The p-version of the boundary element method for a three-dimensional crack problem

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Abstract

We study the p-version of the boundary element method for a crack problem in linear elasticity with Dirichlet boundary conditions. The unknown jump of the traction has strong edge singularities and is approximated by solving an integral equation of the first kind with weakly singular operator. We prove a quasi-optimal a priori error estimate in the energy norm. For sufficiently smooth given data this gives a convergence like $cp^{-1+\varepsilon}$ with $\varepsilon > 0$. Here, p denotes the polynomial degree of the piecewise polynomial functions used to approximate the unknown.

Key words: p-version, boundary element method, linear elasticity, singularities
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1 Introduction and formulation of the problem

We analyze the convergence of the p-version of the boundary element method (BEM) with weakly singular integral operator for problems in $\mathbf{R}^3$. That is we study approximation properties of piecewise polynomial functions on surfaces in a negative order Sobolev space (order $-1/2$). To the knowledge of the authors this is the first paper dealing with this case. The p-version of the finite element method and the p-version of the BEM on curves have been widely studied. For the p-version of the BEM dealing

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with problems in three dimensions, however, there are very few results. The case of hypersingular operators on polyhedral surfaces (the energy space is $H^{1/2}$) is analyzed in [7]. There, using H^1-regularity of the solution, the optimal convergence of the p-version has been shown. In [2] we consider hypersingular operators on open surface, where no H^1-regularity can be assumed, and prove optimal a priori error estimates. The case of weakly singular integral operators on surfaces has been an open problem so far. Here we study this situation for the model problem of linear elasticity with a crack that has a smooth boundary. The solution exhibits in general strong edge singularities not being L_2-regular.

Let us recall the Sobolev spaces used. Then we formulate the model problem. Let Γ be an open smooth surface in $\mathbf{R}^3$ with smooth boundary curve γ. Taking a closed smooth surface $\tilde{\Gamma}$ which contains Γ, we consider Sobolev spaces $H^t(\tilde{\Gamma})$ for $t > 0$ being the restriction of $H^{t+1/2}(\mathbf{R}^3)$ to $\tilde{\Gamma}$ and for $t < 0$ by duality: $H^t(\tilde{\Gamma}) = (H^{-t}(\tilde{\Gamma}))'$. Using these spaces we define the Sobolev spaces on the open surface Γ: $\tilde{H}^t(\Gamma) = \{u \in H^t(\tilde{\Gamma});\ \operatorname{supp} u \subset \bar{\Gamma}\}$ and $H^t(\Gamma) = \{u|_\Gamma;\ u \in H^t(\tilde{\Gamma})\}$ for any real t. We use these notations for scalar functions as well as for vector functions, using the norms and inner products componentwise. In the sequel vector functions will be denoted by bold face symbols.

We consider the Dirichlet boundary value problem for the displacement field $\mathbf{u} = (u_1, u_2, u_3)$ of a homogeneous, isotropic, elastic material covering the domain $\Omega_\Gamma := \mathbf{R}^3 \setminus \bar{\Gamma}$: *For given* $\mathbf{u}_1,\ \mathbf{u}_2 \in H^{1/2}(\Gamma)$ *with* $\mathbf{u}_1 - \mathbf{u}_2 \in \tilde{H}^{1/2}(\Gamma)$ *find* $\mathbf{u}$ *satisfying*

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_\Gamma, \tag{1.1}$$

$$\mathbf{u}|_{\Gamma_1} = \mathbf{u}_1,\ \mathbf{u}|_{\Gamma_2} = \mathbf{u}_2, \tag{1.2}$$

$$\mathbf{u}(x) = o(1),\ \frac{\partial}{\partial x_j}\mathbf{u}(x) = o(|x|^{-1}),\ j = 1,2,3, \quad |x| \to \infty. \tag{1.3}$$

Here, Γ_i, $i = 1, 2$, are the two sides of Γ and $\mu > 0$, $\lambda > -2/3\mu$ are the given Lamé constants. The corresponding Neumann data of the linear elasticity problem are the tractions

$$\mathbf{T}(\mathbf{u}) := \lambda(\operatorname{div}\mathbf{u})\mathbf{n} + 2\mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} + \mu\mathbf{n} \times \operatorname{curl}\mathbf{u} \quad \text{on } \Gamma_i,\ i = 1, 2,$$

where $\mathbf{n}$ is the normal vector exterior to the bounded domain enclosed by $\tilde{\Gamma}$.

