The p-version of the boundary element method for mixed boundary value problems on polyhedra

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Abstract

We study the p-version of the boundary element method for a mixed boundary value problem. The discretising scheme is based upon a system of boundary integral operators that uniquely defines the Cauchy data of the problem, the trace on one part of the boundary in $\tilde{H}^{1/2}$ and the normal derivative on the other part in $\tilde{H}^{-1/2}$. We consider polyhedral domains where singularities at vertices and edges appear. These singularities are very strong at edges where different boundary conditions meet, i.e., the trace is not necessarily in H^1 and the normal derivative not in L_2. For this situation we prove an optimal a priori error estimate.

Keywords. p-version, boundary element method, singularities, mixed boundary conditions.

1 Introduction and model problem

It is well-known that solutions to elliptic problems on domains with corners and edges behave singularly. For the simplest case, the Laplacian with Dirichlet or Neumann boundary conditions in two dimensions on a polygon, the strongest corner singularities behave like $r^{\pi/\omega}$ with r being the distance to a specific corner and ω the interior angle at that corner. For mixed problems, however, where Dirichlet and Neumann boundary conditions at a specific corner meet, the behaviour is more singular like $r^{\pi/2\omega}$, in general. For details in two dimensions see [6]. In three dimensions, on polyhedra, the situation is more involved as edge, vertex, and vertex-edge singularities appear. As in the two-dimensional situation, singularities of mixed boundary value problems are in general stronger than those of Dirichlet or Neumann boundary value problems. Here, the edge and vertex-edge singularities are stronger at the intersections of faces where different types of conditions meet.

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The appearance of singularities reduces convergence orders of approximating schemes such as finite elements or boundary elements. Here, we study the p-version of the boundary element method (BEM) that uses a fixed mesh and improves approximations by increasing polynomial degrees. Our model problem will be the Laplacian on a polyhedral domain with mixed boundary conditions. That means there is a part of the boundary (let us say Γ_2) where the trace of an H^1 function must be approximated (thus a function in $H^{1/2}(\Gamma_2)$) and on the remaining part of Γ, Γ_1, the Neumann datum is unknown (thus a function in $H^{-1/2}(\Gamma_1)$). There are some theoretical results on the p-version of the BEM on polyhedral domains [8]. In that paper, however, only the Neumann problem is considered and approximation theory is presented in $H^{1/2}$. Additionally, for problems on closed surfaces as in [8] one has H^1-regularity of the trace on the boundary such that L^2 and H^1 results lead to the analysis in $H^{1/2}$ by interpolation [1]. For mixed boundary conditions one cannot assume H^1 regularity. We use our result for screen problems [3] to deal with the approximation of these more singular parts in $H^{1/2}$. The approximation analysis for the Neumann unknown in $H^{-1/2}(\Gamma_1)$ is based upon our recent papers [2, 4], for Dirichlet problems on open surfaces. The final result of this paper is an optimal a priori error estimate for the p-version of the BEM dealing with a system of boundary integral operators to approximate the above mentioned mixed boundary value problem for the Laplacian. The generalisation to other elliptic problems of second order is straightforward since the whole analysis just assumes a specific regularity result (decomposition of the solution into several singularities and a regular remainder) and Cea's lemma (quasi-optimal convergence in the energy norm).

Before presenting the model problem let us recall some Sobolev spaces. Let Ω be a polyhedral domain in $\mathbf{R}^3$ with boundary $\Gamma = \cup_{j=1}^{J}\bar{\Gamma}^j$ (Γ^j being the faces of Γ). For non-negative s we define

$$H^s(\Omega) = \{u|_\Omega;\ u \in H^s(\mathbf{R}^3)\}, \quad H^{-s}(\Omega) = (\tilde{H}^s(\Omega))' \quad \text{(dual space)},$$

$$\tilde{H}^s(\Omega) = \{u|_\Omega;\ u \in H^s(\mathbf{R}^3),\ \operatorname{supp} u \subset \bar{\Omega}\},$$

