) \\
&\ge\; \left(\frac{1}{\Delta t} \,-\, \tilde{C}_1 \epsilon_2^{-C_2} C_B^{C_3} \mathcal{K}^{C_3} \right) \|u_h^n\|^2 \,+\, (C_c \,-\, \epsilon_2)\, \|u_h^n\|_\alpha^2 \,. \quad (3.12)
\end{aligned}
$$

Hence, for Δt chosen sufficiently small we have that (3.5) is uniquely solvable for u_h^n.

■

The discrete Gronwall's lemma plays an important role in the following analysis.

Lemma 5 (Discrete Gronwall's Lemma) *[11] Let* Δt*,* H*, and* a_n*,* b_n*,* c_n*,* γ_n *(for integers* $n \ge 0$*) be nonnegative numbers such that*

$$
a_l \,+\, \Delta t \sum_{n=0}^{l} b_n \;\le\; \Delta t \sum_{n=0}^{l} \gamma_n \, a_n \;+\; \Delta t \sum_{n=0}^{l} c_n \;+\; H \quad \text{for } l \ge 0 \,.
$$

Suppose that $\Delta t \, \gamma_n < 1$*, for all* n*, and set* $\sigma_n = (1 - \Delta t \, \gamma_n)^{-1}$*. Then,*

$$
a_l \,+\, \Delta t \sum_{n=0}^{l} b_n \;\le\; \exp\left(\Delta t \sum_{n=0}^{l} \sigma_n \, \gamma_n \right) \left\{ \Delta t \sum_{n=0}^{l} c_n \,+\, H \right\} \quad \text{for } l \ge 0 \,. \qquad (3.13)
$$

4 A Priori Error Estimate

In this section we analyze the error between the finite element approximation given by (3.5) and the true solution. A priori error estimates for the approximation are given in Theorem 2.

Theorem 2 *Assume that (1.1)-(1.3) has a solution* u *satisfying* $u_t \in L^2(0,T; H^{k+1}(\Omega))$*,* $u_{tt} \in L^2(0,T; L^2(\Omega))$*, with* $u^0 \in H^{k+1}(\Omega)$*. In addition assume that* $\Delta t \le c\,h$*. Then, the finite element* *approximation (3.5) is convergent to the solution of (1.1)-(1.3) on the interval* $(0,T)$ *as* $\Delta t,\ h \to 0$. *The approximation* u_h *satisfies the following error estimates:*

$$\| u - u_h \|_{0,\alpha} \ \leq \ C \left(h^{k+1} \|u_t\|_{0,k+1} \ + \ h^{(k+1-\alpha)} \, \| u \|_{0,k+1} \ + \ \Delta t \, \| u_t \|_{0,1} \ + \ \Delta t \| u_{tt} \|_{0,0} \right) \quad (4.1)$$

$$\| u - u_h \|_{\infty,0} \ \leq \ C \left(h^{k+1} \|u_t\|_{0,k+1} \ + \ h^{(k+1-\alpha)} \, \| u \|_{0,k+1} \ + \ \Delta t \, \| u_t \|_{0,1} \ + \ \Delta t \| u_{tt} \|_{0,0} \right.$$
$$\left. + \, h^{k+1} \, \| u \|_{\infty,k+1} \right) . \quad (4.2)$$

Remarks: 1. $u_t \in L^2(0,T;H^{k+1}(\Omega))$, $u_0 \in H^{k+1}(\Omega)$ implies that $u \in L^2(0,T;H^{k+1}(\Omega)) \cap L^\infty(0,T;H^{k+1}(\Omega))$.
2. As previously defined in (3.1), k is the polynomial order of the approximation functions u_h^n.

In order to establish the estimates (4.1),(4.2), we begin by introducing the following notation. Let $u^n = u(t_n)$ represent the solution of (1.1)-(1.3), and u_h^n denote the solution of (3.5).

For $\mathcal{U}^n \in X_h$, define $\Lambda^n, E^n, \epsilon_u$, as

$$\Lambda^n = u^n - \mathcal{U}^n, \qquad E^n = \mathcal{U}^n - u_h^n, \qquad \epsilon_u = u^n - u_h^n \, .$$

The proof of Theorem 2 is established in three steps.

1. Prove a lemma, assuming the induction hypothesis.
2. Show that the induction hypothesis is true.
3. Prove the error estimates given in (4.1),(4.2).

Step 1. We prove the following lemma.

Lemma 6 *Under the induction hypothesis* $\|u_h^j\|_1 \leq \mathcal{K}$ *for* $j = 0, 1, \ldots, l-1$*, we have that*

$$\|E^l\|^2 \leq G(\Delta t, h), \quad (4.3)$$

where

$$G(\Delta t, h) \ = \ C \left(h^{2(k+1)} \|u_t\|_{0,k+1}^2 \ + \ h^{2(k+1-\alpha)} \, \| u \|_{0,k+1}^2 \ + \ (\Delta t)^2 \, \| u_t \|_{0,1}^2 \ + \ (\Delta t)^2 \| u_{tt} \|_{0,0}^2 \right) .$$

Proof of Lemma 6: From (1.1),(1.2) we have that the true solution u satisfies

$$(d_t u^n, v) \ + \ \langle \mathcal{D}^{2\alpha} u^n \, , \, v \rangle \ + \ (u^n B(u_h^{n-1}) \, , \, \nabla v) \ = \ (f^n, v) \ - \ (u_t - d_t u^n, v)$$
$$- \, (u^n B(u^n - u_h^{n-1}) \, , \, \nabla v), \ \ v \in X_h \, . \quad (4.4)$$