The problem (1.1)–(1.3) can be formulated as an integral equation of the first kind, see, e.g., [8, 4]: $\mathbf{u} \in H^1_{\mathrm{loc}}(\mathbf{R}^3 \setminus \bar{\Gamma})$ is the solution of the Dirichlet problem (1.1)–(1.3) if and only if the jump of the traction $\mathbf{t} := \mathbf{T}(\mathbf{u})|_{\Gamma_1} - \mathbf{T}(\mathbf{u})|_{\Gamma_2} \in \tilde{H}^{-1/2}(\Gamma)$ solves the weakly singular integral equation

$$\mathbf{V}\mathbf{t}(x) := \int_\Gamma \mathbf{E}(y, x)\mathbf{t}(y)\,ds_y = \mathbf{g}(x), \quad x \in \Gamma \tag{1.4}$$

where

$$\mathbf{g}(x) = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)(x) + \int_\Gamma \mathbf{T}_y \mathbf{E}(y,x)(\mathbf{u}_1 - \mathbf{u}_2)(y)\, ds_y.$$

Here,

$$\mathbf{E}(y,x) = \frac{\lambda + 3\mu}{8\pi\mu(\lambda+2\mu)} \left(\frac{1}{|x-y|} I + \frac{\lambda+\mu}{\lambda+3\mu} \frac{(x-y)(x-y)^T}{|x-y|^3} \right)$$

denotes the fundamental solution of (1.1) with the identity matrix I. The solution $\mathbf{t}$ of (1.4) yields the solution to problem (1.1)–(1.3) via the representation or Betti's formula

$$\mathbf{u}(x) = \int_\Gamma \Big(\mathbf{E}(y,x)\mathbf{t}(y) - (\mathbf{T}_y \mathbf{E}(y,x))^T (\mathbf{u}_1(y) - \mathbf{u}_2(y)) \Big)\, ds_y, \quad x \notin \Gamma.$$

In what follows, together with usual space coordinates $(x_1, x_2, x_3) = x \in \Gamma$ we will use surface coordinates (s, ρ) in a small neighborhood of γ on Γ such that s (respectively, ρ) varies in tangential (respectively, normal) direction to γ. Thus the boundary curve γ is described by the equation $\rho = 0$, and in a sufficiently small neighborhood of γ one has $s = s(x)$ and $\rho = \rho(x)$. Throughout the paper we will specify this small neighborhood of γ as the boundary strip Γ_δ of Γ such that for small $\delta > 0$,

$$\Gamma_\delta = \{x \in \Gamma;\ 0 < \rho(x) < \delta\}.$$

Let us cite the following regularity result from [4].

Proposition 1.1 Let $|\sigma| < 1/2$ and $\mathbf{u}_j \in H^{3/2+\sigma}(\Gamma)$, $j = 1, 2$, with $\mathbf{u}_1 - \mathbf{u}_2 \in \tilde{H}^{3/2+\sigma}(\Gamma)$. Then the solution $\mathbf{t} \in \tilde{H}^{-1/2}(\Gamma)$ of the integral equation (1.4) has the form

$$\mathbf{t} = \boldsymbol{\beta}(s)\rho^{-1/2}\chi(\rho) + \mathbf{t}_0 \tag{1.5}$$

with vector functions $\boldsymbol{\beta} \in H^{1/2+\sigma}(\gamma)$ and $\mathbf{t}_0 \in \tilde{H}^{1/2+\sigma'}(\Gamma)$ for any $\sigma' < \sigma$. Furthermore, $\chi \in C_0^\infty(\mathbf{R})$ denotes a cut-off function with $0 \le \chi \le 1$ and $\chi = 1$ near zero.

In the next section we formulate the p-version of the BEM for the approximate solution of (1.4) and state the main result which proves an almost optimal convergence rate (Theorem 2.1). Technical details and the proof of Theorem 2.1 are given in Section 3.

2 The p-version of the BEM

Below p will always denote a polynomial degree, and C is a generic positive constant independent of p.

