Discontinuous Galerkin finite element methods for dynamic linear solid viscoelasticity problems

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Abstract

We consider the usual linear elastodynamics equations augmented with evolution equations for viscoelastic internal stresses. Both semi- and fully discrete approximations are defined, based on a discontinuous Galerkin finite element method, and error estimates are given.

1 Introduction

This is the second in a series of papers, [12, 11], extending spatially discontinous Galerkin methods to viscoelasticity problems.

We consider a model for the dynamic response of linear viscoelastic solids. This comprises the usual equations of elastodynamics, but augmented with evolution equations for the viscoelastic internal stresses. The spatial discretisation is effected by a discontinuous Galerkin finite element method (DG FEM), which can be taken as either a symmetric or non-symmetric scheme, and the time discretisation is a standard finite difference method of Crank-Nicolson type.

For the analagous quasistatic problem considered in [12] we represented the viscoelasticity through a hereditary integral. Here we have chosen

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the alternative representation through internal variables. The reasons for this are, firstly, to show that the error estimates can be extended to this case and, secondly, because some practitioners prefer to work with internal variables rather than history integrals (see e.g. [6, 5]).

For background to viscoelasticity and the assumptions we make we refer back to [12], and for more general background to the application of DG methods we refer to [9, 8, 3, 13, 10] and, in particular, to the elastic problem studied in [10].

This article is arranged as follows. We finish this section with some notation and then in Section 2 describe the model problem and the spatial discretisation. Estimates for the semidiscrete scheme are then given in Section 3 and for the fully discrete scheme in Section 4. We then conclude with Section 5.

Let $\Omega \subset \mathbb{R}^d, d = 2, 3$, be an open bounded domain with polygonal/polyhedral boundary and let $I = (0, T)$ be a finite time interval. The following notation is standard. For $\omega \subseteq \bar{\Omega}$,

$$(\boldsymbol{v}, \boldsymbol{w})_\omega := \int_\omega \boldsymbol{v} \cdot \boldsymbol{w} \, d\omega,$$

but we drop the subscript when $\omega = \Omega$, We use $\|\cdot\|_{p,\omega}$ to denote the $\boldsymbol{H}^p(\omega) := (H^p(\omega))^d$ norm and again abbreviate, $\|\cdot\|_m = \|\cdot\|_{m,\Omega}$, when $\omega = \Omega$. Since we are dealing with time dependent functions we take the usual approach of treating these as maps from time into a Banach space and set,

$$\|v\|_{L_p(0,t;X)} := \left(\int_0^t \|v(t)\|_X^p \, dt \right)^{1/p},$$

for $t \leqslant T$, $1 \leqslant p < \infty$ and with the obvious modification for $p = \infty$. When $t = T$ we abbreviate: $\|\cdot\|_{L_2(L_2)} := \|\cdot\|_{L_2(0,T;L_2(\Omega))}$ and so on.

We need also to deal with scalar- and tensor-valued functions and, to ease notation, we make no distinction with the inner products and norms in these cases.

2 Model problem and spatial discretisation

The basic equations are,

$$\rho \boldsymbol{u}_{tt}(\boldsymbol{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}; \boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, t) \quad \text{in} \quad \Omega \times I, \tag{1}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \bar{\boldsymbol{u}}(\boldsymbol{x}) \quad \text{in} \quad \bar{\Omega}, \tag{2}$$

$$\boldsymbol{u}_t(\boldsymbol{x}, 0) = \bar{\boldsymbol{z}}(\boldsymbol{x}) \quad \text{in} \quad \bar{\Omega}, \tag{3}$$

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \quad \text{on} \quad \Gamma_D \times \bar{I}, \tag{4}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}; \boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}, t), \quad \text{on} \quad \Gamma_N \times \bar{I}. \tag{5}$$

In these $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \varnothing$, and we assume that Γ_D is a closed set with positive surface measure. We do not explicitly display the $\boldsymbol{x}$ dependence in most of what follows. Also, to ease notation, we denote partial time differentiation with either a subscript, as above, or a dot. Thus $\dot{\boldsymbol{u}} = \boldsymbol{u}_t$, $\ddot{\boldsymbol{u}} = \boldsymbol{u}_{tt}$, and so on. We also assume that the boundary and initial data are compatible at $t = 0$.

The symmetric second-order stress tensor satisfies the constitutive relation,

$$\boldsymbol{\sigma}(\boldsymbol{u}(t)) = \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \sum_{i=1}^{N_\varphi} \gamma_i {}^*\boldsymbol{\sigma}_i(t),$$

where: $\varepsilon_{ij}(\boldsymbol{u}) := (u_{i,j} + u_{j,i})/2$; for $i = 1, \dots, N_\varphi$,

$$^*\boldsymbol{\sigma}_i(t) = \int_0^t \gamma_i e^{-(t-s)/\tau_i} \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds : \tag{6}$$

and, the fourth order Hooke's tensor, $\boldsymbol{D}$, satisfies the symmetries,

$$D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij},$$

and is positive definite over symmetric second order tensors. Also,

$$\gamma_i = \left(\frac{\varphi_i}{\tau_i}\right)^{1/2},$$

where the φ_i and τ_i are positive constants, and we impose the normalisation,

$$\sum_{i=0}^{N_\varphi} \varphi_i = 1$$

with, additionally (see (52) later), $\varphi_0 > 0$ (note that we have a φ_0 but not a τ_0). Then it follows that,

$$\sum_{i=1}^{N_\varphi} \gamma_i^2 \tau_i = 1 - \varphi_0 > 0. \tag{7}$$

From (6), we see that each of the internal stress tensors, $^*\boldsymbol{\sigma}_i$, satisfies an initial value problem,

$$^*\dot{\boldsymbol{\sigma}}_i(t) + \frac{1}{\tau_i} {}^*\boldsymbol{\sigma}_i(t) = \gamma_i \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{8}$$

$$^*\boldsymbol{\sigma}_i(0) = \boldsymbol{0}, \tag{9}$$

and on the other hand, if we eliminate the viscous stresses, our basic equation becomes a second-order hyperbolic partial differential equation with a fading memory Volterra integral. For the well-posedness of these types of equations we refer to [2], and for numerical analysis we cite, for example, [14, 7, 15, 4, 1]. All of these deal with the Volterra form of the problem whereas, here, we include the viscoelasticity through the evolution equations for the internal variables, (8). We are not aware of literature containing error estimates for this approach.

From our definitions we obtain the following regularity estimates.

Lemma 2.1 *For each $i = 1, \dots, N_\varphi$ we have,*

$$\left\| \frac{\partial^{n*}\boldsymbol{\sigma}_i}{\partial t^n} \right\|_{L_2(0,t;\boldsymbol{L}_2(\Omega))} \leqslant C \sum_{j=0}^{n-1} \left\| \frac{\partial^j \boldsymbol{u}}{\partial t^j} \right\|_{L_2(0,t;\boldsymbol{H}^1(\Omega))},$$

for $n = 1, 2, \dots$.

Proof. Taking norms in (6) and using Hölder's inequality for convolutions gives, $\|{}^*\boldsymbol{\sigma}_i\|_{L_2(0,t;\boldsymbol{L}_2(\Omega))} \leqslant C\|\boldsymbol{u}\|_{L_2(0,t;\boldsymbol{H}^1(\Omega))}$. Now use successive differentiation on (8) and recursively apply the estimates obtained. □

We also have the following.

Lemma 2.2 *For each $i = 1, \dots, N_\varphi$ we have,*

$$\|{}^*\boldsymbol{\sigma}_i(t)\|_r \leqslant C\|\boldsymbol{u}(t)\|_{r+1}$$

for $r \geqslant 0$.

The first step towards spatial discretisation is to establish some more notation. Let $\mathcal{E}_h = \{E_1, E_2, \dots, E_{N_h}\}$ be a nondegenerate quasiuniform subdivision of Ω, where E_j is a triangle if $d = 2$, or a tetrahedron if $d = 3$. The nondegeneracy requirement is that there exists $\rho > 0$ such that if $h_j = \operatorname{diam}(E_j)$, then E_j contains a ball of radius ρh_j in its interior. Let $h = \max\{h_j : 1 \leqslant j \leqslant N_h\}$, the quasiuniformity requirement is that there exists $\tau > 0$ such that $h/h_j \leqslant \tau$ for all $j \in \{1, \dots, N_h\}$. We denote the set of interior edges (faces for $d = 3$) of $\mathcal{E}_h$ by Γ_h. With each edge (or face) e, we associate a unit normal vector $\boldsymbol{n}_e$. For a boundary edge (or face), $\boldsymbol{n}_e$ is taken to be the unit outward vector normal to $\partial\Omega$.

We now define the average and the jump operators. For each of the interior edges suppose the neighbouring elements of e are E_e^1 and E_e^2 so that $e = \partial E_e^1 \cap \partial E_e^2$, and for a boundary edge suppose that E_e is the neighbouring element. We define the averaging operator $\{\cdot\}$ by,

$$\{\boldsymbol{w}\} := \begin{cases} \frac{1}{2}(\boldsymbol{w}|_{E_e^1})|_e + \frac{1}{2}(\boldsymbol{w}|_{E_e^2})|_e & \text{if } e \subset \Omega, \\ (\boldsymbol{w}|_{E_e})|_e & \text{if } e \subset \partial\Omega. \end{cases}$$

and the jump operator $[\cdot]$ by,

$$[\boldsymbol{w}] := \begin{cases} (\boldsymbol{w}|_{E_e^1})|_e - (\boldsymbol{w}|_{E_e^2})|_e & \text{if } e \subset \Omega, \\ \\ (\boldsymbol{w}|_{E_e})|_e & \text{if } e \subset \partial\Omega. \end{cases}$$

The distinction between $[\cdot]$ and $-[\cdot]$ can be made because each edge e has a unit normal associated with it. The "direction" in which the jump takes place is unimportant.