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma;\ u \in H^{s+1/2}(\Omega)\}, & s > 0, \\ L_2(\Gamma), & s = 0, \\ (H^{-s}(\Gamma))', & s < 0, \end{cases}$$

and

$$H^s(\Gamma^j) = \{u|_{\Gamma^j};\ u \in H^s(\Gamma)\}, \quad \tilde{H}^s(\Gamma^j) = \{u|_{\Gamma^j};\ u \in H^s(\Gamma),\ \operatorname{supp} u \subset \bar{\Gamma}^j\}.$$

Now, to introduce our model problem, let $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ be split into two parts ($\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$). For simplicity we assume that Γ_1 and Γ_2 are unions of entire faces of Γ. Our model problem is: *For given g_1, g_2 find u such that*

$$\Delta u = 0 \ \text{in}\ \Omega, \qquad u = g_1 \ \text{on}\ \Gamma_1, \qquad \frac{\partial u}{\partial n} = g_2 \ \text{on}\ \Gamma_2. \tag{1.1}$$

Here, $\partial u/\partial n$ is the normal derivative with respect to the outward unit normal n on Γ. We note that, for functions $v \in \{v \in H^1(\Omega);\ \Delta v \in L_2(\Omega)\}$, the normal derivative $\partial v/\partial n \in H^{-1/2}(\Gamma)$ is weakly defined by Green's formula:

$$\langle \frac{\partial v}{\partial n}, w|_\Gamma \rangle = -\int_\Omega \Delta v\, w\, dx - \int_\Omega \nabla v\, \nabla w\, dx \quad \forall w \in H^1(\Omega).$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ (for closed surfaces the spaces $\tilde{H}^s(\Gamma)$ and $H^s(\Gamma)$ coincide).

In order to solve (1.1) by the BEM let us recall from [9] an equivalent formulation by a system of boundary integral operators. Using the fundamental solution of the Laplacian

$$\phi(x,y) = -\frac{1}{4\pi|x-y|}, \qquad x,y \in \mathbf{R}^3,$$

we define, for $w \in C^\infty(\Omega)$ and $x \in \Omega$,

$$\begin{aligned} \mathcal{V}w(x) &= -2\int_\Gamma w(y)\phi(x,y)\,ds_y \qquad \text{(single layer potential)},\\ \mathcal{K}w(x) &= -2\int_\Gamma w(y)\frac{\partial}{\partial n_y}\phi(x,y)\,ds_y \qquad \text{(double layer potential)},\end{aligned}$$

and, for $w \in C^\infty(\Gamma_j)$ and $x \in \Gamma_k$ $(j,k = 1,2)$,

$$\begin{aligned} V_{jk}w(x) &= -2\textstyle\int_{\Gamma_j} w(y)\phi(x,y)\,ds_y,\\ K_{jk}w(x) &= -2\textstyle\int_{\Gamma_j} w(y)\frac{\partial}{\partial n_y}\phi(x,y)\,ds_y,\\ K'_{jk}w(x) &= -2\textstyle\int_{\Gamma_j} w(y)\frac{\partial}{\partial n_x}\phi(x,y)\,ds_y,\\ W_{jk}w(x) &= 2\frac{\partial}{\partial n_x}\textstyle\int_{\Gamma_j} w(y)\frac{\partial}{\partial n_y}\phi(x,y)\,ds_y.\end{aligned} \tag{1.2}$$

By continuity, these operators are well-defined for functions of Sobolev spaces, as used below. We extend the given functions g_1 on Γ_1 and g_2 on Γ_2 in an arbitrary way to functions $g_1^* \in H^{1/2}(\Gamma)$ and $g_2^* \in H^{-1/2}(\Gamma)$, respectively. Using these extensions, the trace $v = u|_\Gamma$ and the normal derivative $\psi = (\partial u/\partial n)|_\Gamma$ of the solution u to (1.1) can be written in the following form:

$$v = v^0 + g_1^* \text{ with } v^0 \in \tilde{H}^{1/2}(\Gamma_2), \qquad \psi = \psi^0 + g_2^* \text{ with } \psi^0 \in \tilde{H}^{-1/2}(\Gamma_1).$$

The system of boundary integral equations then reads

$$\begin{pmatrix} W_{22} & K'_{12} \\ -K_{21} & V_{11}\end{pmatrix}\begin{pmatrix} v^0 \\ \psi^0\end{pmatrix} = \begin{pmatrix} -(W)_2 & (1-K')_2 \\ (1+K)_1 & -(V)_1\end{pmatrix}\begin{pmatrix} g_1^* \\ g_2^*\end{pmatrix}. \tag{1.3}$$