In order to define finite dimensional subspaces of $\tilde{H}^{-1/2}(\Gamma)$ we use a regular parameter representation $x = X(u)$, $u \in U$, U being a compact region in $\mathbf{R}^2$ whose boundary is mapped onto γ. On U we use a fixed regular mesh $\mathcal{T} = \{U_j;\ j = 1, \ldots, J\}$ of quadrilaterals and triangles which are in general curvilinear such that U is completely discretized. We assume that for each $j = 1, \ldots, J$ there exists a smooth one-to-one mapping M_j such that $\bar{U}_j = M_j(\bar{K})$ with $K = Q$ or T (here, $Q = (-1, 1)^2$ and $T = \{\xi = (\xi_1, \xi_2);\ 0 < \xi_1 < 1, 0 < \xi_2 < \xi_1\}$ denote the reference square and triangle, respectively). The Jacobians of M_j are assumed to be bounded above and below by positive constants independent of j.

Using the parameter representation X we have a fixed regular mesh $\Delta = \{\Gamma_j = X(U_j);\ j = 1, \ldots, J\}$ on Γ. The union of the elements of Δ touching the boundary curve γ will be denoted by A_γ, i.e., $\bar{A}_\gamma = \cup\{\bar{\Gamma}_j;\ \bar{\Gamma}_j \cap \gamma \neq \emptyset\}$. We assume that, close to the γ, the mesh is fine enough such that $\bar{A}_\gamma \subset (\Gamma_{\delta/2} \cup \gamma)$. We also assume that the cut-off function χ in (1.5) is chosen such that $\operatorname{supp}(\boldsymbol{\beta}(s)\rho^{-1/2}\chi(\rho)) \subset \bar{A}_\gamma$.

Now for given integer p we define the space $S_p(\Gamma)$ of piecewise polynomials on Γ. For $K = Q$ or $K = T$ let $\mathcal{Q}_p(K)$ be the set of polynomials of degree $\leq p$ (in each variable for $K = Q$ and of total degree $\leq p$ on T). Furthermore, for $K = I$ an interval, $\mathcal{Q}_p(I)$ denotes the set of polynomials of degree $\leq p$ on I. We will also use the set $\mathcal{R}_p(\Gamma_j)$ of polynomials of degree $\leq p$ in each variable s and ρ on the elements $\Gamma_j \subset A_\gamma \subset \Gamma_{\delta/2}$. Then using the notation $\mathbf{v}_j = \mathbf{v}|_{\Gamma_j}$ we define

$$S_p(\Gamma) := \{\mathbf{v};\ \mathbf{v}_j \in [\mathcal{R}_p(\Gamma_j)]^3 \text{ if } \Gamma_j \subset A_\gamma, \text{ and}$$

$$(\mathbf{v}_j \circ X \circ M_j) \in [\mathcal{Q}_p(K)]^3,\ K = Q \text{ or } T,\ \text{if } \Gamma_j \subset (\Gamma \backslash A_\gamma)\}$$

(here, we denote by $[\cdot]^3$ the sets of vector functions with corresponding polynomial components).

One has $S_p(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$, and the p-version of the boundary element Galerkin method is as follows: *For given p find $\mathbf{t}_p \in S_p(\Gamma)$ such that*

$$\langle \mathbf{V}\mathbf{t}_p, \mathbf{v}\rangle = \langle \mathbf{g}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in S_p(\Gamma). \tag{2.1}$$

As it is well known, this method converges quasi-optimally, see [3], i.e., there exists a constant $C > 0$ such that for all polynomial degrees p there holds

$$\|\mathbf{t} - \mathbf{t}_p\|_{\tilde{H}^{-1/2}(\Gamma)} \leq C \inf\{\|\mathbf{t} - \mathbf{v}\|_{\tilde{H}^{-1/2}(\Gamma)};\ \mathbf{v} \in S_p(\Gamma)\}. \tag{2.2}$$

We now present the main result giving an a priori error estimate.

Theorem 2.1 Let $|\sigma| < 1/2$ and $\mathbf{u}_j \in H^{3/2+\sigma}(\Gamma)$, $j = 1, 2$, with $\mathbf{u}_1 - \mathbf{u}_2 \in \tilde{H}^{3/2+\sigma}(\Gamma)$. Then there holds the a priori error estimate

$$\|\mathbf{t} - \mathbf{t}_p\|_{\tilde{H}^{-1/2}(\Gamma)} \leq Cp^{-\alpha}, \quad \alpha = 1/2 + \sigma - \varepsilon,\ \varepsilon > 0, \tag{2.3}$$

where $C > 0$ depends on ε but not on p. Here, $\mathbf{t}$ is the solution of (1.4) and $\mathbf{t}_p$ is the boundary element approximation to $\mathbf{t}$ given by (2.1).