These operators are well defined if $\boldsymbol{w}|_{E_e^i} \in (H^{\frac{1}{2}+\epsilon}(E_e^i))^d$ for $i = 1, 2$ and $\epsilon > 0$. Below, we use $|e|$ to denote the $(d-1)$-dimensional surface measure of the edge/face e. We also frequently use the estimate, $|e| \leqslant Ch^{d-1}$ which arises as a consequence of our assumptions.

Define the broken spaces for any integer $r \geqslant 0$,

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L_2(\Omega) : v|_E \in I\!P_r(E) \quad \forall E \in \mathcal{E}_h\}, \tag{10}$$

$$\mathbf{D}_r(\mathcal{E}_h) = \mathcal{D}_r(\mathcal{E}_h)^d, \tag{11}$$

$$\mathbf{L}_r(\mathcal{E}_h) = \mathcal{D}_r(\mathcal{E}_h)^{d\times d}. \tag{12}$$

For these finite element spaces we have the following interpolation-error estimates. If $\boldsymbol{v} \in \boldsymbol{H}^n(\mathcal{E}_h) \cap C(\bar{\Omega})^d$ and $\mu = \min\{r+1, n\}$ then there is an interpolant $\hat{\boldsymbol{v}} \in \mathbf{D}_r(\mathcal{E}_h) \cap C(\bar{\Omega})^d$ such that for each $E \in \mathcal{E}_h$,

$$\|\boldsymbol{v} - \hat{\boldsymbol{v}}\|_{m,E} \leqslant Ch_E^{\mu-m}\|\boldsymbol{v}\|_{n,E} \qquad \text{for } n \geqslant m \geqslant 0, \tag{13}$$

$$\|\boldsymbol{v} - \hat{\boldsymbol{v}}\|_{m,\gamma} \leqslant Ch_E^{\mu-m-1/2}\|\boldsymbol{v}\|_{n,E} \qquad \text{for } m = 0, 1 \text{ and } n \geqslant m, \tag{14}$$

where $\gamma \subseteq \partial E$.

For positive constants, δ and β, define the bilinear forms,

$$J_0^{\delta,\beta}(\boldsymbol{w}, \boldsymbol{v}) = \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{\delta}{|e|^\beta}\int_e [\boldsymbol{w}]\cdot[\boldsymbol{v}], \tag{15}$$

$$\begin{aligned} A(\boldsymbol{w}, \boldsymbol{v}) = {} & \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w}) \colon \boldsymbol{\varepsilon}(\boldsymbol{v}) - \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{w})\boldsymbol{n}_e\}\cdot[\boldsymbol{v}] \\ & + \kappa \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n}_e\}\cdot[\boldsymbol{w}] + J_0^{\delta,\beta}(\boldsymbol{w}, \boldsymbol{v}). \end{aligned} \tag{16}$$

Here κ is a switch: we set $\kappa = 1$ to obtain the non-symmetric DG scheme, and $\kappa = -1$ to obtain the symmetric scheme.

Defining $\boldsymbol{z}(t) := \boldsymbol{u}_t(t)$, we first note that if $\boldsymbol{z}(t)$, $\boldsymbol{u}(t) \in C(\bar{\Omega})^d$ for each

t, then we have,

$$
\begin{aligned}
&(\rho\dot{\boldsymbol{z}}(t),\boldsymbol{v}) + A(\boldsymbol{u}(t),\boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{z}(t),\boldsymbol{v}) \\
&+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \gamma_i\{{}^*\boldsymbol{\sigma}_i(t)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}] - \sum_{i=1}^{N_\varphi}\sum_E\int_E \gamma_i{}^*\boldsymbol{\sigma}_i(t):\boldsymbol{\varepsilon}(\boldsymbol{v}) \\
&= L(t;\boldsymbol{v}) \qquad \forall \boldsymbol{v}\in\mathbf{D}_r(\mathcal{E}_h),
\end{aligned} \tag{17}
$$

where

$$
L(t;\boldsymbol{v}) := (\boldsymbol{f}(t),\boldsymbol{v}) + (\boldsymbol{g}(t),\boldsymbol{v})_{\Gamma_N},
$$

and, for each $i = 1,\ldots,N_\varphi$,

$$
\begin{aligned}
\sum_E({}^*\dot{\boldsymbol{\sigma}}(t) + \frac{1}{\tau_i}{}^*\boldsymbol{\sigma}(t),\boldsymbol{w}_i)_E &= \sum_E \gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{w}_i)_E \\
&- \sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{u}(t)] \qquad \forall \boldsymbol{w}_i\in\mathbf{L}_{r-1}(\mathcal{E}_h),
\end{aligned} \tag{18}
$$

and,

$$
(\rho\boldsymbol{z}(t),\boldsymbol{v})_E = (\rho\dot{\boldsymbol{u}}(t),\boldsymbol{v})_E \qquad \forall\boldsymbol{v}\in\mathbf{D}_r(\mathcal{E}_h). \tag{19}
$$

Equations (17) and (18) arise from elementwise integration by parts, see [12], and 'adding zero'.

We will also use the following norm and semi-norm,

$$
\begin{aligned}
\|\boldsymbol{v}\|_{\mathcal{A}} &:= \Big(|\boldsymbol{v}|_{\mathcal{E}}^2 + J_0^{\delta,\beta}(\boldsymbol{v},\boldsymbol{v})\Big)^{\frac{1}{2}}, \\
|\boldsymbol{v}|_{\mathcal{E}} &:= \left(\sum_{E\in\mathcal{E}_h}\int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v})\right)^{\frac{1}{2}}.
\end{aligned}
$$

3 Semidiscrete error estimates

For $r > 0$ the semidiscrete DG scheme is as follows: find $\boldsymbol{u}^h : I \to \mathbf{D}_r(\mathcal{E}_h)$ and ${}^*\boldsymbol{\sigma}_i^h : I \to \mathbf{L}_{r-1}(\mathcal{E}_h)$ such that,

$$
\begin{aligned}
&(\rho\boldsymbol{u}_{tt}^h(t),\boldsymbol{v}) + A(\boldsymbol{u}^h(t),\boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{u}_t^h(t),\boldsymbol{v}) \\
&+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{{}^*\boldsymbol{\sigma}_i^h(t)\boldsymbol{n}_e\}\cdot[\boldsymbol{v}] - \sum_{i=1}^{N_\varphi}\sum_E\gamma_i\int_E{}^*\boldsymbol{\sigma}_i^h(t):\boldsymbol{\varepsilon}(\boldsymbol{v}) \\
&= L(t;\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\mathbf{D}_r(\mathcal{E}_h),
\end{aligned} \tag{20}
$$

along with, for $i = 1, \dots, N_\varphi$,

$$({}^*\dot{\boldsymbol{\sigma}}_i^h(t) + \frac{1}{\tau_i}{}^*\boldsymbol{\sigma}_i^h(t), \boldsymbol{w}) = \sum_E \gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}^h(t)), \boldsymbol{w})_E$$
$$- \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{D}\boldsymbol{w}\boldsymbol{n}_e\} \cdot [\boldsymbol{u}^h(t)] \quad \forall \boldsymbol{w} \in \mathbf{L}_{r-1}(\mathcal{E}_h). \tag{21}$$

The initial conditions are given by ${}^*\boldsymbol{\sigma}_i(0) = \mathbf{0}$ and,

$$A(\boldsymbol{u}^h(0), \boldsymbol{v}) = A(\bar{\boldsymbol{u}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h), \tag{22}$$
$$(\rho\boldsymbol{u}_t^h(0), \boldsymbol{v}) = (\rho\bar{\boldsymbol{z}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{23}$$

We first prove an optimal error estimate for the natural ‘energy’ norm, and then follow it with an error estimate in the for the $L_2(\Omega)$ norm.

Theorem 3.1 (semidiscrete ‘energy’ error estimate) *If $h \leqslant \hat{h}$, $\beta \geqslant (d-1)^{-1}$, $\bar{\boldsymbol{u}} \in \boldsymbol{H}^{r+1}(\Omega)$, $\bar{\boldsymbol{z}} \in \boldsymbol{H}^r(\Omega)$ and $\boldsymbol{u} \in H^2(\boldsymbol{H}^{r+1}) \cap C^1(C(\bar{\Omega})^d)$ then, for $\kappa = \pm 1$ and $\hat{h}^{(d-1)\beta-1}/\delta$ small enough, there exists a constant, C, independent of h, such that,*

$$\|\rho^{1/2}(\boldsymbol{u}_t(\tau) - \boldsymbol{u}_t^h(\tau))\|_0 + \|\boldsymbol{u}(\tau) - \boldsymbol{u}^h(\tau)\|_{\mathcal{A}}$$
$$+ \left(\varphi_0 \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\boldsymbol{D}^{-1/2}({}^*\boldsymbol{\sigma}_i(\tau) - {}^*\boldsymbol{\sigma}_i^h(\tau))\|_0^2\right)^{1/2} \leqslant Ch^r,$$

for all $\tau \in I$.

Proof: Set,

$$\boldsymbol{\chi} = \boldsymbol{u}^h - \tilde{\boldsymbol{u}}, \qquad \boldsymbol{\eta}_i = {}^*\boldsymbol{\sigma}_i^h - {}^*\tilde{\boldsymbol{\sigma}}_i,$$
$$\boldsymbol{\xi} = \boldsymbol{u} - \tilde{\boldsymbol{u}}, \qquad \boldsymbol{\theta}_i = {}^*\boldsymbol{\sigma}_i - {}^*\tilde{\boldsymbol{\sigma}}_i,$$

then $\boldsymbol{u}^h - \boldsymbol{u} = \boldsymbol{\chi} - \boldsymbol{\xi}$ and ${}^*\boldsymbol{\sigma}_i^h - {}^*\boldsymbol{\sigma}_i = \boldsymbol{\eta}_i - \boldsymbol{\theta}_i$.