Here, W, K', K, and V are defined as in (1.2), where the integrations are performed over the whole boundary Γ, and an index i means that the operator is evaluated on Γ_i $(i = 1,2)$. The mixed boundary value problem (1.1) and the system of integral equations (1.3) are equivalent:

Theorem 1.1 [9, Theorem 1.3] *Let $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in H^{-1/2}(\Gamma_2)$ be given. Then, the boundary value problem* (1.1) *and the system of integral equations* (1.3) *are uniquely solvable and equivalent. If $u \in H^1(\Omega)$ solves* (1.1) *then $v^0 = u|_{\Gamma_2} - g_1^*|_{\Gamma_2}$, $\psi^0 = (\partial u/\partial n)|_{\Gamma_1} - g_2^*|_{\Gamma_1}$ solve* (1.3), *where g_1^* and g_2^* are arbitrary extensions to elements in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively. Conversely, if (v^0,ψ^0) is the solution to* (1.3), *then*

$$u = -\frac{1}{2}(\mathcal{K}\tilde{v} - \mathcal{V}\tilde{\psi}) \tag{1.4}$$

solves (1.1) *with*

$$\tilde{v} = \begin{cases} g_1 & \text{on } \Gamma_1 \\ v^0 + g_1^* & \text{on } \Gamma_2 \end{cases} \qquad \text{and} \qquad \tilde{\psi} = \begin{cases} \psi^0 + g_2^* & \text{on } \Gamma_1 \\ g_2 & \text{on } \Gamma_2 \end{cases}. \tag{1.5}$$

Here, g_1^* and g_2^* are extensions as before.

2 The p-version of the BEM

We present the p-version of the BEM for the solution of (1.3) and give an optimal a priori convergence estimate (Theorem 2.1), depending on a specific regularity result.

In the following, p will always denote a polynomial degree. We consider a fixed mesh on Γ, $\{\Gamma^{ji};\ i = 1, \ldots, J_j,\ j = 1, \ldots, J\}$, consisting of triangles and parallelograms. Here, Γ^{ji} are elements on the face Γ^j $(i = 1, \ldots, J_j)$ and we require that the mesh is regular (there are no hanging nodes).

Let $Q = (-1, 1)^2$ and $T = \{(x_1, x_2);\ 0 < x_1 < 1, 0 < x_2 < x_1\}$ be the reference square and triangle, respectively. For $K = Q$ or T, let $\mathcal{Q}_p(K)$ ($\mathcal{P}_p(T)$) denote the set of polynomials on K of degree less than or equal to p in each variable (in total). For given integer $p > 0$ we define spaces of piecewise polynomials on the mesh introduced before,

$$V_0^p(\Gamma_1) := \{v \in L_2(\Gamma_1);\ v|_{\Gamma^{ji}} \circ T_{ji} \in \mathcal{Q}_{p-1}(Q) \text{ or } \mathcal{P}_{p-1}(T) \text{ for } \Gamma^{ji} \subset \Gamma_1\}$$

and

$$V_1^p(\Gamma_2) := \{v \in C^0(\Gamma_2);\ v = 0 \text{ on } \partial\Gamma_2,\ v|_{\Gamma^{ji}} \circ T_{ji} \in \mathcal{Q}_p(Q) \text{ or } \mathcal{P}_p(T) \text{ for } \Gamma^{ji} \subset \Gamma_2\}.$$

Here, T_{ji} is an affine mapping with $T_{ji}(K) = \Gamma^{ji}$, $K = Q$ or T as appropriate. Note that $V_0^p(\Gamma_1) \subset \tilde{H}^{-1/2}(\Gamma_1)$ and $V_1^p(\Gamma_2) \subset \tilde{H}^{1/2}(\Gamma_2)$ and thus, the following p-version of the BEM is conforming:

For given $g_1^* \in H^{1/2}(\Gamma)$ and $g_2^* \in H^{-1/2}(\Gamma)$ find $\psi_p^0 \in V_0^p(\Gamma_1)$ and $v_p^0 \in V_1^p(\Gamma_2)$ such that

$$\langle W_{22}\, v_p^0 + K'_{12}\, \psi_p^0, w\rangle_{\Gamma_2} = \langle -W g_1^* + (1 - K')g_2^*, w\rangle_{\Gamma_2} \quad \forall w \in V_1^p(\Gamma_2), \tag{2.6}$$

$$\langle -K_{21}\, v_p^0 + V_{11}\, \psi_p^0, \phi\rangle_{\Gamma_1} = \langle (1 + K)g_1^* - V g_2^*, \phi\rangle_{\Gamma_1} \quad \forall \phi \in V_0^p(\Gamma_1). \tag{2.7}$$

Here, $\langle\cdot,\cdot\rangle_{\Gamma_2}$ and $\langle\cdot,\cdot\rangle_{\Gamma_1}$ denote the duality pairings between $H^{-1/2}(\Gamma_2)$ and $\tilde{H}^{1/2}(\Gamma_2)$, and between $H^{1/2}(\Gamma_1)$ and $\tilde{H}^{-1/2}(\Gamma_1)$, respectively.

An approximate solution to (1.1) is then obtained by the representation (1.4), (1.5), where (v^0, ψ^0) is to be substituted by (v_p^0, ψ_p^0). By the strong ellipticity of the system of boundary integral operators in (1.3) (see [9], [5]) and by the conformity of our method (2.6), (2.7) we directly obtain the quasi-optimal error estimate (Cea's lemma):

$$\begin{aligned} &\|v^0 - v_p^0\|_{\tilde{H}^{1/2}(\Gamma_2)} + \|\psi^0 - \psi_p^0\|_{\tilde{H}^{-1/2}(\Gamma_1)} \\ &\leq C \left\{ \inf_{w \in V_1^p(\Gamma_2)} \|v^0 - w\|_{\tilde{H}^{1/2}(\Gamma_2)} + \inf_{\phi \in V_0^p(\Gamma_1)} \|\psi^0 - \phi\|_{\tilde{H}^{-1/2}(\Gamma_1)} \right\} \end{aligned} \tag{2.8}$$

where the constant C is independent of p.

Before presenting our main result on the convergence of the BEM for our model problem we need to recall results on the regularity of v^0 and ψ^0, see [9].

For $j \in \{1, \dots, J\}$ let V_j and E_j denote the sets of vertices and edges of Γ^j, respectively. For $\nu \in V_j$, let $E_j(\nu)$ denote the set of edges of Γ^j with ν as an end point. Then, for sufficiently smooth given functions g_1^* and g_2^* the solution (v^0, ψ^0) of (1.3) has the following form.

1. For ν being a vertex of a face $\Gamma^j \subset \Gamma_2$ with Neumann boundary condition there holds on Γ^j

$$v^0 = v_{\text{reg}} + \sum_{e \in E_j} v^e + \sum_{\nu \in V_j} v^\nu + \sum_{\nu \in V_j} \sum_{e \in E_j(\nu)} v^{e\nu}, \tag{2.9}$$

where, using local coordinate systems (r_ν, θ_ν) and (x_{e1}, x_{e2}) with origin ν, we have the following representations:

(i) The *regular part* $v_{\text{reg}} \in H^k(\Gamma^j)$, $k > 3/2$, or as large as needed.

(ii) The *edge singularities* v^e have the form

$$v^e = \sum_{j=1}^{m_e} \left(\sum_{s=0}^{s_j^e} b_{js}^e(x_{e1}) |\log x_{e2}|^s \right) x_{e2}^{\gamma_j^e} \chi_1^e(x_{e1}) \chi_2^e(x_{e2}),$$

where $\gamma_{j+1}^e \geq \gamma_j^e > \frac{1}{4}$, and m_e, s_j^e are integers. Here, χ_1^e, χ_2^e are C^∞ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of e and $\chi_1^e = 0$ in a neighborhood of these vertices. Moreover, for a $\rho_e > 0$, $\chi_2^e = 1$ for $0 \leq x_{e2} \leq \rho_e$ and $\chi_2^e = 0$ for $x_{e2} \geq 2\rho_e$. The functions $b_{js}^e \chi_1^e \in H^m(e)$ for m as large as required.