This error estimate is quasi-optimal for sufficiently smooth given data. More precisely, if σ is large enough then there exists for any $\varepsilon > 0$ a constant $c > 0$ such that the p-version converges like $cp^{-1+\varepsilon}$. A convergence like cp^{-1} would be optimal, cf. the results in [7, 2]. The sub-optimality of (2.3) is due to Proposition 1.1 which states the regularity of the term $\boldsymbol{\beta}$ in the representation of the exact solution only in standard Sobolev spaces, which are not appropriate to obtain optimal results. For numerical results (dealing with the scalar version of the Laplace operator) which underline the a priori error estimate we refer to [6].

The proof of Theorem 2.1 is given in the next section.

3 Technical details

Before proving Theorem 2.1 we collect several auxiliary results.

Lemma 3.1 Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain. If $u \in \tilde{H}^t(\Omega)$ with $0 \leq t \leq 1$, then for $i = 1, 2$, $\partial u/\partial x_i \in \tilde{H}^{t-1}(\Omega)$, and

$$\|\partial u/\partial x_i\|_{\tilde{H}^{t-1}(\Omega)} \leq C\|u\|_{\tilde{H}^t(\Omega)},$$

where $C > 0$ is independent of u.

On an interval, this statement is proved in [9, Lemma 3.5]. In two dimensions the proof is similar and is skipped.

Lemma 3.2 Let Ω, Ω_1 be two Lipschitz domains in $\mathbf{R}^n$ $(n = 1, 2, 3)$, and $\Omega_1 \subset \Omega$. Then, for $0 \leq t < 1/2$, there holds

$$\|u\|_{\tilde{H}^{-t}(\Omega_1)} \leq C\|u\|_{\tilde{H}^{-t}(\Omega)} \quad \forall u \in \tilde{H}^{-t}(\Omega), \tag{3.1}$$

where the constant $C > 0$ is independent of u.

Proof. For $0 \leq t < 1/2$, the identity $H_0^t(\Omega_1) = H^t(\Omega_1)$ holds (see, e.g., [5]). Let us consider the function $v \in H^t(\Omega_1) = H_0^t(\Omega_1)$ and denote by $\overline{v}$ the extension of v by zero outside Ω_1. Then $\overline{v} \in H^t(\Omega) = H_0^t(\Omega)$,

$$\|\overline{v}\|_{H^t(\Omega)} \leq C\left(\|\overline{v}\|_{H^t(\Omega_1)} + \|\overline{v}\|_{H^t(\Omega\setminus\Omega_1)}\right) = C\|v\|_{H^t(\Omega_1)},$$

and (3.1) follows from the definition of the norm in $\tilde{H}^{-t}(\Omega_1)$. □

Lemma 3.3 Let $f \in H^t(K)$ for real $t > 0$ with $K = I \subset \mathbf{R}$ (respectively, $K = Q$ or $K = T$ in $\mathbf{R}^2$). Then there exists a sequence $f_p \in \mathcal{Q}_p(K)$, $p = 0, 1, 2, \ldots$, such that

$$\|f - f_p\|_{L_2(K)} \le C\, p^{-t}\, \|f\|_{H^t(K)}.$$

For a proof of Lemma 3.3 we refer to [1].

Lemma 3.4 [10, Lemma 3.3] Let $f(x) \in \tilde{H}^{-t_1}(I_1)$ and $g(y) \in \tilde{H}^{-t_2}(I_2)$ with $0 \le t_1,\, t_2 \le 1$. Then $f(x)g(y) \in \tilde{H}^{-t_1-t_2}(I_1 \times I_2)$ and

$$\|f(x)g(y)\|_{\tilde{H}^{-t_1-t_2}(I_1\times I_2)} \le c\|f(x)\|_{\tilde{H}^{-t_1}(I_1)}\|g(y)\|_{\tilde{H}^{-t_2}(I_2)}.$$

The constant c is independent of f and g.