Subtracting (3.5) from (4.4) we obtain the following equation for ϵ_u:

$$(d_t \epsilon_u, v) \ + \ \langle \mathcal{D}^{2\alpha} \epsilon_u \, , \, v \rangle \ + \ (\epsilon_u B(u_h^{n-1}) \, , \, \nabla v) \ = \ - \, (u_t - d_t u^n, v)$$
$$- \, (u^n B(u^n - u_h^{n-1}) \, , \, \nabla v), \ \ v \in X_h \, . \quad (4.5)$$

Substituting $\epsilon_u = E^n + \Lambda^n$, $v = E^n$ into (4.5), we obtain

$$(d_t E^n, E^n) \;+\; \langle \mathcal{D}^{2\alpha} E^n \,,\, E^n \rangle \;+\; (E^n B(u_h^{n-1}) \,,\, \nabla E^n) \;=\; F(E^n)\,, \tag{4.6}$$

where,

$$\begin{aligned} F(E^n) \;&:=\; -(d_t \Lambda^n, E^n) \;-\; \langle \mathcal{D}^{2\alpha} \Lambda^n \,,\, E^n \rangle \;-\; (\Lambda^n B(u_h^{n-1}) \,,\, \nabla E^n) \\ &\quad\; -(u_t - d_t u^n, E^n) \;-\; (u^n B(u^n - u_h^{n-1}) \,,\, \nabla E^n)\,. \end{aligned}$$

Note that

$$\begin{aligned} (d_t E^n, E^n) \;&=\; \frac{1}{\Delta t}\big((E^n, E^n) - (E^{n-1}, E^n)\big) \\ &\geq\; \frac{1}{\Delta t}\big(\|E^n\|^2 \;-\; \|E^n\|\,\|E^{n-1}\|\big) \\ &\geq\; \frac{1}{2\Delta t}\left(\|E^n\|^2 - \|E^{n-1}\|^2\right)\,, \end{aligned}$$

and from (2.32)

$$(E^n B(u_h^{n-1}) \,,\, \nabla E^n) \leq \epsilon_2 \,\|E^n\|_\alpha^2 \;+\; C_1 \epsilon_2^{-C_2} \|\nabla \cdot B(u_h^{n-1})\|^{C_3} \|E^n\|^2\,. \tag{4.7}$$

Multiplying (4.6) by $2\,\Delta t$, summing from $n = 1$ to l, and using (2.3) we have:

$$\begin{aligned} &\Big(\|E^l\|^2 - \|E^0\|^2\Big) \;+\; 2\,(C_c - \epsilon_2) \sum_{n=1}^{l} \Delta t \|E^n\|_\alpha^2 \\ &\qquad\leq\; 2\Delta t \sum_{n=1}^{l} C_1 \epsilon_2^{-C_2} \|\nabla \cdot B(u_h^{n-1})\|^{C_3} \|E^n\|^2 \;+\; 2\Delta t \sum_{n=1}^{l} F(E^n)\,. \end{aligned} \tag{4.8}$$

We now estimate each term in F(E^n).

$$\begin{aligned} (d_t \Lambda^n, E^n) \;&\leq\; \|E^n\|\,\|d_t \Lambda^n\| \\ &\leq\; \frac{1}{2}\|E^n\|^2 \;+\; \frac{1}{2}\|d_t \Lambda^n\|^2\,. \end{aligned} \tag{4.9}$$

Using (2.2)

$$\begin{aligned} \langle \mathcal{D}^{2\alpha} \Lambda^n \,,\, E^n \rangle \;&\leq\; C_t \|E^n\|_\alpha \,\|\Lambda^n\|_\alpha \\ &\leq\; \epsilon_4 \|E^n\|_\alpha^2 \;+\; \frac{C_t^2}{4\epsilon_4}\|\Lambda^n\|_\alpha^2\,. \end{aligned} \tag{4.10}$$

Using duality with respect to the L^2 inner-product

$$\begin{aligned} (\Lambda^n B(u_h^{n-1}) \,,\, \nabla E^n) \;&\leq\; \|\nabla E^n\|_{-(1-\alpha)} \,\|\Lambda^n B(u_h^{n-1})\|_{(1-\alpha)} \\ &\leq\; \epsilon_5 \|E^n\|_\alpha^2 \;+\; \frac{C_4}{4\epsilon_5}\|\Lambda^n B(u_h^{n-1})\|_{(1-\alpha)}^2\,. \end{aligned} \tag{4.11}$$

For the next term in $F(E^n)$ we use

$$(u_t - d_t u^n, E^n) \;\leq\; \|E^n\|\,\|u_t - d_t u^n\| \;\leq\; \frac{1}{2}\|E^n\|^2 \;+\; \frac{1}{2}\|u_t - d_t u^n\|^2\,. \tag{4.12}$$

The remaining term is rewritten as the sum of three terms.

$$
\begin{aligned}
(u^n B(u^n - u_h^{n-1})\,,\,\nabla E^n) &= (u^n B(u^n - u^{n-1})\,,\,\nabla E^n) \,+\, (u^n B(u^{n-1} - u_h^{n-1})\,,\,\nabla E^n) \\
&= (u^n B(u^n - u^{n-1})\,,\,\nabla E^n) \,+\, (u^n B(\Lambda^{n-1})\,,\,\nabla E^n) \\
&\quad + (u^n B(E^{n-1})\,,\,\nabla E^n)\ .
\end{aligned}
$$

Each of these terms are rewritten in a similar fashion as in (4.11).