(iii) The *vertex singularities* v^ν have the form

$$v^\nu = \chi^\nu(r_\nu) \sum_{i=1}^{n_\nu} \sum_{t=0}^{q_i^\nu} B_{it}^\nu |\log r_\nu|^t r_\nu^{\lambda_i^\nu} w_{it}^\nu(\theta_\nu),$$

where $\lambda_{i+1}^\nu \geq \lambda_i^\nu > 0$, n_ν, $q_i^\nu \geq 0$ are integers, and B_{it}^ν are real numbers. Here, χ^ν is a C^∞ cut-off function with $\chi^\nu = 1$ for $0 \leq r_\nu \leq \tau_\nu$ and $\chi^\nu = 0$ for $r_\nu \geq 2\tau_\nu$ for some $\tau_\nu > 0$. The functions $w_{it}^\nu \in H^q(0, \omega_\nu)$ for q as large as required. Here, ω_ν denotes the interior angle (on Γ^j) between the edges meeting at ν.

(iv) The *edge-vertex singularities* $v^{e\nu}$ have the form

$$v^{e\nu} = v_1^{e\nu} + v_2^{e\nu},$$

where

$$v_1^{e\nu} = \sum_{j=1}^{m_e} \sum_{i=1}^{n_\nu} \left(\sum_{s=0}^{s_j^e} \sum_{t=0}^{q_i^\nu} \sum_{l=0}^{s} B_{ijlts}^{e\nu} |\log x_{e1}|^{s+t-l} |\log x_{e2}|^l \right) x_{e1}^{\lambda_i^\nu - \gamma_j^e} x_{e2}^{\gamma_j^e} \chi^\nu(r_\nu) \chi^{e\nu}(\theta_\nu)$$

and

$$v_2^{e\nu} = \sum_{j=1}^{m_e} \sum_{s=0}^{s_j^e} B_{js}^{e\nu}(r_\nu) |\log x_{e2}|^s x_{e2}^{\gamma_j^e} \chi^\nu(r_\nu) \chi^{e\nu}(\theta_\nu)$$

with

$$B_{js}^{e\nu}(r_\nu) = \sum_{l=0}^{s} B_{jsl}^{e\nu}(r_\nu) |\log r_\nu|^l.$$

Here, q_i^ν, s_j^e, λ_i^ν, γ_j^e, χ^ν are as above, $B_{ijlts}^{e\nu}$ are real numbers, and $\chi^{e\nu}$ is a C^∞ cut-off function with $\chi^{e\nu} = 1$ for $0 \le \theta_\nu \le \beta$ and $\chi^{e\nu} = 0$ for $\frac{3}{2}\beta \le \theta_\nu \le \omega_\nu$ for some $0 < \beta \le \min\{\omega_\nu/2, \pi/8\}$. The functions $B_{jsl}^{e\nu}$ may be chosen such that

$$B_{js}^{e\nu}(r_\nu) \chi^\nu(r_\nu) \chi^{e\nu}(\theta_\nu) = \xi_{js}(x_{e1}, x_{e2}) \chi_2^e(x_{e2}),$$

where the extension of ξ_{js} by zero onto $\mathbf{R}^{2+} := \{(x_{e1}, x_{e2});\ x_{e2} > 0\}$ lies in $H^m(\mathbf{R}^{2+})$, with m as in (ii). Here, χ_2^e is a C^∞ cut-off function as in (ii).

2. For ν being a vertex of a face $\Gamma^j \subset \Gamma_1$ with Dirichlet boundary condition there holds on Γ^j

$$\psi^0 = \psi_{\text{reg}} + \sum_{e \in E_j} \psi^e + \sum_{\nu \in V_j} \psi^\nu + \sum_{\nu \in V_j} \sum_{e \in E_j(\nu)} \psi^{e\nu} \tag{2.10}$$

where, by using the same notation as before for v^0, we have the following representations.

(i) For the regular part there holds $\psi_{\text{reg}} \in H^k(\Gamma_1)$, $k > 1/2$, or as large as needed.