To analyze the approximation of the singular part of $\mathbf{t}$ in (1.5) we first study singularities on an interval. Let us consider the singular function

$$\psi(x) = (1+x)^{\lambda-1}\chi(x), \quad x \in I = (-1,1), \tag{3.2}$$

where $\lambda > 0$ is real, $\chi \in C^\infty(I)$ is a cut-off function with $\chi(x) = 1$ for $x \in (-1, -1+d]$ and $\chi(x) = 0$ for $x \ge -1+2d$ $(0 < d \le 1/4)$.

Observe that $\psi \in \tilde{H}^t(I)$ for $-1 \le t < \min\{0, \lambda - 1/2\}$.

Theorem 3.1 Let $\psi(x)$ be given by (3.2) with $\lambda > 0$. Then there exists a sequence $\psi_p \in \mathcal{Q}_p(I)$, $p = 1, 2, \ldots$, such that for $-1 \le t < \min\{0, \lambda - 1/2\}$,

$$\|\psi - \psi_p\|_{\tilde{H}^t(\tilde{I})} \le C\, p^{-2(\lambda-1/2-t)}, \quad \tilde{I} = (-1,0). \tag{3.3}$$

Proof. Introducing a C^∞ cut-off function $\tilde{\chi}(x)$ such that

$$\tilde{\chi}(x) = 1 \;\text{ for }\; x \in [-1,0] \;\text{ and }\; \tilde{\chi}(x) = 0 \;\text{ for }\; x \ge 1/2, \tag{3.4}$$

we define

$$\Psi(x) := \tilde{\chi}(x)\int_{-1}^{x} \psi(\xi)d\xi, \quad \hat{\Psi}(x) := (1-x)^{-1}\Psi(x), \quad x \in I = (-1,1).$$

Then $\Psi(-1) = \hat{\Psi}(-1) = 0$, $\Psi(x) = \hat{\Psi}(x) = 0$ for $x \in [1/2, 1]$, and

$$\Psi'(x) = \psi(x) \;\text{ for }\; x \in \tilde{I} = (-1,0). \tag{3.5}$$

Further, using integration by parts we obtain

$$\hat{\Psi}(x) = \frac{(1+x)^\lambda\chi(x)\tilde{\chi}(x)}{\lambda(1-x)} - \frac{\tilde{\chi}(x)}{\lambda(1-x)}\int_{-1}^{x}(1+\xi)^\lambda\chi'(\xi)d\xi =: F(x) - G(x). \tag{3.6}$$

Referring to [2, Theorem 3.1] if $0 < \lambda \le 1/2$ and to [7, Theorem 5.1] if $\lambda > 1/2$, we find a polynomial $F_p \in \mathcal{Q}_p(I)$ such that $F_p(-1) = F(-1) = 0$ and

$$\|F - F_p\|_{H^t(I)} \le C\, p^{-2(\lambda+1/2-t)}, \quad 0 \le t < \min\{1, \lambda + 1/2\}. \tag{3.7}$$

For the function $G \in C_0^\infty(I)$ there exists by Lemma 3.3 a polynomial $G_p \in \mathcal{Q}_p(I)$ such that $G_p(\pm 1) = G(\pm 1) = 0$, and for arbitrary $\tau > 0$,

$$\|G - G_p\|_{H^t(I)} \le C\, p^{-\tau}, \quad 0 \le t \le 1. \tag{3.8}$$

Let us define $\Psi_p(x) := (1 - x)(F_p(x) - G_p(x))$. Then $\Psi_p \in \mathcal{Q}_{p+1}(I)$, $\Psi_p(\pm 1) = 0$, and for $0 \le t < \min\{1, \lambda + 1/2\}$ we deduce from (3.6)–(3.8)

$$\|\Psi - \Psi_p\|_{H^t(I)} \le C\|\hat{\Psi} - (F_p - G_p)\|_{H^t(I)} \le C\, p^{-2(\lambda+1/2-t)}. \tag{3.9}$$

Hence

$$\|\Psi - \Psi_p\|_{\tilde{H}^t(I)} \le C\, p^{-2(\lambda+1/2-t)}, \quad t \in [0, \min\{1, \lambda + 1/2\}) \backslash \{1/2\}, \tag{3.10}$$

because $(\Psi - \Psi_p) \in H_0^t(I) = \tilde{H}^t(I)$ for these values of t.