$$
\begin{aligned}
(u^n B(u^n - u^{n-1})\,,\,\nabla E^n) &\le \epsilon_7 \|E^n\|_\alpha^2 \,+\, \frac{C_4}{4\epsilon_7}\,\|u^n B(u^n - u^{n-1})\|_{(1-\alpha)}^2\,, && (4.13)\\
(u^n B(\Lambda^{n-1})\,,\,\nabla E^n) &\le \epsilon_8 \|E^n\|_\alpha^2 \,+\, \frac{C_4}{4\epsilon_8}\,\|u^n B(\Lambda^{n-1})\|_{(1-\alpha)}^2\,, && (4.14)\\
(u^n B(E^{n-1})\,,\,\nabla E^n) &\le \epsilon_9 \|E^n\|_\alpha^2 \,+\, \frac{C_4}{4\epsilon_9}\,\|u^n B(E^{n-1})\|_{(1-\alpha)}^2\,. && (4.15)
\end{aligned}
$$

Combining (4.8)-(4.15), for $\epsilon_1, \ldots, \epsilon_9$ appropriately chosen, there exist constants $C_j > 0$ such that

$$
\begin{aligned}
\left(\|E^l\|^2 - \|E^0\|^2\right) \;+\; & C_{12} \sum_{n=1}^{l} \Delta t \|E^n\|_\alpha^2 \\
&\le \Delta t \sum_{n=1}^{l} \left(C_{13}\|\nabla\cdot B(u_h^{n-1})\|^{C_3} + C_{14}\right)\|E^n\|^2 \\
&+ \Delta t \sum_{n=1}^{l} C_{15}\|u^n B(E^{n-1})\|_{(1-\alpha)}^2 \\
&+ \Delta t \sum_{n=1}^{l} C_{16}\|d_t\Lambda^n\|^2 \,+\, \Delta t \sum_{n=1}^{l} C_{17}\|\Lambda^n\|_\alpha^2 \\
&+ \Delta t \sum_{n=1}^{l} C_{18}\|\Lambda^n B(u_h^{n-1})\|_{(1-\alpha)}^2 \\
&+ \Delta t \sum_{n=1}^{l} C_{19}\|u^n B(u^n - u^{n-1})\|_{(1-\alpha)}^2 \\
&+ \Delta t \sum_{n=1}^{l} C_{20}\|u^n B(\Lambda^{n-1})\|_{(1-\alpha)}^2 \\
&+ \Delta t \sum_{n=1}^{l} C_{21}\|u_t - d_t u^n\|^2\,. \qquad (4.16)
\end{aligned}
$$

We now apply the interpolation property of the approximation space to estimate the terms on the right hand side of (4.16).

$$
\sum_{n=1}^{l} \Delta t \|d_t\Lambda^n\|^2 \;=\; \sum_{n=1}^{l} \Delta t \left\| \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} 1\,\frac{\partial\Lambda}{\partial t}\,dt \right\|^2
$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{l} \Delta t \left(\frac{1}{\Delta t}\right)^2 \int_\Omega \left(\int_{t_{n-1}}^{t_n} 1\, dt\right)\left(\int_{t_{n-1}}^{t_n} \left(\frac{\partial \Lambda}{\partial t}\right)^2 dt\right) dx \\
&\leq Ch^{2k+2}\|u_t\|_{0,k+1}^2 \, . \qquad (4.17)
\end{aligned}
$$

Note that $(d_t u^n - u_t^n)$ may be expressed as

$$
d_t u^n - u_t^n = \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u_{tt}(\cdot, t)(t_{n-1} - t)\, dt\, .
$$

Also,

$$
\begin{aligned}
\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} u_{tt}(\cdot, t)(t_{n-1} - t)\, dt\right)^2 &\leq \frac{1}{(\Delta t)^2}\int_{t_{n-1}}^{t_n} u_{tt}^2(\cdot, t)\, dt \int_{t_{n-1}}^{t_n} (t_{n-1} - t)^2\, dt \\
&= \frac{1}{3}\Delta t \int_{t_{n-1}}^{t_n} u_{tt}^2(\cdot, t)\, dt.
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
\sum_{n=1}^{l} \Delta t\, \|u_t - d_t u^n\|^2 &\leq \sum_{n=1}^{l} \Delta t \int_\Omega \frac{1}{3}\,\Delta t \int_{t_{n-1}}^{t_n} u_{tt}^2(\cdot, t)\, dt\, dx \\
&= \frac{1}{3}\,(\Delta t)^2 \|u_{tt}\|_{0,0}^2\, . \qquad (4.18)
\end{aligned}
$$

Next we estimate the terms in (4.16) involving $B(\cdot)$ using (2.4),(2.19) and (3.8). These estimates for $B(\cdot)$ are dimension specific (i.e. $\acute{d} = 2$).

For the first term on the RHS,

$$
\|\nabla \cdot B(u_h^{n-1})\| \; \leq \; C\|B(u_h^{n-1})\|_1 \; \leq \; C\|u_h^{n-1}\|_1 \; \leq \; C\mathcal{K}\, . \qquad (4.19)
$$

Using interpolation between L^2 and H^α, and Young's inequality we obtain, for $\delta \in (0, 2\alpha - 1)$,

$$
\begin{aligned}
\|u^n B(E^{n-1})\|_{(1-\alpha)}^2 &\leq C\, \|B(E^{n-1})\|_{(1-\alpha+\delta)}^2\, \|u^n\|_1^2 \\
&\leq C\, \|E^{n-1}\|_{(1-\alpha+\delta)}^2\, \|u^n\|_1^2 \\
&\leq C\|E^{n-1}\|^{2(2\alpha-1-\delta)/\alpha}\, \|E^{n-1}\|_\alpha^{2(1-\alpha+\delta)/\alpha}\, \|u^n\|_1^2 \\
&\leq \epsilon_{10}\|E^{n-1}\|_\alpha^2 \, + \, C\, \|u^n\|_1^{2(\alpha/(2\alpha-1-\delta))}\|E^{n-1}\|^2\, . \qquad (4.20)
\end{aligned}
$$