We will define later what $\tilde{\boldsymbol{u}} \in \mathbf{D}_r(\mathcal{E}_h)$ and ${}^*\tilde{\boldsymbol{\sigma}}_i \in \boldsymbol{L}_{r-1}(\mathcal{E}_h)$ are. The error equations follow by subtracting (17), (18) from (20), (21). They are,

$$(\rho\boldsymbol{\chi}_{tt}^h, \boldsymbol{v}) + A(\boldsymbol{\chi}, \boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{v})$$
$$+ \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i\boldsymbol{n}_e\}[\boldsymbol{v}] - \sum_{i=1}^{N_\varphi}\sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\boldsymbol{v})$$
$$= (\rho\boldsymbol{\xi}_{tt}, \boldsymbol{v}) + A(\boldsymbol{\xi}, \boldsymbol{v}) + J_0^{\delta,\beta}(\boldsymbol{\xi}_t, \boldsymbol{v}) + \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{\theta}_i\boldsymbol{n}_e\}[\boldsymbol{v}]$$
$$- \sum_{i=1}^{N_\varphi}\sum_E \gamma_i \int_E \boldsymbol{\theta}_i : \boldsymbol{\varepsilon}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{24}$$

And, for each i,

$$\sum_E(\dot{\boldsymbol{\eta}}_i+\frac{1}{\tau_i}\boldsymbol{\eta}_i,\boldsymbol{w}_i)_E-\sum_E\gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}),\boldsymbol{w}_i)_E+\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}]$$
$$=\sum_E(\dot{\boldsymbol{\theta}}_i+\frac{1}{\tau_i}\boldsymbol{\theta}_i,\boldsymbol{w}_i)_E-\sum_E\gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}),\boldsymbol{w}_i)_E$$
$$+\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\xi}]\quad\forall\boldsymbol{w}_i\in\mathbf{L}_{r-1}(\mathcal{E}_h).\tag{25}$$

Choose $\boldsymbol{v}=\boldsymbol{\chi}_t$ in (24) and $\boldsymbol{w}_i=\boldsymbol{D}^{-1}\dot{\boldsymbol{\eta}}_i$ in (25) and add the resulting equations to get,

$$\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2+\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\chi}\|_{\mathcal{A}}^2+J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t)$$
$$+\sum_{i=1}^{N_\varphi}\|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2+\sum_{i=1}^{N_\varphi}\frac{1}{2\tau_i}\frac{d}{dt}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2$$
$$=-\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{\eta}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_t]+\sum_{i=1}^{N_\varphi}\sum_E\gamma_i\int_E\boldsymbol{\eta}_i:\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t)$$
$$+\sum_{i=1}^{N_\varphi}\sum_E(\dot{\boldsymbol{\theta}}_i+\frac{1}{\tau_i}\boldsymbol{\theta}_i,\boldsymbol{D}^{-1}\dot{\boldsymbol{\eta}}_i)_E-\sum_{i=1}^{N_\varphi}\sum_E\gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\xi}),\dot{\boldsymbol{\eta}}_i)_E$$
$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\dot{\boldsymbol{\eta}}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\xi}]+(\rho\boldsymbol{\xi}_{tt},\boldsymbol{\chi}_t)+A(\boldsymbol{\xi},\boldsymbol{\chi}_t)+J_0^{\delta,\beta}(\boldsymbol{\xi}_t,\boldsymbol{\chi}_t)$$
$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{\theta}_i\boldsymbol{n}_e\}[\boldsymbol{\chi}_t]-\sum_{i=1}^{N_\varphi}\sum_E\gamma_i\int_E\boldsymbol{\theta}_i:\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t)$$
$$+\sum_{i=1}^{N_\varphi}\sum_E\gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi}),\dot{\boldsymbol{\eta}}_i)_E-\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\dot{\boldsymbol{\eta}}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}]$$
$$+\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_t]-\kappa\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}].$$

Now, if we choose $\tilde{\boldsymbol{u}}$ to be the elliptic projection:

$$A(\boldsymbol{\xi},\boldsymbol{v})=0,\quad\forall\boldsymbol{v}\in\mathbf{D}_r(\mathcal{E}_h),$$

and if we choose ${}^*\tilde{\boldsymbol{\sigma}}_i$ to be the L_2 projection into $\mathbf{L}_{r-1}(\mathcal{E}_h)$ we have,

$$A(\boldsymbol{\xi}, \boldsymbol{\chi}_t) = 0,$$
$$\sum_{i=1}^{N_\varphi} \sum_E \gamma_i \int_E \boldsymbol{\theta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) = 0,$$
$$\sum_{i=1}^{N_\varphi} \sum_E (\dot{\boldsymbol{\theta}}_i + \frac{1}{\tau_i}\boldsymbol{\theta}_i, \boldsymbol{D}^{-1}\dot{\boldsymbol{\eta}}_i)_E = 0.$$

Thus, we are left with

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2 + \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) \\
&+ \sum_{i=1}^{N_\varphi} \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i}\frac{d}{dt}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2 \\
= &-\sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{\eta}_i \boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_t] + \sum_{i=1}^{N_\varphi}\sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) \\
&- \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\xi}), \dot{\boldsymbol{\eta}}_i)_E + \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \boldsymbol{n}_e\}\cdot[\boldsymbol{\xi}] \\
&+ (\rho\boldsymbol{\xi}_{tt}, \boldsymbol{\chi}_t) + J_0^{\delta,\beta}(\boldsymbol{\xi}_t, \boldsymbol{\chi}_t) + \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{\theta}_i \boldsymbol{n}_e\}[\boldsymbol{\chi}_t] \\
&+ \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi}), \dot{\boldsymbol{\eta}}_i)_E - \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\dot{\boldsymbol{\eta}}_i \boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}] \\
&+ \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_t] - \kappa \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}].
\end{aligned}$$

We now integrate the equation over time from 0 to τ, and certain terms simplify by integration by parts:

$$\begin{aligned}
&\int_0^\tau \sum_{i=1}^{N_\varphi}\sum_E \gamma_i \int_E \boldsymbol{\eta}_i : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_t) + \int_0^\tau \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi}), \dot{\boldsymbol{\eta}}_i)_E \\
= &\sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau), \boldsymbol{\eta}_i(\tau))_E - \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0), \boldsymbol{\eta}_i(0))_E,
\end{aligned}$$

and

$$-\kappa\int_0^\tau \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi}_t)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}] = \kappa\int_0^\tau \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_t]$$
$$- \kappa \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}(\tau)]$$
$$+ \kappa \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}(0)].$$

Thus, the error equation becomes

$$\frac{1}{2}\|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + \frac{1}{2}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t)$$
$$+ \int_0^\tau \sum_{i=1}^{N_\varphi}\|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{1}{2\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2$$
$$= \frac{1}{2}\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + \frac{1}{2}\|\boldsymbol{\chi}(0)\|_{\mathcal{A}}^2$$
$$+ \sum_{i=1}^{N_\varphi}\frac{1}{2\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(0)\|_0^2 - \int_0^\tau \sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e \{\boldsymbol{\eta}_i\boldsymbol{n}_e\}[\boldsymbol{\chi}_t]$$
$$+ \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau),\boldsymbol{\eta}_i(\tau))_E - \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0),\boldsymbol{\eta}_i(0))_E$$
$$+ (1+\kappa)\int_0^\tau \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_t]$$
$$- \kappa \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}(\tau)]$$
$$+ \kappa \sum_{e\in\Gamma_h\cup\Gamma_D}\int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(0)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}(0)]$$
$$+ \int_0^\tau (\rho\boldsymbol{\xi}_{tt},\boldsymbol{\chi}_t) + \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\xi}_t,\boldsymbol{\chi}_t)$$
$$+ \int_0^\tau \sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e \{\boldsymbol{\theta}_i\boldsymbol{n}_e\}[\boldsymbol{\chi}_t] - \int_0^\tau \sum_{i=1}^{N_\varphi}\sum_E \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\xi}),\dot{\boldsymbol{\eta}}_i)_E$$
$$+ \int_0^\tau \sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e \{\dot{\boldsymbol{\eta}}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\xi}] - \int_0^\tau \sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e \{\dot{\boldsymbol{\eta}}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}]$$
$$= T_1 + \cdots + T_{15}.$$

We now bound each of the terms T_i, beginning with T_4,

$$
\begin{aligned}
|T_4| &\leqslant \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \|\{\boldsymbol{\eta}_i \boldsymbol{n}_e\}\|_{0,e} \frac{|e|^{\beta/2}}{\delta^{1/2}} \|[\boldsymbol{\chi}_t]\|_{0,e} \frac{\delta^{1/2}}{|e|^{\beta/2}}, \\
&\leqslant C\int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t)^{1/2} h^{(d-1)\beta/2-1/2} \|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0, \\
&\leqslant \frac{C}{\epsilon} h^{(d-1)\beta-1} \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2 + \frac{\epsilon}{2}\int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t),
\end{aligned}
$$

where we used (7). For T_5 we have,

$$
\begin{aligned}
|T_5| &\leqslant \sum_{i=1}^{N_\varphi} \gamma_i \|\boldsymbol{D}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau)\|_0 \|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0, \\
&\leqslant \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon}\gamma_i^2}{2\varphi_i} \|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{\varphi_i}{2\bar{\epsilon}} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2, \\
&\leqslant \sum_{i=1}^{N_\varphi} \frac{\bar{\epsilon}}{2\tau_i} \|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 + \frac{1-\varphi_0}{2\bar{\epsilon}} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2.
\end{aligned}
$$

If $\kappa = -1$, then the term T_7 vanishes. Otherwise if $\kappa = 1$, we have,

$$
\begin{aligned}
|T_7| &\leqslant 2\int_0^\tau \sum_{e\in\Gamma_h\cup\Gamma_D} \|\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})\boldsymbol{n}_e\}\|_{0,e} \frac{|e|^{\beta/2}}{\delta^{1/2}} \|[\boldsymbol{\chi}_t]\|_{0,e} \frac{\delta^{1/2}}{|e|^{\beta/2}}, \\
&\leqslant Ch^{(d-1)\beta/2-1/2} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}} J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t)^{1/2}, \\
&\leqslant \frac{C}{2\epsilon} h^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 dt + \frac{\epsilon}{2}\int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t)dt.
\end{aligned}
$$