Now we set $\psi_p(x) := \Psi_p'(x)$ for $x \in I$. Then $\psi_p \in \mathcal{Q}_p(I)$, and recalling (3.5) we have $\psi - \psi_p = (\Psi - \Psi_p)'$ on $\tilde{I}$. Therefore, using sequentially the one-dimensional versions of Lemmas 3.2, 3.1, and then estimate (3.10) we obtain for any fixed $t' \in (1/2, \min\{1, \lambda + 1/2\})$

$$\begin{aligned} \|\psi - \psi_p\|_{\tilde{H}^{t'-1}(\tilde{I})} &= \|(\Psi - \Psi_p)'\|_{\tilde{H}^{t'-1}(\tilde{I})} \le C\, \|(\Psi - \Psi_p)'\|_{\tilde{H}^{t'-1}(I)} \\ &\le C\, \|\Psi - \Psi_p\|_{\tilde{H}^{t'}(I)} \le C\, p^{-2(\lambda+1/2-t')}. \end{aligned} \tag{3.11}$$

Thus we have proved (3.3) for $t \in (-1/2, \min\{0, \lambda - 1/2\})$.

On the other hand, applying Lemma 3.1 and inequality (3.9) we have

$$\|\psi - \psi_p\|_{\tilde{H}^{-1}(\tilde{I})} = \|(\Psi - \Psi_p)'\|_{\tilde{H}^{-1}(\tilde{I})} \le C\, \|\Psi - \Psi_p\|_{H^0(\tilde{I})} \le C\, p^{-2(\lambda+1/2)}.$$

Since $-1/2 < t' - 1 < \min\{0, \lambda - 1/2\}$ in (3.11), the interpolation between $\tilde{H}^{-1}(\tilde{I})$ and $\tilde{H}^{t'-1}(\tilde{I})$ gives (3.3) for any $t \in [-1, -1/2]$. □

Remark 3.1 When proving Theorem 3.1 we have also established the following inequality (see (3.9))

$$\|\Psi - \Psi_p\|_{L_2(I)} \le C\, p^{-2(\lambda+1/2)}, \tag{3.12}$$

where $\Psi(x) = \tilde{\chi}(x) \int\limits_{-1}^{x} \psi(\xi) d\xi$, $\Psi_p(x) = \int\limits_{-1}^{x} \psi_p(\xi) d\xi$, the function $\psi(x)$ is given by (3.2), and $\psi_p(x)$ is a polynomial approximation to $\psi(x)$.

Moreover, $\Psi(x) \in L_2(I)$, and (3.12) yields

$$\|\Psi_p\|_{L_2(I)} \le C. \tag{3.13}$$

Now we prove the main result of the paper.

Proof of Theorem 2.1. Due to the regularity result of Proposition 1.1 and the quasi-optimal convergence (2.2) of the BEM, one only needs to find a piecewise polynomial function that approximates $\mathbf{t}$ in (1.5) with the upper bound stated by (2.3).

For elements at the boundary γ we need covering rectangles in surface coordinates. Let $\Gamma_j \subset A_\gamma$ be an element touching the boundary γ. Since $A_\gamma \subset (\Gamma_{\delta/2} \cup \gamma)$, there exist two points on γ with coordinates $(s_1, 0)$ and $(s_2, 0)$ such that

$$\Gamma_j \subset Q_j = \{(s,\rho) \in \Gamma_{\delta/2};\ s_1 < s < s_2,\ 0 < \rho < \delta/2\}.$$

First, we define an approximation $\mathbf{t}_{0,p}$ to the vector function $\mathbf{t}_0 \in \tilde{H}^\alpha(\Gamma) \subset H^\alpha(\Gamma)$ (hereafter, $\alpha = 1/2 + \sigma - \varepsilon > 0$ with sufficiently small $\varepsilon > 0$). If $\Gamma_j \subset (\Gamma \backslash A_\gamma)$, we apply Lemma 3.3 componentwise on the square (or triangle) K such that $\Gamma_j = X(M_j(K))$. However, if $\Gamma_j \subset A_\gamma$, we apply Lemma 3.3 on $Q_j \supset \Gamma_j$. Since Γ is smooth, the function $\mathbf{t}_0$ on $\Gamma_\delta \supset A_\gamma$ has the same regularity in terms of coordinates (s,ρ) as in terms of space variables $x = X(u)$. Therefore, recalling the definition of $S_p(\Gamma)$ and applying Lemma 3.3 as indicated above, we find $\mathbf{t}_{0,p} \in S_p(\Gamma)$ such that