With the interpolation error bound $\|\Lambda^n\|_{(1-\alpha+\delta)} \leq Ch^{k+\alpha-\delta}\, \|u^n\|_{k+1}$,

$$
\begin{aligned}
\|\Lambda^n B(u_h^{n-1})\|_{(1-\alpha)} &\leq C\, \|\Lambda^n\|_{(1-\alpha+\delta)}\|B(u_h^{n-1})\|_1 \\
&\leq C\|\Lambda^n\|_{(1-\alpha+\delta)}\, \|u_h^{n-1}\|_1 \\
&\leq C\mathcal{K}h^{k+\alpha-\delta}\, \|u^n\|_{k+1}\, . \qquad (4.21)
\end{aligned}
$$

To estimate $\|u^n B(\Lambda^{n-1})\|_{(1-\alpha)}$ we proceed similarly.

$$
\begin{aligned}
\|u^n B(\Lambda^{n-1})\|_{(1-\alpha)} &\leq C\, \|u^n\|_1\, \|B(\Lambda^{n-1})\|_{(1-\alpha+\delta)} \\
&\leq C\, \|u^n\|_1\, \|\Lambda^{n-1}\|_{(1-\alpha+\delta)} \\
&\leq Ch^{k+\alpha-\delta}\, \|u^{n-1}\|_{k+1}\, \|u^n\|_1. \qquad (4.22)
\end{aligned}
$$

Using,

$$\|u^n - u^{n-1}\|_1^2 \ \leq \ \Delta t \int_{t_{n-1}}^{t_n} \|u_t\|_1^2 \, dt \, ,$$

we have that

$$\begin{aligned}
\|u^n B(u^n - u^{n-1})\|_{(1-\alpha)}^2 \ &\leq \ C \, \|u^n\|_{(1-\alpha+\delta)}^2 \, \|B(u^n - u^{n-1})\|_1^2 \\
&\leq \ C \|u^n\|_{(1-\alpha+\delta)}^2 \, \|u^n - u^{n-1}\|_1^2 \\
&\leq \ C \Delta t \, \|u^n\|_{(1-\alpha+\delta)}^2 \int_{t_{n-1}}^{t_n} \|u_t\|_1^2 \, dt \, . \qquad (4.23)
\end{aligned}$$

From (4.17)-(4.23), and $\|E^0\| = 0$, estimate (4.16) becomes (using $\|u\|_1$ is bounded for $t \in [0,T]$)

$$\begin{aligned}
\|E^l\|^2 \, + \, C_{12} \sum_{n=1}^{l} \Delta t \|E^n\|_\alpha^2 \ \leq \ & \Delta t \sum_{n=1}^{l} C_{22} \|E^n\|^2 \\
& + \, C h^{2(k+1)} \|u_t\|_{0,k+1}^2 \, + \, C h^{2(k+1-\alpha)} \, \||u\||_{0,k+1}^2 \\
& + C (\Delta t)^2 \, \|u_t\|_{0,1}^2 \\
& + C \, h^{2(k+\alpha-\delta)} \, \||u\||_{0,k+1}^2 \\
& + \, C \, (\Delta t)^2 \|u_{tt}\|_{0,0}^2 \, . \qquad (4.24)
\end{aligned}$$

Finally, as $\alpha > 0.5$; with $\Delta t < 1/C_{22}$, and the associations $a_l = \|E^l\|^2$, $b_n = C_{12}\|E^n\|_\alpha^2$, $\gamma_n = C_{22}$, $c_n = 0$, $H = \tilde{C} \left(h^{2(k+1)} \|u_t\|_{0,k+1}^2 \, + \, h^{2(k+1-\alpha)} \, \||u\||_{0,k+1}^2 \, + \, (\Delta t)^2 \, \|u_t\|_{0,1}^2 \, + \, (\Delta t)^2 \|u_{tt}\|_{0,0}^2 \right)$, applying Gronwall's lemma we obtain the bound given in (4.3) where $C = \tilde{C} \exp(T C_{22}/(1 - \Delta t C_{22}))$.

■

Step 2. We show that the induction hypothesis (**IH1**) is true.
Assume that $\|u_h^j\|_1 \leq \mathcal{K}$ for $j = 0, 1, \ldots, l-1$. Using the interpolation property, inverse estimate (3.3), and (4.3), we have that

$$\begin{aligned}
\|\nabla u_h^l\| \ &\leq \ \|\nabla (u_h^l - u^l)\| \, + \, \|\nabla u^l\| \\
&\leq \ \|\nabla E^l\| \, + \|\nabla \Lambda^l\| \, + \, \|\nabla u^l\| \\
&\leq \ C \left(h^{-1} \|E^l\| \, + \, \|\nabla u^l\| \right) \\
&\leq \ C h^{-1} (h^{k+1-\alpha} + \Delta t) + \, C \|\nabla u^l\| \, . \qquad (4.25)
\end{aligned}$$

Thus as C is independent of l, $u \in L^\infty(0,T; H^1(\Omega))$, for $\Delta t \leq ch$, we have that $\|\nabla u_h^l\|$ is bounded. An analogous argument shows that $\|u_h^l\|$ is also bounded independent of h and l.

■

Step 3. We derive the error estimates in (4.1) and (4.2).
Proof of the Theorem 2.
To establish (4.1), from (4.24) and (4.3), and using $T = N \Delta t$,

$$\||E\||_{0,\alpha}^2 \ = \ \sum_{n=1}^{N} \Delta t \, \|E^n\|_\alpha^2 \ \leq \ C \, (T+1) G(\Delta t, h) \, .$$

Hence, using the interpolation property and that

$$|||u - u_h|||_{0,\alpha} \; \leq \; |||E|||_{0,\alpha} \, + \, |||\Lambda|||_{0,\alpha}$$

estimate (4.1) then follows.