For T_8,

$$
\begin{aligned}
|T_8| &\leqslant \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \|\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\chi})(\tau)\boldsymbol{n}_e\}\|_{0,e} \|[\boldsymbol{\chi}(\tau)]\|_{0,e} \left(\frac{\delta|e|^\beta}{\delta|e|^\beta}\right)^{1/2}, \\
&\leqslant Ch^{(d-1)\beta/2-1/2} \|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}} J_0^{1,\beta}(\boldsymbol{\chi}(\tau),\boldsymbol{\chi}(\tau))^{1/2}, \\
&\leqslant \frac{\hat{\epsilon}}{2}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \frac{C}{2\hat{\epsilon}} h^{(d-1)\beta-1} J_0^{1,\beta}(\boldsymbol{\chi}(\tau),\boldsymbol{\chi}(\tau)).
\end{aligned}
$$

Because we are using the elliptic projection on the initial value of $\boldsymbol{u}$ we have that $\boldsymbol{\chi}(0) = \mathbf{0}$, and $T_9 = 0$. For T_{10},

$$\begin{aligned}
|T_{10}| &\leqslant \int_0^\tau \|\rho^{1/2}\boldsymbol{\xi}_{tt}\|_0 \|\rho^{1/2}\boldsymbol{\chi}_t\|_0, \\
&\leqslant \frac{1}{2}\int_0^\tau \|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2 + Ch^{2r}\int_0^\tau \|\boldsymbol{u}_{tt}\|_{r+1}^2,
\end{aligned}$$

where we used the estimate,

$$\|\rho^{\frac{1}{2}}\boldsymbol{\xi}_{tt}\|_0^2 \leqslant C(\rho)h^{2r}\|\boldsymbol{u}_{tt}\|_{r+1}^2.$$

For T_{11} we have,

$$\begin{aligned}
|T_{11}| &\leqslant \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t)^{1/2} J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t)^{1/2}, \\
&\leqslant \frac{\epsilon}{2}\int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) + \frac{C}{2\epsilon}h^{2r}\int_0^\tau \|\boldsymbol{u}_t\|_{r+1}^2,
\end{aligned}$$

where we used,

$$|J_0^{\delta,\beta}(\boldsymbol{\xi}_t, \boldsymbol{\xi}_t)| \leqslant Ch^{2r}\|\boldsymbol{u}_t\|_{r+1}^2.$$

For T_{12},

$$\begin{aligned}
|T_{12}| &\leqslant C\int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i h^{(d-1)\beta/2-1/2}\|\boldsymbol{\theta}_i\|_0 \; J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t)^{1/2}, \\
&\leqslant \frac{\epsilon}{2}\int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t, \boldsymbol{\chi}_t) + C\int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i \frac{h^{(d-1)\beta-1+2r}}{2\epsilon}\|\boldsymbol{u}\|_{r+1}^2,
\end{aligned}$$

where we used,

$$\|\boldsymbol{\theta}_i\|_0 \leqslant Ch^r \|^*\boldsymbol{\sigma}_i\|_r,$$

and Lemma 2.2. For T_{13},

$$\begin{aligned}
|T_{13}| &\leqslant \int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i \|\boldsymbol{D}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{\xi})\|_0 \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0, \\
&\leqslant \frac{C}{2\epsilon^*}\int_0^\tau h^{2r}\|\boldsymbol{u}\|_{r+1}^2 + \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{\epsilon^*}{2}\|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2.
\end{aligned}$$

For T_{14} we use,

$$|J_0^{\delta,\beta}(\boldsymbol{\xi}, \boldsymbol{\xi})| \leqslant \|\boldsymbol{\xi}\|_{\mathcal{A}}^2,$$

and obtain,

$$\begin{aligned}|T_{14}| &\leqslant \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \|\{\dot{\boldsymbol{\eta}}_i \boldsymbol{n}_e\}\|_{0,e} \|[\boldsymbol{\xi}]\|_{0,e}, \\ &\leqslant \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{\epsilon^*}{2} \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + \int_0^\tau \frac{C}{2\epsilon^*} h^{2r} |\boldsymbol{u}|_{r+1}^2.\end{aligned}$$

Finally, for T_{15} we get,

$$\begin{aligned}|T_{15}| &\leqslant \int_0^\tau \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \|\{\dot{\boldsymbol{\eta}}_i \boldsymbol{n}_e\}\|_{0,e} \|[\boldsymbol{\chi}]\|_{0,e}, \\ &\leqslant \int_0^\tau \sum_{i=1}^{N_\varphi} \frac{\epsilon^*}{2} \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + \int_0^\tau \frac{C}{2\epsilon^*} J_0^{1,\beta}(\boldsymbol{\chi},\boldsymbol{\chi}).\end{aligned}$$

Combining all bounds above and using the initial condition $\boldsymbol{\chi}(0) = \mathbf{0}$, we are left with,

$$\begin{aligned}&\frac{1}{2}\|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + (\frac{1}{2} - \frac{1-\varphi_0}{2\epsilon})\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 \\ &\quad - \frac{C}{2\hat{\epsilon}} h^{(d-1)\beta-1} J_0^{1,\beta}(\boldsymbol{\chi}(\tau),\boldsymbol{\chi}(\tau)) \\ &\quad - \frac{\hat{\epsilon}}{2}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + (1-2\epsilon)\int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t) \\ &+ (1-\frac{3\epsilon^*}{2})\int_0^\tau \sum_{i=1}^{N_\varphi} \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + (1-\bar{\epsilon})\sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 \\ &\leqslant \frac{1}{2}\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{2\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(0)\|_0^2 \\ &\quad + \frac{C}{\epsilon} h^{(d-1)\beta-1} \int_0^\tau (1-\varphi_0)\sum_{i=1}^{N_\varphi} \frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2 \\ &\quad + \frac{C}{2\epsilon} h^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 dt + \frac{1}{2}\int_0^\tau \|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2 \\ &+ \int_0^\tau \frac{C}{2\epsilon^*} J_0^{1,\beta}(\boldsymbol{\chi},\boldsymbol{\chi}) + Ch^{2r}\int_0^\tau \|\boldsymbol{u}_{tt}\|_{r+1}^2 + \frac{C}{2\epsilon}\int_0^\tau h^{2r}\|\boldsymbol{u}_t\|_{r+1}^2 \\ &\quad + C\int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i \frac{h^{(d-1)\beta-1+2r}}{2\epsilon}\|\boldsymbol{u}\|_{r+1}^2 + \frac{C}{\epsilon^*}\int_0^\tau h^{2r}\|\boldsymbol{u}\|_{r+1}^2.\end{aligned}$$

Choosing $\epsilon = 1/4$, $\epsilon^* = 1/3$ and $\bar{\epsilon} = 1 - (\varphi_0/2)$ we then obtain,

$$\frac{1}{2}\|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + \frac{\varphi_0}{2(2-\varphi_0)}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{C}{2\hat{\epsilon}}h^{(d-1)\beta-1}J_0^{1,\beta}(\boldsymbol{\chi}(\tau),\boldsymbol{\chi}(\tau))$$
$$- \frac{\hat{\epsilon}}{2}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \frac{1}{2}\int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t) + \frac{1}{2}\int_0^\tau \sum_{i=1}^{N_\varphi}\|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2$$
$$+ \frac{\varphi_0}{2}\sum_{i=1}^{N_\varphi}\frac{1}{2\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 \leqslant \frac{1}{2}\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{1}{2\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(0)\|_0^2$$
$$+ Ch^{(d-1)\beta/2-1/2}\int_0^\tau (1-\varphi_0)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2$$
$$+ Ch^{(d-1)\beta-1}\int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + \frac{1}{2}\int_0^\tau \|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2 + C\int_0^\tau J_0^{1,\beta}(\boldsymbol{\chi},\boldsymbol{\chi})$$
$$+ Ch^{2r}\left(\|\boldsymbol{u}\|_{L_2(H^{r+1})}^2 + \|\boldsymbol{u}_t\|_{L_2(H^{r+1})}^2 + \|\boldsymbol{u}_{tt}\|_{L_2(H^{r+1})}^2\right)$$
$$+ Ch^{2r}\int_0^\tau \sum_{i=1}^{N_\varphi}\gamma_i h^{(d-1)\beta-1}\|\boldsymbol{u}\|_{r+1}^2.$$

Now we choose $\hat{\epsilon} = \varphi_0/(4-2\varphi_0)$ and find a constant, C^*, independent of h, such that for small enough $\hat{h}^{(d-1)\beta-1}/\delta$,

$$\frac{\varphi_0}{2(2-\varphi_0)}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{\hat{\epsilon}}{2}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{Ch^{(d-1)\beta-1}}{2\hat{\epsilon}}J_0^{1,\beta}(\boldsymbol{\chi}(\tau),\boldsymbol{\chi}(\tau))$$
$$= \frac{\varphi_0}{4(2-\varphi_0)}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 - \frac{Ch^{(d-1)\beta-1}(4-2\varphi_0)}{2\delta\varphi_0}J_0^{\delta,\beta}(\boldsymbol{\chi}(\tau),\boldsymbol{\chi}(\tau)),$$
$$\geqslant \left(\frac{\varphi_0}{4(2-\varphi_0)} - \frac{C\hat{h}^{(d-1)\beta-1}(4-2\varphi_0)}{2\delta\varphi_0}\right)\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2,$$
$$= \frac{C^*}{2}\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2.$$