(ii) The edge singularities ψ^e have the form

$$\psi^e = \sum_{j=1}^{m_e} \left(\sum_{s=0}^{s_j^e} b_{js}^e(x_{e1}) |\ln x_{e2}|^s \right) x_{e2}^{\gamma_j^e - 1} \chi_1^e(x_{e1}) \chi_2^e(x_{e2}).$$

(iii) The vertex singularities ψ^ν have the form

$$\psi^\nu = \chi^\nu(r_\nu) \sum_{i=1}^{n_\nu} \sum_{t=0}^{q_i^\nu} B_{it}^\nu |\ln r_\nu|^t r_\nu^{\lambda_i^\nu - 1} w_{it}^\nu(\theta_\nu).$$

(iv) The edge-vertex singularities $\psi^{e\nu}$ have the form

$$\psi^{e\nu} = \psi_1^{e\nu} + \psi_2^{e\nu},$$

where

$$\psi_1^{e\nu} = \sum_{j=1}^{m_e} \sum_{i=1}^{n_\nu} \left(\sum_{s=0}^{s_j^e} \sum_{t=0}^{q_i^\nu} \sum_{l=0}^{s} B_{ijlts}^{e\nu} |\ln x_{e1}|^{s+t-l} |\ln x_{e2}|^l \right) x_{e1}^{\lambda_i^\nu - \gamma_j^e} x_{e2}^{\gamma_j^e - 1} \chi^\nu(r_\nu) \chi^{e\nu}(\theta_\nu)$$

and

$$\psi_2^{e\nu} = \sum_{j=1}^{m_e} \sum_{s=0}^{s_j^e} B_{js}^{e\nu}(r_\nu) |\ln x_{e2}|^s x_{e2}^{\gamma_j^e - 1} \chi^\nu(r_\nu) \chi^{e\nu}(\theta_\nu).$$

Remark 2.1 *For a pure Neumann boundary condition one has the regularity result* 1.(i)–(iv) *with* $\gamma_j^e > 1/2$ *in* (ii) *and* (iv) *(and* $\gamma_j^e \geq 1/2$ *for open surfaces). The same holds for a pure Dirichlet boundary condition and the parameter* γ_j^e *in* 2.(ii),(iv). *For mixed boundary conditions, and at edges where faces of different boundary conditions meet, stronger edge and vertex-edge singularities are possible (*$\gamma_j^e > 1/4$ *as above). Note that in these singular cases there does not hold* $v^0 \in H^1(\Gamma_2)$ *or* $\psi^0 \in L_2(\Gamma_1)$ *in general.*

Our main result is as follows.

Theorem 2.1 *For sufficiently smooth given functions* g_1^* *and* g_2^* *on* Γ *let* (v^0, ψ^0) *denote the solution of* (1.3). *Moreover, assume that the characterisations* 1.(i)–(iv) *and* 2.(i)–(iv) *hold with* $\nu_0 \in V_j$, $e_0 \in E_j(\nu_0)$ $(j \in \{1, \dots, J\})$ *such that*

$$\min\{\lambda_1^{\nu_0} + 1/2, \gamma_1^{e_0}\} = \min_{j=1,\dots,J} \min_{\nu \in V_j, e \in E_j(\nu)} \min\{\lambda_1^{\nu} + 1/2, \gamma_1^{e}\}.$$

Then denote

$$\beta = \begin{cases} s_1^{e_0} + q_1^{\nu_0} + 1/2 & \textit{if } \lambda_1^{\nu_0} = \gamma_1^{e_0} - 1/2, \\ s_1^{e_0} + q_1^{\nu_0} & \textit{otherwise}, \end{cases}$$

where the numbers $s_1^{e_0}$, $q_1^{\nu_0}$ *are given in* 1.(iv).

Then the BE approximation (v_p^0, ψ_p^0) *defined by* (2.6), (2.7) *satisfies*

$$\|v^0 - v_p^0\|_{\tilde{H}^{1/2}(\Gamma_2)} + \|\psi^0 - \psi_p^0\|_{\tilde{H}^{-1/2}(\Gamma_1)} \leq C\, |\log p|^{\beta} p^{-2\min\{\gamma_1^{e_0}, \lambda_1^{\nu_0} + \frac{1}{2}\}},$$

where $C > 0$ *is a constant which is independent of* p.