Using estimates (4.3) and approximation properties, we have

$$\begin{aligned} |||u - u_h|||_{\infty,0}^2 \;&\leq\; |||E|||_{\infty,0}^2 + |||\Lambda|||_{\infty,0}^2 \\ &\leq\; G(\Delta t, h) + h^{2k+2} \, |||u|||_{\infty,k+1}^2 \ , \end{aligned}$$

which yields estimate (4.2).

■

5 Numerical Results

In this section we illustrate the predicted convergence results given in Theorem 2 with numerical computations for $\Omega \subset \mathbb{R}^2$. For points $x, \; y \in \mathbb{R}^2$ we use $x = (x_1, x_2)$ and $y = (y_1, y_2)$. For ease of notation, for $\theta \in [0, 2\pi)$ we let $D_\theta^{-\sigma} u := D_\nu^{-\sigma} u$, where $\nu = [\cos\theta, \sin\theta]^T$.

The fractional differential operator used in our computations was (see 2.7)

$$D_M^{2\alpha} u(x) \; := \; -\frac{1}{\pi} \int_{\theta=0}^{2\pi} D_\theta^{2\alpha} u(x) \, d\theta \ , \tag{5.1}$$

which we approximate as:

$$D_M^{2\alpha} u(x) \; \approx \; -\frac{1}{2} D_0^{2\alpha} u(x) \; - \frac{1}{2} D_{\pi/2}^{2\alpha} u(x) \; - \; \frac{1}{2} D_\pi^{2\alpha} u(x) \; - \frac{1}{2} D_{3\pi/2}^{2\alpha} u(x) \ . \tag{5.2}$$

The value of α we used was $\alpha = 0.75$.

The approximation space X_h was taken to be the space of continuous piecewise linear functions, i.e. $k = 1$.

For a discussion on the implementation of the FEM approximation for the fractional diffusion operator (5.1) in $\mathbb{R}^2$ see [14].

From Theorem 2, (4.1),(4.2), we have the predicted rates of convergence for $\Delta t = C h^{k+1-\alpha}$ ($= C h^{1.25}$ for $k = 1, \; \alpha = 0.75$) of

$$|||u - u_h|||_{0,\alpha} \; \sim O(h^{1.25}) \;\; , \;\; |||u - u_h|||_{\infty,0} \; \sim O(h^{1.25}) \ . \tag{5.3}$$

In Tables 5.1–5.6 we give the results for $|||u - u_h|||_{0,0}$ which from (4.1) and (5.3) are predicted to satisfy

$$|||u - u_h|||_{0,0} \; \sim O(h^{1.25}) \;\; .$$

For comparison, computations were also performed with the usual diffusion operator in place of $D_M^{2\alpha} u$, namely on the equation

$$u_t \; - \; \Delta u \; - \; \nabla \cdot (u \, B(u)) \; = \; f(x) \ . \tag{5.4}$$

For the usual diffusion operator Δt was chosen as $\Delta t = Ch$. From Theorem 2, the predicted rate of convergence is then

$$|||u - u_h|||_{0,0} \sim O(h) \quad , \quad |||u - u_h|||_{\infty,0} \sim O(h) \ . \tag{5.5}$$

Example 1: For the problem described in (1.1)–(1.3) we take $\Omega = (0,1) \times (0,1)$, and a known solution $u(x_1, x_2, t) = (4t^2 - 4t + 1)(x_1 - x_1^2)(x_2 - x_2^2)$, with $u^0(x_1, x_2) = u(x_1, x_2, 0)$. The RHS of (1.1) was computed using the true solution and the approximation to $D_M^{2\alpha} u(x)$ given in (5.2).

Computations were performed for $B(u)$ given by (2.8) with
(a) $b(x, y) = 0$ i.e. $B(u) = 0$, (see Tables 5.1, 5.4),
(b) $b(x, y) = x - y$ i.e. a smooth operator B, (see Tables 5.2, 5.5),
(c) $b(x, y) = (x - y)/|x - y|$, (see Tables 5.3, 5.6).

The results presented in Tables 5.1–5.6 are consistent with those predicted by Theorem 2, given in (5.3).

h	$\|\|\|u - u_h\|\|\|_{\infty,0}$	cvge. rate	$\|\|\|u - u_h\|\|\|_{0,0}$	cvge. rate
1/4	$7.44283 \cdot 10^{-3}$		$5.735602 \cdot 10^{-3}$	
1/8	$2.991413 \cdot 10^{-3}$	1.32	$2.281621 \cdot 10^{-3}$	1.33
1/12	$1.784701 \cdot 10^{-3}$	1.27	$1.365371 \cdot 10^{-3}$	1.27
1/16	$1.232144 \cdot 10^{-3}$	1.29	$9.449112 \cdot 10^{-4}$	1.28
1/20	$9.411762 \cdot 10^{-4}$	1.21	$7.232338 \cdot 10^{-4}$	1.20
1/24	$7.470209 \cdot 10^{-4}$	1.27	$5.748151 \cdot 10^{-4}$	1.26

Table 5.1: Experimental error results for Example 1 for the fractional diffusion operator and no B term.

h	$\|\|\|u - u_h\|\|\|_{\infty,0}$	cvge. rate	$\|\|\|u - u_h\|\|\|_{0,0}$	cvge. rate
1/4	$7.160147 \cdot 10^{-3}$		$5.603630 \cdot 10^{-3}$	
1/8	$2.907621 \cdot 10^{-3}$	1.30	$2.242255 \cdot 10^{-3}$	1.32
1/12	$1.729328 \cdot 10^{-3}$	1.28	$1.336439 \cdot 10^{-3}$	1.28
1/16	$1.185318 \cdot 10^{-3}$	1.31	$9.186409 \cdot 10^{-4}$	1.30
1/20	$8.977671 \cdot 10^{-4}$	1.25	$6.978723 \cdot 10^{-4}$	1.23
1/24	$7.053820 \cdot 10^{-4}$	1.32	$5.498338 \cdot 10^{-4}$	1.31

Table 5.2: Experimental error results for Example 1 for the fractional diffusion operator and $b(x, y) = (x - y)$.