Multiplying the error equation by 2 and using this lower bound then yields,

$$
\begin{aligned}
&\|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + C^*\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t) \\
&+ \int_0^\tau \sum_{i=1}^{N_\varphi} \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 + \varphi_0 \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 \\
&\leqslant \|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(0)\|_0^2 \\
&+ Ch^{(d-1)\beta/2-1/2} \int_0^\tau (1-\varphi_0) \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i\|_0^2 \\
&+ Ch^{(d-1)\beta-1} \int_0^\tau \|\boldsymbol{\chi}\|_{\mathcal{A}}^2 + \int_0^\tau \|\rho^{1/2}\boldsymbol{\chi}_t\|_0^2 + C\int_0^\tau J_0^{1,\beta}(\boldsymbol{\chi},\boldsymbol{\chi}) \\
&+ Ch^{2r}\left(\|\boldsymbol{u}\|_{L_2(H^{r+1})}^2 + \|\boldsymbol{u}_t\|_{L_2(H^{r+1})}^2 + \|\boldsymbol{u}_{tt}\|_{L_2(H^{r+1})}^2\right) \\
&+ Ch^{2r}\int_0^\tau \sum_{i=1}^{N_\varphi} \gamma_i h^{(d-1)\beta-1}\|\boldsymbol{u}\|_{r+1}^2.
\end{aligned}
$$

Now we apply Gronwall's lemma, the initial condition $\boldsymbol{\eta}_i(0) = \boldsymbol{0}$, and the fact that $(d-1)\beta - 1 \geqslant 0$ and arrive at,

$$
\begin{aligned}
&\|\rho^{1/2}\boldsymbol{\chi}_t(\tau)\|_0^2 + C^*\|\boldsymbol{\chi}(\tau)\|_{\mathcal{A}}^2 + \int_0^\tau J_0^{\delta,\beta}(\boldsymbol{\chi}_t,\boldsymbol{\chi}_t) + \int_0^\tau \sum_{i=1}^{N_\varphi} \|\boldsymbol{D}^{-1/2}\dot{\boldsymbol{\eta}}_i\|_0^2 \\
&+ \varphi_0 \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_i(\tau)\|_0^2 \leqslant C\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0^2 + Ch^{2r}.
\end{aligned}
$$

Now, subtracting like terms from both sides of (23) we have,

$$
(\rho\boldsymbol{\chi}_t(0),\boldsymbol{v}) = (\rho(\boldsymbol{u}_t^h(0) - \tilde{\boldsymbol{u}}_t(0)),\boldsymbol{v}) = (\rho(\boldsymbol{u}_t(0) - \tilde{\boldsymbol{u}}_t(0)),\boldsymbol{v}) \ \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h).
$$

Taking $\boldsymbol{v} = \boldsymbol{\chi}_t(0)$ and using (13) we then get,

$$
\|\rho^{1/2}\boldsymbol{\chi}_t(0)\|_0 \leqslant \|\rho^{1/2}(\boldsymbol{u}_t(0) - \tilde{\boldsymbol{u}}_t(0))\|_0 \leqslant Ch^r\|\boldsymbol{u}_t(0)\|_r.
$$

The theorem is obtained by using approximation results for $\boldsymbol{\xi}$ and the $\boldsymbol{\theta}_i$, and using the triangle inequality. □

The next result is concerned with the error in $\boldsymbol{L}_2(\Omega)$. It applies only to the symmetric formulation where $\kappa = -1$, and relies on error representation through the following dual problem:

$$\rho\boldsymbol{\Phi}_{tt} - \nabla\cdot\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}) = \boldsymbol{g} - \sum_{i=1}^{N_\varphi}\gamma_i\nabla\cdot\boldsymbol{\Psi}_i^* \qquad \text{in} \quad \Omega\times I, \tag{26}$$

$$\boldsymbol{\Phi} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_D, \tag{27}$$

$$\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi})\boldsymbol{n} - \sum_{i=1}^{N_\varphi}\gamma_i\boldsymbol{\Psi}_i^*\boldsymbol{n} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_N, \tag{28}$$

$$-\dot{\boldsymbol{\Psi}}_i^* + \frac{1}{\tau_i}\boldsymbol{\Psi}_i^* = \gamma_i\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}) \qquad \text{in} \quad \Omega\times I, \tag{29}$$

$$\boldsymbol{\Psi}_i^*(T) = \boldsymbol{0}, \quad \boldsymbol{\Phi}(T) = \boldsymbol{0}, \quad \boldsymbol{\Phi}_t(T) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{30}$$

for $i = 1, \ldots, N_\varphi$.

Theorem 3.2 (semidiscrete $\boldsymbol{L}_2(\Omega)$ error estimate) *Let Theorem 3.1 hold and assume that in the dual problem we have $\boldsymbol{\Phi}(t) \in C(\bar{\Omega})^d$ for each $t \in \bar{I}$ and also that,*

$$\|\boldsymbol{\Phi}(t)\|_2 + \|\boldsymbol{\Phi}_t(t)\|_1 \leqslant C\|\boldsymbol{g}(t)\|_0 \qquad \textit{and} \qquad \|\boldsymbol{\Psi}_i^*(t)\|_1 \leqslant C\|\boldsymbol{g}(t)\|_0,$$

for each $i = 1, \ldots, N_\varphi$. Assume further that the dual stress is in equilibrium across each element edge,

$$\left[\left(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}(t)) - \sum_{i=1}^{N_\varphi}\gamma_i\boldsymbol{\Psi}_i^*\right)\boldsymbol{n}_e\right] = \boldsymbol{0} \qquad \textit{for each } e \in \Gamma_h.$$

Also assume that $\bar{\boldsymbol{u}} \in \mathbf{D}_r(\mathcal{E}_h) \cap C(\bar{\Omega})^d$. Then, for the symmetric formulation (where $\kappa = -1$),

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_{L_2(L_2)} \leqslant Ch^{r+1}$$

Proof: Apply the dual problem with $\boldsymbol{g} = \boldsymbol{\zeta} = \boldsymbol{u} - \boldsymbol{u}^h$ and use the fact that $[\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi})\boldsymbol{n} - \sum_{i=1}^{N_\varphi}\gamma_i\boldsymbol{\Psi}_i^*\boldsymbol{n}] = \boldsymbol{0}$ to get,

$$\begin{aligned}
\|\boldsymbol{\zeta}\|_0^2 &= (\boldsymbol{\zeta}, \rho\boldsymbol{\Phi}_{tt} - \nabla\cdot(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}) - \sum_{i=1}^{N_\varphi}\gamma_i\boldsymbol{\Psi}_i^*)), \\
&= (\boldsymbol{\zeta}, \rho\boldsymbol{\Phi}_{tt}) + \sum_E((\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}) - \sum_{i=1}^{N_\varphi}\gamma_i\boldsymbol{\Psi}_i^*), \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E \\
&\qquad - \sum_{e\in\Gamma_h\cup\partial\Omega}\int_e\{(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}) - \sum_{i=1}^{N_\varphi}\gamma_i\boldsymbol{\Psi}_i^*)\boldsymbol{n}_e\}\cdot[\boldsymbol{\zeta}].
\end{aligned}$$

Thus, using the fact that $[\boldsymbol{\Phi}] = \mathbf{0}$ and $\boldsymbol{\Phi} = 0$ on Γ_D:

$$\begin{aligned}
\|\boldsymbol{\zeta}\|_0^2 \;=\;& (\boldsymbol{\zeta}, \rho\boldsymbol{\Phi}_{tt}) + \sum_E (\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}), \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E - \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Psi}_i^*, \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E \\
&- \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi})\boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}] + \kappa^* \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\boldsymbol{n}_e\} \cdot [\boldsymbol{\Phi}] \\
&+ J_0^{\delta,\beta}(\boldsymbol{\zeta}, \boldsymbol{\Phi}) + J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \boldsymbol{\Phi}) + \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\partial\Omega} \int_e \{\boldsymbol{\Psi}_i^* \boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}].
\end{aligned}$$

Using the boundary conditions (27) and (28) and integrating over time, we obtain,

$$\begin{aligned}
\int_0^T \|\boldsymbol{\zeta}\|_0^2 = \int_0^T (\boldsymbol{\zeta}, \rho\boldsymbol{\Phi}_{tt}) + \int_0^T \sum_E (\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}), \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E \\
- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Psi}_i^*, \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E - \int_0^T \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi})\boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}] \\
+ \kappa^* \int_0^T \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\boldsymbol{n}_e\} \cdot [\boldsymbol{\Phi}] \\
+ \int_0^T J_0^{\delta,\beta}(\boldsymbol{\zeta}, \boldsymbol{\Phi}) + \int_0^T J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \boldsymbol{\Phi}) + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{\Psi}_i^* \boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}].
\end{aligned}$$

Let us consider the symmetric case ($\kappa = \kappa^* = -1$) then, using the definition of the bilinear form A,

$$\begin{aligned}
\int_0^T \|\boldsymbol{\zeta}\|_0^2 = \int_0^T (\boldsymbol{\zeta}, \rho\boldsymbol{\Phi}_{tt}) + \int_0^T A(\boldsymbol{\zeta}, \boldsymbol{\Phi}) + \int_0^T J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \boldsymbol{\Phi}) \\
- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Psi}_i^*, \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{\Psi}_i^* \boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}].
\end{aligned}$$

Integrating by parts twice and using the conditions $\boldsymbol{\Phi}(T) = \boldsymbol{\Phi}_t(T) = 0$ results in,

$$\begin{aligned}
\int_0^T (\boldsymbol{\zeta}, \rho\boldsymbol{\Phi}_{tt}) &= (\boldsymbol{\zeta}(T), \rho\boldsymbol{\Phi}(T)) - (\boldsymbol{\zeta}(0), \rho\boldsymbol{\Phi}_t(0)) - \int_0^T (\boldsymbol{\zeta}_t, \rho\boldsymbol{\Phi}_t) \\
&= -(\boldsymbol{\zeta}(0), \rho\boldsymbol{\Phi}_t(0)) + (\boldsymbol{\zeta}_t(0), \rho\boldsymbol{\Phi}(0)) + \int_0^T (\rho\boldsymbol{\zeta}_{tt}, \boldsymbol{\Phi}),
\end{aligned}$$

and so,

$$\int_0^T \|\boldsymbol{\zeta}\|_0^2 = \int_0^T ((\boldsymbol{\zeta}_{tt}, \rho\boldsymbol{\Phi}) + A(\boldsymbol{\zeta}, \boldsymbol{\Phi}) + J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \boldsymbol{\Phi}))$$
$$- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Psi}_i^*, \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{\Psi}_i^* \boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}]$$
$$- (\boldsymbol{\zeta}(0), \rho\boldsymbol{\Phi}_t(0)) + (\boldsymbol{\zeta}_t(0), \rho\boldsymbol{\Phi}(0)). \quad (31)$$