$$\|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{\tilde{H}^{-1/2}(\Gamma_j)} \le \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{L_2(\Gamma_j)} \le Cp^{-\alpha}\|\mathbf{t}_0\|_{H^\alpha(\Gamma_j)} \le Cp^{-\alpha} \quad (3.14)$$

if $\Gamma_j \subset (\Gamma \backslash A_\gamma)$, and

$$\begin{aligned} \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{\tilde{H}^{-1/2}(\Gamma_j)} &\le \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{L_2(\Gamma_j)} \le \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{L_2(Q_j)} \\ &\le Cp^{-\alpha}\|\mathbf{t}_0\|_{H^\alpha(Q_j)} \le Cp^{-\alpha} \end{aligned} \quad (3.15)$$

if $\Gamma_j \subset A_\gamma$.

Now we consider the singular term $\boldsymbol{\beta}(s)\psi(\rho) = \boldsymbol{\beta}(s)\rho^{-1/2}\chi(\rho)$ in (1.5). Let $\Gamma_j \subset A_\gamma$, and $\Gamma_j \subset Q_j$ as above. Then using the one-dimensional version of Lemma 3.3 we approximate the function $\boldsymbol{\beta}(s) \in H^{1/2+\sigma}(\gamma)$: there exists $\boldsymbol{\beta}_p(s) \in [\mathcal{Q}_p(s_1, s_2)]^3$ satisfying

$$\|\boldsymbol{\beta} - \boldsymbol{\beta}_p\|_{L_2(s_1,s_2)} \le Cp^{-(1/2+\sigma)}\|\boldsymbol{\beta}\|_{H^{1/2+\sigma}(s_1,s_2)} \le Cp^{-(1/2+\sigma)}\|\boldsymbol{\beta}\|_{H^{1/2+\sigma}(\gamma)}. \quad (3.16)$$

For the singular function $\psi(\rho)$ we apply Theorem 3.1, scaled to the interval $(0,\delta)$, with $\lambda = 1/2$: there exists a polynomial $\psi_p(\rho) \in \mathcal{Q}_p(0,\delta)$ satisfying

$$\|\psi - \psi_p\|_{\tilde{H}^{-t}(0,\delta/2)} \le Cp^{-2t}, \qquad 0 < t \le 1. \quad (3.17)$$

Since $\psi(\rho) \in \tilde{H}^{-t}(0,\delta/2)$ with $t \in (0,1]$, we estimate by (3.17)

$$\|\psi_p\|_{\tilde{H}^{-t}(0,\delta/2)} \le C, \qquad 0 < t \le 1. \quad (3.18)$$

Furthermore, introducing a C^∞ cut-off function $\tilde{\chi}(\rho)$ such that (cf. (3.4))

$$\tilde{\chi}(\rho) = 1 \quad \text{for} \quad \rho \in [0, \delta/2] \quad \text{and} \quad \tilde{\chi}(\rho) = 0 \quad \text{for} \quad \rho \geq 3\delta/4$$

and arguing as in the proof of Theorem 3.1 we obtain (cf. (3.12), (3.13))

$$\|\Psi - \Psi_p\|_{L_2(0,\delta)} \leq C\, p^{-2}, \quad \|\Psi_p\|_{L_2(0,\delta)} \leq C, \tag{3.19}$$

where $\Psi(\rho) = \tilde{\chi}(\rho) \int\limits_0^\rho \psi(r)dr$ and $\Psi_p(\rho) = \int\limits_0^\rho \psi_p(r)dr$.