Example 2:
In order to demonstrate the influence of the non-local quadratic non-linearity $\nabla \cdot (uB(u))$, we present in Figure 5.1–5.10 the plots of the time evolution of the approximation u_h for the initial value $u^0(x_1, x_2) = 16(x_1 - x_1^2)(x_2 - x_2^2)$. Plots for the fractional diffusion equation are displayed on the left, the usual diffusion equation on the right. Note that u^0 has height 1 at $(1/2\,,\,1/2)$ and is symmetric with respect to x_1 and x_2. The profiles given are along the line segment $[x_1\,,\,1/2]$, $1/2 \le x_1 \le 1$. The operator $B(u)$ was chosen as in (2.8) with $b(x, y)$ given by (2.9). Values for $c = 0$ (Figures 5.1, 5.2), $c = \pm 1$ (Figures 5.3–5.6), and $c = \pm 5$ (Figures 5.7–5.10) were used.

h	$\|\|\|u-u_h\|\|\|_{\infty,0}$	cvge. rate	$\|\|\|u-u_h\|\|\|_{0,0}$	cvge. rate
1/4	$6.940074 \cdot 10^{-3}$		$5.532543 \cdot 10^{-3}$	
1/8	$2.735276 \cdot 10^{-3}$	1.34	$2.160750 \cdot 10^{-3}$	1.36
1/12	$1.564182 \cdot 10^{-3}$	1.38	$1.248073 \cdot 10^{-3}$	1.35
1/16	$1.041509 \cdot 10^{-3}$	1.41	$8.274240 \cdot 10^{-4}$	1.43
1/20	$7.742380 \cdot 10^{-4}$	1.33	$6.060513 \cdot 10^{-4}$	1.40
1/24	$5.960361 \cdot 10^{-4}$	1.50	$4.586714 \cdot 10^{-4}$	1.53

Table 5.3: Experimental error results for Example 1 for the fractional diffusion operator and $b(x,y) = (x-y)/|x-y|$.

h	$\|\|\|u-u_h\|\|\|_{\infty,0}$	cvge. rate	$\|\|\|u-u_h\|\|\|_{0,0}$	cvge. rate
1/4	$1.478687 \cdot 10^{-3}$		$1.228611 \cdot 10^{-3}$	
1/8	$8.125659 \cdot 10^{-4}$	0.86	$6.938199 \cdot 10^{-4}$	0.82
1/12	$5.538746 \cdot 10^{-4}$	0.95	$4.874202 \cdot 10^{-4}$	0.87
1/16	$4.180683 \cdot 10^{-4}$	0.98	$3.751933 \cdot 10^{-4}$	0.91
1/20	$3.353536 \cdot 10^{-4}$	0.99	$3.048365 \cdot 10^{-4}$	0.93
1/24	$2.798554 \cdot 10^{-4}$	0.99	$2.566538 \cdot 10^{-4}$	0.94

Table 5.4: Experimental error results for Example 1 for the usual diffusion operator and no B term.

h	$\|\|\|u-u_h\|\|\|_{\infty,0}$	cvge. rate	$\|\|\|u-u_h\|\|\|_{0,0}$	cvge. rate
1/4	$1.470130 \cdot 10^{-3}$		$1.223586 \cdot 10^{-3}$	
1/8	$8.119423 \cdot 10^{-4}$	0.87	$6.874692 \cdot 10^{-4}$	0.83
1/12	$5.527071 \cdot 10^{-4}$	0.95	$4.812465 \cdot 10^{-4}$	0.88
1/16	$4.172209 \cdot 10^{-4}$	0.98	$3.690746 \cdot 10^{-4}$	0.92
1/20	$3.346173 \cdot 10^{-4}$	0.99	$2.987263 \cdot 10^{-4}$	0.95
1/24	$2.791593 \cdot 10^{-4}$	0.99	$2.505369 \cdot 10^{-4}$	0.96

Table 5.5: Experimental error results for Example 1 for the usual diffusion operator and $b(x,y) = (x-y)$.

h	$\|\|\|u-u_h\|\|\|_{\infty,0}$	cvge. rate	$\|\|\|u-u_h\|\|\|_{0,0}$	cvge. rate
1/4	$1.481271 \cdot 10^{-3}$		$1.203806 \cdot 10^{-3}$	
1/8	$8.093125 \cdot 10^{-4}$	0.87	$6.612167 \cdot 10^{-4}$	0.86
1/12	$5.477165 \cdot 10^{-4}$	0.96	$4.556000 \cdot 10^{-4}$	0.92
1/16	$4.135933 \cdot 10^{-4}$	0.98	$3.436790 \cdot 10^{-4}$	0.98
1/20	$3.314616 \cdot 10^{-4}$	0.99	$2.734522 \cdot 10^{-4}$	1.02
1/24	$2.761739 \cdot 10^{-4}$	1.00	$2.253575 \cdot 10^{-4}$	1.06

Table 5.6: Experimental error results for Example 1 for the usual diffusion operator and $b(x,y) = (x-y)/|x-y|$.

For the negative values of c the diffusion of u away from the maximum at $(1/2\,,\,1/2)$ is enhanced. For positive values of c the $\nabla\cdot(uB(u))$ term acts "against the diffusion operator" to try and concentrate u at $(1/2\,,\,1/2)$. This behavior is consistent with the case $c<0$ modeling Brownian diffusion and $c>0$ being used to model mutual gravitational attraction of particles in clouds (see [4]).