But the error equation (by the Galerkin orthogonality resulting from (17) and (20)), with $\boldsymbol{\Lambda}_i = {}^*\boldsymbol{\sigma}_i - {}^*\boldsymbol{\sigma}_i^h$ is,

$$(\rho\boldsymbol{\zeta}_{tt}, \boldsymbol{\Phi}_h) + A(\boldsymbol{\zeta}, \boldsymbol{\Phi}_h) + J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \boldsymbol{\Phi}_h) - \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Lambda}_i, \boldsymbol{\varepsilon}(\boldsymbol{\Phi}_h))_E$$
$$+ \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{\Lambda}_i \boldsymbol{n}_e\} \cdot [\boldsymbol{\Phi}_h] = 0, \quad \forall \boldsymbol{\Phi}_h \in \mathbf{D}_r(\mathcal{E}_h).$$

Integrate this from 0 to T and subtract the result from (31) and we get, for all $\boldsymbol{\Phi}_h \in \mathbf{D}_r(\mathcal{E}_h)$

$$\|\boldsymbol{\zeta}\|_{L_2(L_2)}^2 = \int_0^T ((\rho\boldsymbol{\zeta}_{tt}, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h) + A(\boldsymbol{\zeta}, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h) + J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h))$$
$$- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Psi}_i^*, \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E + \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{\Psi}_i^* \boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}]$$
$$- (\boldsymbol{\zeta}(0), \rho\boldsymbol{\Phi}_t(0)) + (\boldsymbol{\zeta}_t(0), \rho\boldsymbol{\Phi}(0))$$
$$+ \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\Lambda}_i, \boldsymbol{\varepsilon}(\boldsymbol{\Phi}_h))_E - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\boldsymbol{\Lambda}_i \boldsymbol{n}_e\} \cdot [\boldsymbol{\Phi}_h]. \quad (32)$$

Now multiply (29) by a test function $\boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h)$ and integrate by parts,

$$-(\boldsymbol{\Psi}_i^*(T), \boldsymbol{w}_i(T)) + (\boldsymbol{\Psi}_i^*(0), \boldsymbol{w}_i(0)) + \int_0^T (\boldsymbol{\Psi}_i^*, \dot{\boldsymbol{w}}_i + \frac{1}{\tau_i}\boldsymbol{w}_i)$$
$$= \int_0^T \gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi}), \boldsymbol{w}_i).$$

Take $\boldsymbol{w}_i = \boldsymbol{D}^{-1}\boldsymbol{\eta}_i$ where $\boldsymbol{\eta}_i := {}^*\boldsymbol{\sigma}_i^h - {}^*\tilde{\boldsymbol{\sigma}}_i$, and ${}^*\tilde{\boldsymbol{\sigma}}_i$ is the $\boldsymbol{L}_2(\Omega)$ projection of ${}^*\boldsymbol{\sigma}_i$ onto $\boldsymbol{L}_{r-1}(\mathcal{E}_h)$, sum over i, and use the condition $\boldsymbol{\Psi}_i^*(T) = \mathbf{0}$,

$$\sum_{i=1}^{N_\varphi}(\boldsymbol{\Psi}_i^*(0), \boldsymbol{D}^{-1}\boldsymbol{\eta}_i(0)) + \int_0^T \sum_{i=1}^{N_\varphi}(\boldsymbol{D}^{-1}\boldsymbol{\Psi}_i^*, \dot{\boldsymbol{\eta}}_i + \frac{1}{\tau_i}\boldsymbol{\eta}_i) - \sum_{i=1}^{N_\varphi}\int_0^T \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\Phi}), \boldsymbol{\eta}_i) = 0. \tag{33}$$

From (18) and (21) the error $\boldsymbol{\Lambda}_i$ satisfies:

$$(\dot{\boldsymbol{\Lambda}}_i + \frac{1}{\tau_i}\boldsymbol{\Lambda}_i, \boldsymbol{w}_i) - \gamma_i(\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\zeta}), \boldsymbol{w}_i) + \sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot[\boldsymbol{\zeta}] = 0 \quad \forall \boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h).$$

Choose $\boldsymbol{w}_i = \boldsymbol{D}^{-1}\boldsymbol{\Psi}_i^h$, where $\boldsymbol{\Psi}_i^h$ is the L^2 projection of $\boldsymbol{\Psi}_i^*$, sum over i and integrate from 0 to T:

$$\sum_{i=1}^{N_\varphi}\int_0^T(\dot{\boldsymbol{\Lambda}}_i + \frac{1}{\tau_i}\boldsymbol{\Lambda}_i, \boldsymbol{D}^{-1}\boldsymbol{\Psi}_i^h) - \sum_{i=1}^{N_\varphi}\int_0^T \gamma_i(\boldsymbol{\varepsilon}(\boldsymbol{\zeta}), \boldsymbol{\Psi}_i^h) + \sum_{i=1}^{N_\varphi}\int_0^T \sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{\Psi}_i^h\boldsymbol{n}_e\}\cdot[\boldsymbol{\zeta}] = 0. \tag{34}$$

Combining (32), (34) and (33), setting $\boldsymbol{\theta}_i := {}^*\boldsymbol{\sigma}_i - {}^*\tilde{\boldsymbol{\sigma}}_i$, and noting that

$\mathbf{\Lambda}_i = \boldsymbol{\theta}_i - \boldsymbol{\eta}_i$, we obtain,

$$
\begin{aligned}
\|\boldsymbol{\zeta}\|^2_{L_2(L_2)} = \int_0^T ((\rho\boldsymbol{\zeta}_{tt}, \mathbf{\Phi} - \mathbf{\Phi}_h) + A(\boldsymbol{\zeta}, \mathbf{\Phi} - \mathbf{\Phi}_h) + J_0^{\delta,\beta}(\boldsymbol{\zeta}_t, \mathbf{\Phi} - \mathbf{\Phi}_h))& \\
- \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E ((\mathbf{\Psi}_i^* - \mathbf{\Psi}_i^h), \boldsymbol{\varepsilon}(\boldsymbol{\zeta}))_E& \\
+ \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{(\mathbf{\Psi}_i^* - \mathbf{\Psi}_i^h)\boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}]& \\
- (\boldsymbol{\zeta}(0), \rho\mathbf{\Phi}_t(0)) + (\boldsymbol{\zeta}_t(0), \rho\mathbf{\Phi}(0)) - \sum_{i=1}^{N_\varphi} (\mathbf{\Psi}_i^*(0), \boldsymbol{D}^{-1}\boldsymbol{\eta}_i(0))& \\
+ \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\eta}_i, \boldsymbol{\varepsilon}(\mathbf{\Phi} - \mathbf{\Phi}_h))_E - \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \{\mathbf{\Lambda}_i \boldsymbol{n}_e\} \cdot [\mathbf{\Phi}_h]& \\
- \int_0^T \sum_{i=1}^{N_\varphi} (\boldsymbol{D}^{-1}\mathbf{\Psi}_i^h, \dot{\boldsymbol{\theta}}_i + \frac{1}{\tau_i}\boldsymbol{\theta}_i) + \int_0^T \sum_{i=1}^{N_\varphi} (\boldsymbol{D}^{-1}(\mathbf{\Psi}_i^h - \mathbf{\Psi}_i^*), \dot{\boldsymbol{\eta}}_i + \frac{1}{\tau_i}\boldsymbol{\eta}_i)& \\
+ \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \sum_E (\boldsymbol{\theta}_i, \boldsymbol{\varepsilon}(\mathbf{\Phi}_h))_E = R_1 + \cdots + R_{13}.&
\end{aligned}
$$

We now bound each term R_i. First, we integrate by parts in R_1, combine it with R_7 and use the fact that $(\mathbf{\Phi} - \mathbf{\Phi}_h)(T) = \mathbf{0}$. Thus,

$$
\begin{aligned}
R_1 + R_7 &= (\rho\boldsymbol{\zeta}_t(T), (\mathbf{\Phi} - \mathbf{\Phi}_h)(T)) - (\rho\boldsymbol{\zeta}_t(0), (\mathbf{\Phi} - \mathbf{\Phi}_h)(0)) \\
&\quad + (\rho\boldsymbol{\zeta}_t(0), \mathbf{\Phi}(0)) - \int_0^T (\rho\boldsymbol{\zeta}_t, (\mathbf{\Phi} - \mathbf{\Phi}_h)_t) \\
&= (\rho\boldsymbol{\zeta}_t(0), \mathbf{\Phi}_h(0)) - \int_0^T (\rho\boldsymbol{\zeta}_t, (\mathbf{\Phi} - \mathbf{\Phi}_h)_t) \\
&= -\int_0^T (\rho\boldsymbol{\zeta}_t, (\mathbf{\Phi} - \mathbf{\Phi}_h)_t),
\end{aligned}
$$

because $(\rho\boldsymbol{\zeta}_t(0), \boldsymbol{v}) = 0, \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h)$ and so, if $\mathbf{\Phi}_h$ is a continuous interpolant of $\mathbf{\Phi}$, we have

$$
\begin{aligned}
|R_1| &\leqslant \|\rho^{1/2}\boldsymbol{\zeta}_t\|_{L_2(L_2)} \|\rho^{1/2}(\mathbf{\Phi} - \mathbf{\Phi}_h)_t\|_{L_2(L_2)}, \\
&\leqslant Ch\|\mathbf{\Phi}_t\|_{L_2(H^1)} \|\rho^{1/2}\boldsymbol{\zeta}_t\|_{L_2(L_2)}, \\
&\leqslant Ch\|\boldsymbol{\zeta}\|_{L_2(L_2)} \|\rho^{1/2}\boldsymbol{\zeta}_t\|_{L_2(L_2)}.
\end{aligned}
$$