Then making use of Lemma 3.2 (which remains valid with $\Omega_1 = \Gamma_j \subset Q_j = \Omega$), Lemma 3.4, the triangle inequality, and estimates (3.16)–(3.18), we derive for some fixed $t' \in (0, 1/2)$

$$\begin{aligned}
&\|\boldsymbol{\beta}\psi - \boldsymbol{\beta}_p\psi_p\|_{\tilde{H}^{-t'}(\Gamma_j)} \\
&\quad \leq C\Big(\|\boldsymbol{\beta}(\psi - \psi_p)\|_{\tilde{H}^{-t'}(Q_j)} + \|(\boldsymbol{\beta} - \boldsymbol{\beta}_p)\psi_p\|_{\tilde{H}^{-t'}(Q_j)}\Big) \\
&\quad \leq C\Big(\|\boldsymbol{\beta}\|_{L_2(s_1,s_2)}\|\psi-\psi_p\|_{\tilde{H}^{-t'}(0,\delta/2)} + \|\boldsymbol{\beta}-\boldsymbol{\beta}_p\|_{L_2(s_1,s_2)}\|\psi_p\|_{\tilde{H}^{-t'}(0,\delta/2)}\Big) \\
&\quad \leq Cp^{-\min\{1/2+\sigma, 2t'\}}\|\boldsymbol{\beta}\|_{H^{1/2+\sigma}(\gamma)}.
\end{aligned} \tag{3.20}$$

On the other hand, using the above notation for $\Psi(\rho)$ and $\Psi_p(\rho)$ we write

$$\|\boldsymbol{\beta}\psi - \boldsymbol{\beta}_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} = \Big\|\frac{\partial}{\partial\rho}\Big(\boldsymbol{\beta}(s)\Psi(\rho) - \boldsymbol{\beta}_p(s)\Psi_p(\rho)\Big)\Big\|_{\tilde{H}^{-1}(\Gamma_j)}.$$

Then applying Lemma 3.1 in terms of coordinates $(s, \rho) \in \Gamma_j$ we have

$$\begin{aligned}
\|\boldsymbol{\beta}\psi - \boldsymbol{\beta}_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} &\leq C\,\|\boldsymbol{\beta}(s)\Psi(\rho) - \boldsymbol{\beta}_p(s)\Psi_p(\rho)\|_{H^0(\Gamma_j)} \\
&\leq C\,\|\boldsymbol{\beta}(s)\Psi(\rho) - \boldsymbol{\beta}_p(s)\Psi_p(\rho)\|_{H^0(Q_j)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\boldsymbol{\beta}\psi - \boldsymbol{\beta}_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} \leq\; & C\Big(\|\boldsymbol{\beta}\|_{L_2(s_1,s_2)}\|\Psi-\Psi_p\|_{L_2(0,\delta/2)} \\
& + \|\boldsymbol{\beta}-\boldsymbol{\beta}_p\|_{L_2(s_1,s_2)}\|\Psi_p\|_{L_2(0,\delta/2)}\Big),
\end{aligned}$$

and we estimate by using (3.16), (3.19)

$$\|\boldsymbol{\beta}\psi - \boldsymbol{\beta}_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} \leq Cp^{-\min\{1/2+\sigma,\, 2\}}\|\boldsymbol{\beta}\|_{H^{1/2+\sigma}(\gamma)}.$$

Since $|\sigma| < 1/2$, we may take t' in (3.20) such that $0 < 1/2 + \sigma \leq 2t' < 1$. Then interpolating between $\tilde{H}^{-1}(\Gamma_j)$ and $\tilde{H}^{-t'}(\Gamma_j)$ we prove for any $\Gamma_j \subset A_\gamma$

$$\|\boldsymbol{\beta}\psi - \boldsymbol{\beta}_p\psi_p\|_{\tilde{H}^{-1/2}(\Gamma_j)} \leq Cp^{-(1/2+\sigma)}\|\boldsymbol{\beta}\|_{H^{1/2+\sigma}(\gamma)}. \tag{3.21}$$

Now let us define the approximating function $\mathbf{v}_p$ on Γ as follows: $\mathbf{v}_p|_{\Gamma_j} = \boldsymbol{\beta}_p\psi_p + \mathbf{t}_{0,p}|_{\Gamma_j}$ if $\Gamma_j \subset A_\gamma$, and $\mathbf{v}_p|_{\Gamma_j} = \mathbf{t}_{0,p}|_{\Gamma_j}$ if $\Gamma_j \subset (\Gamma \backslash A_\gamma)$. Then $\mathbf{v}_p \in S_p(\Gamma)$, and, due to (3.14), (3.15), (3.21), the error $\|\mathbf{t} - \mathbf{v}_p\|_{\tilde{H}^{-1/2}(\Gamma)}$ satisfies the upper bound in (2.3). This proves the theorem. □

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