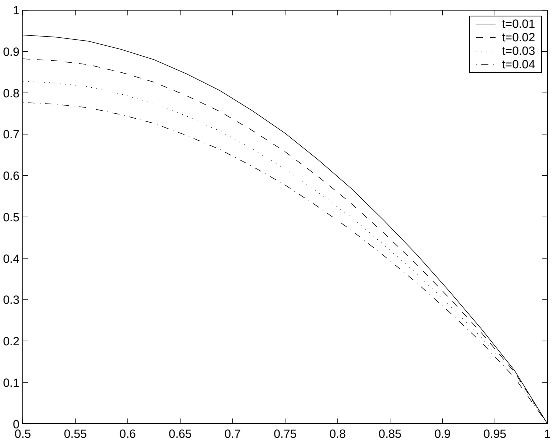

Figure 5.1: Time evolution of (1.1) for $c=0$.

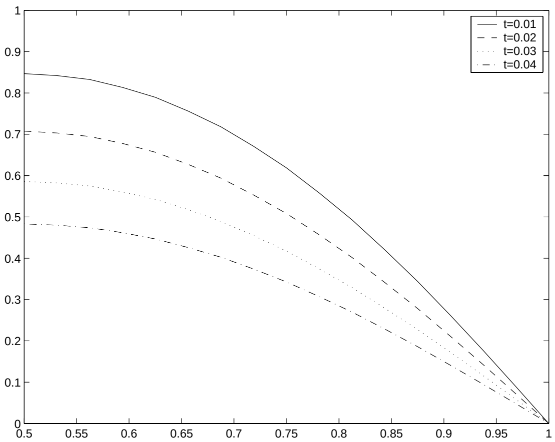

Figure 5.2: Time evolution of (5.4) for $c=0$.

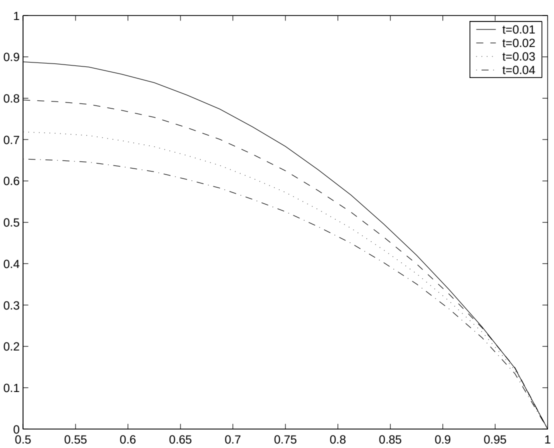

Figure 5.3: Time evolution of (1.1) for $c=-1$.

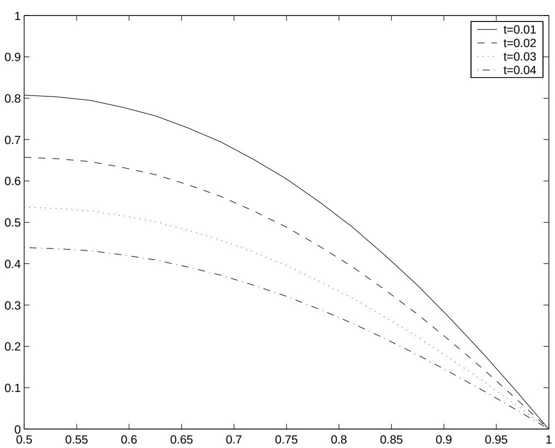

Figure 5.4: Time evolution of (5.4) for $c=-1$.

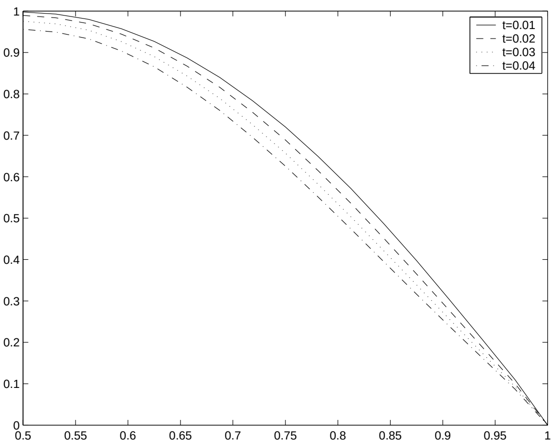

Figure 5.5: Time evolution of (1.1) for $c = 1$.

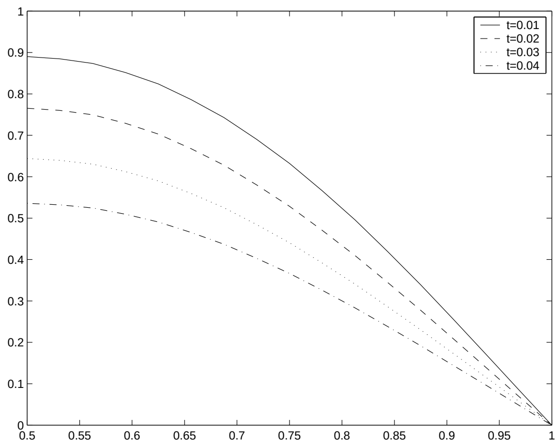

Figure 5.6: Time evolution of (5.4) for $c = 1$.

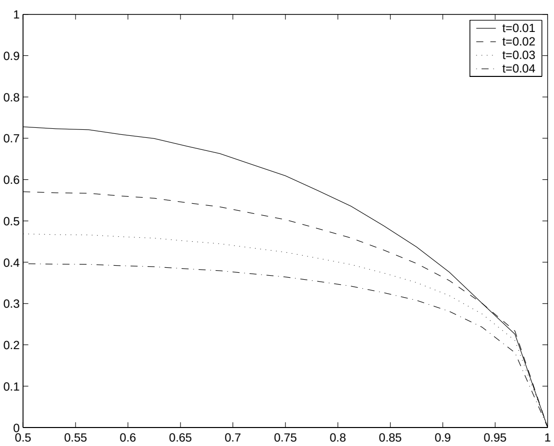

Figure 5.7: Time evolution of (1.1) for $c = -5$.