We first expand R_2,

$$
\begin{aligned}
R_2 = \int_0^T \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\zeta}) : \boldsymbol{\varepsilon}(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h) - \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\zeta})\boldsymbol{n}_e\} \cdot [\boldsymbol{\Phi} - \boldsymbol{\Phi}_h] \\
+ \kappa \int_0^T \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)\boldsymbol{n}_e\} \cdot [\boldsymbol{\zeta}] + \int_0^T J_0^{\delta,\beta}(\boldsymbol{\zeta}, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h), \\
= R_{21} + \cdots + R_{24},
\end{aligned}
$$

and note that R_{22} and R_{24} are zero because $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}_h$ are continuous and vanish on Γ_D.

$$
\begin{aligned}
|R_{21} + R_{23}| &\leqslant C \int_0^T \|\boldsymbol{\zeta}\|_{\mathcal{A}} \|\boldsymbol{\Phi} - \boldsymbol{\Phi}_h\|_1, \\
&\leqslant C \|\boldsymbol{\zeta}\|_{L_2(\mathcal{A})} h \|\boldsymbol{\Phi}\|_{L_2(H^2)}, \\
&\leqslant Ch \|\boldsymbol{\zeta}\|_{L_2(\mathcal{A})} \|\boldsymbol{\zeta}\|_{L_2(L_2)}.
\end{aligned}
$$

The terms R_3 and R_{10} vanish. For R_4 we have, since $\boldsymbol{\Psi}_i^h$ approximates $\boldsymbol{\Psi}_i^*$ optimally

$$
\begin{aligned}
|R_4| &\leqslant C \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \|\boldsymbol{\Psi}_i^* - \boldsymbol{\Psi}_i^h\|_0 \|\boldsymbol{\zeta}\|_{\mathcal{A}}, \\
&\leqslant Ch \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \|\boldsymbol{\Psi}_i^*\|_1 \|\boldsymbol{\zeta}\|_{\mathcal{A}}, \\
&\leqslant C(\gamma_i) h \|\boldsymbol{\zeta}\|_{L_2(L_2)} \|\boldsymbol{\zeta}\|_{L_2(\mathcal{A})}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|R_5| &\leqslant C \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i (\|\boldsymbol{\Psi}_i^* - \boldsymbol{\Psi}_i^h\|_0 + h|\boldsymbol{\Psi}_i^* - \boldsymbol{\Psi}_i^h|_1) \|\boldsymbol{\zeta}\|_{\mathcal{A}}, \\
&\leqslant Ch \int_0^T \sum_{i=1}^{N_\varphi} \gamma_i \|\boldsymbol{\Psi}_i^*\|_1 \|\boldsymbol{\zeta}\|_{\mathcal{A}}, \\
&\leqslant C(\gamma_i) h \|\boldsymbol{\zeta}\|_{L_2(L_2)} \|\boldsymbol{\zeta}\|_{L_2(\mathcal{A})}.
\end{aligned}
$$

Note that if $\bar{\boldsymbol{u}} \in \mathbf{D}_r(\mathcal{E}_h)$, then $\boldsymbol{\zeta}(0) = \mathbf{0}$ and the term R_6 disappears. Also, because $\boldsymbol{\eta}_i(0) = \mathbf{0}$, the term R_8 is zero. For R_9,

$$
\begin{aligned}
|R_9| &\leqslant \int_0^T \sum_{i=1}^{N_\varphi} \|\boldsymbol{\eta}_i\|_0 \|\boldsymbol{\varepsilon}(\boldsymbol{\Phi} - \boldsymbol{\Phi}_h)\|_0, \\
&\leqslant Ch \int_0^T \sum_{i=1}^{N_\varphi} \|\boldsymbol{\eta}_i\|_0 \|\boldsymbol{\Phi}\|_2, \\
&\leqslant Ch \sum_{i=1}^{N_\varphi} \|\boldsymbol{\eta}_i\|_{L_2(L_2)} \|\boldsymbol{\zeta}\|_{L_2(L_2)}.
\end{aligned}
$$

The terms R_{11} and R_{13} vanish because of the property of L_2 projection, and, because $\boldsymbol{\Psi}_i^h$ is the L^2 projection of $\boldsymbol{\Psi}_i^*$, the term R_{12} also disappears.

Thus, combining the terms above and using Theorem 3.1, we conclude that

$$
\|\boldsymbol{\zeta}\|_{L_2(L_2)} \leqslant Ch^{r+1},
$$

and this completes the proof. □

4 Fully discrete estimates

Let us define $k = T/N$ for some positive integer N and set $t_j = jk$. Setting,

$$
L_j(\boldsymbol{v}) := \frac{1}{2}\Big(L(t_j; \boldsymbol{v}) + L(t_{j-1}; \boldsymbol{v})\Big),
$$

our fully discrete approximation of the problem described by (17), (18) and (19) is as follows: for each $j = 1, \dots, N$, find $\{\boldsymbol{z}_j^h, \boldsymbol{u}_j^h, \dots, {}^*\boldsymbol{\sigma}_{ij}^h, \dots\} \in \mathbf{D}_r(\mathcal{E}_h) \times \mathbf{D}_r(\mathcal{E}_h) \times \mathbf{L}_{r-1}(\mathcal{E}_h)^{N_\varphi}$ such that,

$$
\begin{aligned}
&\left(\rho \frac{\boldsymbol{z}_j^h - \boldsymbol{z}_{j-1}^h}{k}, \boldsymbol{v}\right) + A\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}, \boldsymbol{v}\right) + J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \boldsymbol{v}\right) \\
&\qquad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \gamma_i \left\{ \frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2} \boldsymbol{n}_e \right\} \cdot [\boldsymbol{v}] \\
&- \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) = L_j(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h),
\end{aligned}
\tag{35}
$$

with, for each $i = 1, \ldots, N_\varphi$,

$$\sum_E \left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k} + \frac{1}{\tau_i} \frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}, \boldsymbol{w}_i \right)_E$$
$$= \sum_E \gamma_i \left(\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2} \right), \boldsymbol{w}_i \right)_E$$
$$- \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \{\boldsymbol{D}\boldsymbol{w}_i \boldsymbol{n}_e\} \cdot \left[\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2} \right] \qquad \forall \boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \tag{36}$$

and,

$$\left(\rho \frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \boldsymbol{v} \right)_E = \left(\rho \frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}, \boldsymbol{v} \right)_E \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{37}$$

It follows from this last equation that,

$$\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2} = \frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}. \tag{38}$$

For the initial data we set ${}^*\boldsymbol{\sigma}_{i0}^h = \mathbf{0}$, for $i = 1, \ldots, N_\varphi$, and,

$$A(\boldsymbol{u}_0^h, \boldsymbol{v}) = A(\bar{\boldsymbol{u}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h), \tag{39}$$
$$(\rho \boldsymbol{z}_0^h, \boldsymbol{v}) = (\rho \bar{\boldsymbol{z}}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h). \tag{40}$$

The first result is a basic stability estimate. Beforehand, we need some lemmas, and before these we note that if $\bar{\boldsymbol{u}} \in C(\bar{\Omega})^d$ then $J_0^{\delta,\beta}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{u}}) = 0$ and, clearly,

$$\|\bar{\boldsymbol{u}}\|_{\mathcal{A}} \leqslant C \|\bar{\boldsymbol{u}}\|_1. \tag{41}$$

Lemma 4.1 *For the elliptic projection, $\boldsymbol{u}_0^h \in \mathbf{D}_r(\mathcal{E}_h)$, of $\bar{\boldsymbol{u}}$ defined by* (39) *and for $\beta \geqslant (d-1)^{-1}$, $h \leqslant \hat{h}$ and $\kappa = \pm 1$, we have, if $\bar{\boldsymbol{u}} \in \boldsymbol{H}^2(\Omega)$ (hence $u \in C(\bar{\Omega})^d$), that,*

$$\|\boldsymbol{u}_0^h\|_{\mathcal{A}} \leqslant C_{\mathcal{A}} \|\bar{\boldsymbol{u}}\|_2.$$

When $\kappa = -1$ this estimate holds only for δ large enough.

Proof. Since $[\bar{\boldsymbol{u}}] = \mathbf{0}$ we see that,

$$\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2 = \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\boldsymbol{u}_0^h) + J_0^{\delta,\beta}(\bar{\boldsymbol{u}}, \boldsymbol{u}_0^h)$$
$$- \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\boldsymbol{n}_e\} \cdot [\boldsymbol{u}_0^h] + \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\boldsymbol{n}_e\} \cdot [\boldsymbol{u}_0^h]$$
$$- \kappa \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\boldsymbol{n}_e\} \cdot [\boldsymbol{u}_0^h].$$

Now,

$$\left|\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_0^h]\right| \leqslant \frac{1}{2\epsilon}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{|e|^\beta}{\delta}\int_e|\{\boldsymbol{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\boldsymbol{n}_e\}|^2+\frac{\epsilon}{2}\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2.$$

Also, since $|e|^\beta \leqslant Ch^{(d-1)\beta}$, we have by a trace inequality that,

$$\frac{1}{2\epsilon}\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{|e|^\beta}{\delta}\int_e|\{\boldsymbol{D}\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}})\boldsymbol{n}_e\}|^2 \leqslant \frac{Ch^{(d-1)\beta-1}}{2\delta\epsilon}\|\bar{\boldsymbol{u}}\|_2^2,$$

and so, in the case $\kappa = 1$, we arrive at,

$$\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2 \leqslant \frac{1}{2\epsilon}\|\bar{\boldsymbol{u}}\|_{\mathcal{A}}^2+\epsilon\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2+\frac{Ch^{(d-1)\beta-1}}{\delta\epsilon}\|\bar{\boldsymbol{u}}\|_2^2.$$

Choosing $\epsilon = 1/2$ and using (41) then yields the required result for any $\delta > 0$.