Proof. By the quasi-optimal error estimate (2.8) the stated a priori error estimate for the p-version of the BEM boils down to approximation results in the spaces $\tilde{H}^{1/2}(\Gamma_2)$ and $\tilde{H}^{-1/2}(\Gamma_1)$. Using the regularity 1.(i)–(iv) for v^0 and applying the general approximation theorem [3, Theorem 3.8] we obtain a piecewise polynomial $v_p^0 \in V_1^p(\Gamma_2)$ such that

$$\|v^0 - v_p^0\|_{\tilde{H}^{1/2}(\Gamma_2)} \leq C\, |\log p|^{\beta} p^{-2\min\{\gamma_1^{e_0}, \lambda_1^{\nu_0} + \frac{1}{2}\}}.$$

Analogously, using the regularity 2.(i)–(iv) for ψ^0 and applying the general approximation theorem [4, Theorem 3.7], we find a piecewise polynomial $\psi_p^0 \in V_0^p(\Gamma_1)$ such that

$$\|\psi^0 - \psi_p^0\|_{\tilde{H}^{-1/2}(\Gamma_1)} \leq C\, |\log p|^{\beta} p^{-2\min\{\gamma_1^{e_0}, \lambda_1^{\nu_0} + \frac{1}{2}\}}.$$

Combining both estimates we obtain the result. □

We do not present numerical results to underline our error estimate. Indeed, this estimate has been conjectured already some time ago and numerical experiments for the separate approximations in $\tilde{H}^{-1/2}$ and $\tilde{H}^{1/2}$ are also reported, see [7].

Remark 2.2 *For the h-version (using quasi-uniform meshes with elements of diameter h and lowest order piecewise polynomials) the results in* [9] *yield a convergence estimate of the form*

$$\|v^0 - v_h^0\|_{\tilde{H}^{1/2}(\Gamma_2)} + \|\psi^0 - \psi_h^0\|_{\tilde{H}^{-1/2}(\Gamma_1)} \leq C(\epsilon)\, h^{\alpha-\epsilon}$$

for $\epsilon > 0$ *and parameter* $\alpha = \min\{\gamma_1^{e_0}, \lambda_1^{\nu_0} + 1/2\} \geq 1/2$. *Neglecting the dependence on* ϵ *(the constant* C *depends in an unspecified way on* ϵ*) this is, with respect to the number of unknowns, half the rate of convergence of the p-version. For* $\alpha < 1/2$*, the particular case we are interested in, there are no theoretical results known which are close to the optimal ones. In this paper, for the p-version, we proved optimal a priori error estimates for the whole range of possible singularities.*

Since optimal estimates for the h-version are not known so far, also optimal estimates for the hp-version with quasi-uniform meshes are an open problem. We note, however, that the exponential rate of convergence of the hp-version with geometrically graded meshes has been proved, see [7].

References

[1] J. Bergh and J. Löfström, *Interpolation Spaces*, no. 223 in Grundlehren der mathematischen Wissenschaften, Springer-Verlag, Berlin, 1976.

[2] A. Bespalov and N. Heuer, *The p-version of the boundary element method for a three-dimensional crack problem*, Report 04/4, BICOM, Brunel University, UK, 2004. *J. Integral Equations Appl.*, to appear.

[3] ——, *The p-version of the boundary element method for hypersingular operators on piecewise plane open surfaces*, Numer. Math., 100 (2005), pp. 185–209.

[4] ——, *The p-version of the boundary element method for weakly singular operators on piecewise plane open surfaces*, Report 05/2, BICOM, Brunel University, UK, 2005.

[5] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[6] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Publishing Inc., Boston, 1985.

[7] N. Heuer, M. Maischak, and E. P. Stephan, *Exponential convergence of the hp-version for the boundary element method on open surfaces*, Numer. Math., 83 (1999), pp. 641–666.

[8] C. Schwab and M. Suri, *The optimal p-version approximation of singularities on polyhedra in the boundary element method*, SIAM J. Numer. Anal., 33 (1996), pp. 729–759.

[9] T. von Petersdorff and E. P. Stephan, *Regularity of mixed boundary value problems in* $\mathbf{R}^3$ *and boundary element methods on graded meshes*, Math. Methods Appl. Sci., 12 (1990), pp. 229–249.