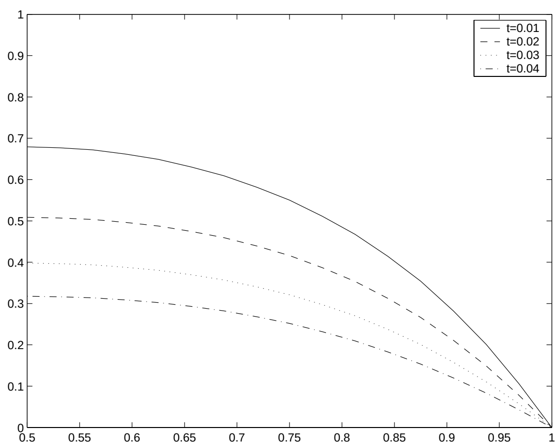

Figure 5.8: Time evolution of (5.4) for $c = -5$.

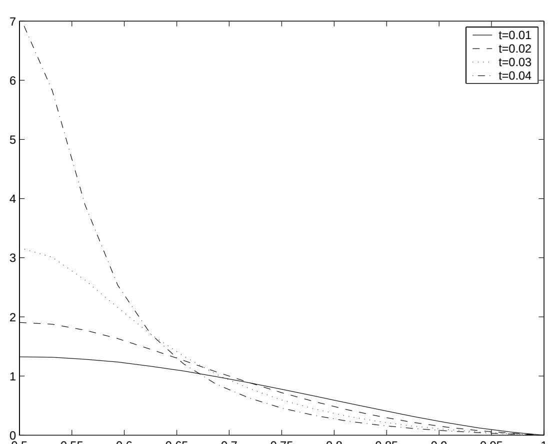

Figure 5.9: Time evolution of (1.1) for $c = 5$.

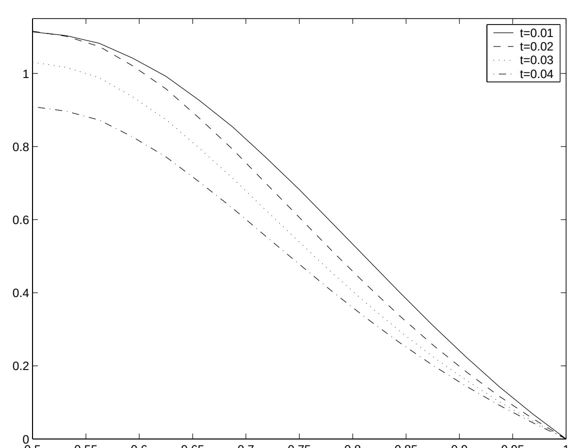

Figure 5.10: Time evolution of (5.4) for $c = 5$.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, (1975).

[2] D.A. Benson, S.W. Wheatcraft and M.M. Meerschaeert, The fractional order governing equations of Lévy motion, *Water Resour. Res.*, **36**, 1413–1423, (2000).

[3] J. Bergh and J. Löfström, *Interpolation Spaces, an Introduction*, Springer-Verlag, (1976).

[4] P. Biler and W.A. Woyczynski, Global and exploding solutions for nonlocal quadratic evolution problems, *SIAM J. Appl. Math.*, **59**, pp. 845-869, (1998).

[5] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer-Verlag, (1994).

[6] B.A. Carreras, V.E. Lynch and G.M. Zaslavsky, Anomalous diffusion and exit time distribution of particle tracers in plasma turbulence models, *Phys. Plasmas*, **8**, 5096-5103, (2001).

[7] V.J. Ervin and J.P. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Methods for Partial Differential Equations*, to appear (2005).

[8] V.J. Ervin and J.P. Roop, Variational solution of fractional advection dispersion equation on bounded domains in $\mathbb{R}^d$, submitted to *Numer. Methods for Partial Differential Equations*, (2005).

[9] I.M. Gelfand and G.E. Shilov, *Generalized Functions*, Vol. I, Academic Press, (1964).

[10] V. Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer-Verlag, (1986).

[11] J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem. Part IV: Error analysis for second-order time discretization, *SIAM J. Numer. Anal.*, **2**, 353–384, (1990).

[12] M.M. Meerschaert, D.A. Benson and B. Baeumer, Multidimensional advection and fractional dispersion, *Phys. Rev. E*, **59**, 5026–5028, (1999).

[13] J.P. Roop, *Variational Solution of the Fractional Advection Dispersion Equation*, Ph.D. thesis, Clemson University, (2004).

[14] J.P. Roop, Computational aspects of FEM approximations of fractional advection dispersion equations on bounded domains in $\mathbb{R}^2$, *J. Comp. Appl. Math*, to appear (2005).

[15] Ch. Schwab, *$p-$ and $hp-$ Finite Element Methods*, Oxford University Press, (1998).

[16] R. Seeley, Topics in pseudo-differential operators, in *Pseudo-Differential Operators*, Ed. L. Nirenberg, C.I.M.E., Roma, Cremonese, 168–305, (1968)

[17] M.F. Shlesinger, B.J. West and J. Klafter, Lévy dynamics of enhanced diffusion: Application to turbulence, *Phys. Rev. Lett.*, **58**, 1100–1103, (1987).

[18] G.M. Zaslavsky, D.Stevens and H. Weitzner, Self-similar transport in incomplete chaos, *Phys. Rev. E*, **48**, 1683–1694, (1993).