When $\kappa = -1$ we have one extra term to deal with:

$$\begin{aligned}
&\left|2\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_0^h]\right| \\
&\leqslant 2\left(\sum_{e\in\Gamma_h\cup\Gamma_D}|e|^\beta\int_e|\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\boldsymbol{n}_e\}|^2\right)^{1/2}\left(\sum_{e\in\Gamma_h\cup\Gamma_D}\frac{1}{|e|^\beta}\int_e|[\boldsymbol{u}_0^h]|^2\right)^{1/2}, \\
&\leqslant \epsilon|\boldsymbol{u}_0^h|_{\mathcal{E}}^2+\frac{Ch^{(d-1)\beta-1}}{\delta\epsilon}J_0^{\delta,\beta}(\boldsymbol{u}_0^h,\boldsymbol{u}_0^h).
\end{aligned}$$

Here we used an 'inverse-trace' inequality to get,

$$\begin{aligned}
&\sum_{e\in\Gamma_h\cup\Gamma_D}|e|^\beta\int_e|\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\boldsymbol{n}_e\}|^2 \\
&\leqslant Ch^{(d-1)\beta-1}\sum_{E\in\mathcal{E}_h}\Big(\|\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\|_{0,E}^2+h_E^2\|\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\|_{1,E}^2\Big), \\
&\leqslant Ch^{(d-1)\beta-1}|\boldsymbol{u}_0^h|_{\mathcal{E}}^2,
\end{aligned}$$

since $h_E^2\|\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\|_{1,E}^2 \leqslant C\|\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\|_{0,E}^2$.

Putting these results together gives,

$$\begin{aligned}
\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2 &\leqslant \frac{1}{2\epsilon}\|\bar{\boldsymbol{u}}\|_{\mathcal{A}}^2+\epsilon\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2+\frac{Ch^{(d-1)\beta-1}}{\delta\epsilon}\|\bar{\boldsymbol{u}}\|_{2,\Omega}^2 \\
&\quad+\epsilon|\boldsymbol{u}_0^h|_{\mathcal{E}}^2+\frac{Ch^{(d-1)\beta-1}}{\delta\epsilon}J_0^{\delta,\beta}(\boldsymbol{u}_0^h,\boldsymbol{u}_0^h),
\end{aligned}$$

and using (41), choosing $\epsilon = 1/4$ and δ such that,

$$\delta \geqslant 4C\hat{h}^{(d-1)\beta-1} \qquad \Longrightarrow \qquad \frac{Ch^{(d-1)\beta-1}}{\delta\epsilon} \leqslant \epsilon,$$

then completes the proof. $\square$

Remark 4.2 *If we appeal to the error estimates given for the elliptic (elasticity) problem in* [12] *then, for $\kappa = 1$, we have,*

$$\|\bar{\boldsymbol{u}} - \boldsymbol{u}_0^h\|_{\mathcal{A}} \leqslant C\hat{h}\|\bar{\boldsymbol{u}}\|_2.$$

The proof of Lemma 4.1 is then very short:

$$\|\boldsymbol{u}_0^h\|_{\mathcal{A}} \leqslant \|\bar{\boldsymbol{u}} - \boldsymbol{u}_0^h\|_{\mathcal{A}} + \|\bar{\boldsymbol{u}}\|_{\mathcal{A}} \leqslant (C\hat{h} + C)\|\bar{\boldsymbol{u}}\|_2,$$

where we used (41)*. For $\kappa = -1$ we can make a similar argument but the error estimate requires that δ be 'large enough'.*

Lemma 4.3 *For every $\epsilon_L',\ \epsilon_L > 0$,*

$$\begin{aligned} 2k\left|\sum_{j=1}^{m} L_j\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)\right| &\leqslant C_{\mathcal{A}}^2\|\bar{\boldsymbol{u}}\|_2^2 + C^2h^{-1}\|\boldsymbol{g}(0)\|_{0,\Gamma_N}^2 \\ &+ \frac{C^2h^{-1}}{\epsilon_L'}\|\boldsymbol{g}(t_m)\|_{0,\Gamma_N}^2 + \frac{C^2h^{-1}k}{\epsilon_L}\sum_{j=1}^{m}\left\|\frac{\boldsymbol{g}(t_j) - \boldsymbol{g}(t_{j-1})}{k}\right\|_{0,\Gamma_N}^2 \\ &+ \frac{k}{\rho\epsilon_L}\sum_{j=1}^{m}\left\|\frac{\boldsymbol{f}(t_j) + \boldsymbol{f}(t_{j-1})}{2}\right\|_0^2 + \epsilon_L k\sum_{j=0}^{m-1}\left(\|\rho^{\frac{1}{2}}\boldsymbol{z}_j^h\|_0^2 + \|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2\right) \\ &+ \epsilon_L k\|\rho^{\frac{1}{2}}\boldsymbol{z}_m^h\|_0^2 + (\epsilon_L' + \epsilon_L k)\|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2, \end{aligned}$$

for $m = 1, 2, \ldots, N$, and where $C_{\mathcal{A}}$ is from Lemma 4.1.

Proof. Using (38) we have,

$$\begin{aligned} 2k\sum_{j=1}^{m} L_j\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right) &= \frac{k}{2}\sum_{j=1}^{m}(\boldsymbol{f}(t_j) + \boldsymbol{f}(t_{j-1}), \boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h) \\ &\quad + \sum_{j=1}^{m}(\boldsymbol{g}(t_j) + \boldsymbol{g}(t_{j-1}), \boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h)_{\Gamma_N}, \\ &= \frac{k}{2}\sum_{j=1}^{m}(\boldsymbol{f}(t_j) + \boldsymbol{f}(t_{j-1}), \boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h) \\ &\qquad - \sum_{j=1}^{m}(\boldsymbol{g}(t_j) - \boldsymbol{g}(t_{j-1}), \boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h)_{\Gamma_N} \\ &\quad + 2(\boldsymbol{g}(t_m), \boldsymbol{u}_m^h)_{\Gamma_N} - 2(\boldsymbol{g}(0), \boldsymbol{u}_0^h)_{\Gamma_N}. \end{aligned}$$

Now, for $\boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h)$ we note that by using inverse estimates and interpolation,

$$
\begin{aligned}
|(\boldsymbol{g}, \boldsymbol{v})_{\Gamma_N}| &\leqslant \|\boldsymbol{g}\|_{0,\Gamma_N} \left(\sum_{e \in \Gamma_N} \|\boldsymbol{v}\|_{0,e}^2 \right)^{1/2}, \\
&\leqslant C\|\boldsymbol{g}\|_{0,\Gamma_N} \left(\sum_{\bar{E} \cap \Gamma_N \neq \varnothing} h^{-1} \|\boldsymbol{v}\|_{1/2,E}^2 \right)^{1/2}, \\
&\leqslant Ch^{-1/2}\|\boldsymbol{g}\|_{0,\Gamma_N} \left(\sum_{\bar{E} \cap \Gamma_N \neq \varnothing} \|\boldsymbol{v}\|_{0,E}\, |\boldsymbol{v}|_{1,E} \right)^{1/2}, \\
&\leqslant Ch^{-1/2}\|\boldsymbol{g}\|_{0,\Gamma_N}\|\boldsymbol{v}\|_{\mathcal{A}}, \qquad (42)
\end{aligned}
$$

by equivalence of norms and where $|\cdot|_{1,E}$ denotes the $H^1(E)^d$ seminorm. Hence,

$$
\begin{aligned}
&2k\left|\sum_{j=1}^{m} L_j\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)\right| \\
&\quad \leqslant k\sum_{j=1}^{m}\left\|\frac{\boldsymbol{f}(t_j)+\boldsymbol{f}(t_{j-1})}{2\rho^{\frac{1}{2}}}\right\|_0 \left(\|\rho^{\frac{1}{2}}\boldsymbol{z}_j^h\|_0 + \|\rho^{\frac{1}{2}}\boldsymbol{z}_{j-1}^h\|_0\right) \\
&\qquad + Ch^{-1/2}k\sum_{j=1}^{m}\left\|\frac{\boldsymbol{g}(t_j)-\boldsymbol{g}(t_{j-1})}{k}\right\|_{0,\Gamma_N} \left(\|\boldsymbol{u}_j^h\|_{\mathcal{A}} + \|\boldsymbol{u}_{j-1}^h\|_{\mathcal{A}}\right) \\
&\qquad + 2Ch^{-1/2}\|\boldsymbol{g}(t_m)\|_{0,\Gamma_N}\|\boldsymbol{u}_m^h\|_{\mathcal{A}} + 2Ch^{-1/2}\|\boldsymbol{g}(0)\|_{0,\Gamma_N}\|\boldsymbol{u}_0^h\|_{\mathcal{A}}.
\end{aligned}
$$

By several applications of Young's inequality and Lemma 4.1 we therefore have,

$$
\begin{aligned}
&2k\left|\sum_{j=1}^{m} L_j\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)\right| \\
&\quad \leqslant \frac{k}{\rho\epsilon_L}\sum_{j=1}^{m}\left\|\frac{\boldsymbol{f}(t_j)+\boldsymbol{f}(t_{j-1})}{2}\right\|_0^2 + \epsilon_L k\sum_{j=0}^{m}\|\rho^{\frac{1}{2}}\boldsymbol{z}_j^h\|_0^2 + C_{\mathcal{A}}^2\|\bar{\boldsymbol{u}}\|_2^2 \\
&\qquad + \frac{C^2h^{-1}k}{\epsilon_L}\sum_{j=1}^{m}\left\|\frac{\boldsymbol{g}(t_j)-\boldsymbol{g}(t_{j-1})}{k}\right\|_{0,\Gamma_N}^2 + \epsilon_L k\sum_{j=0}^{m}\|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2 \\
&\qquad\quad + \frac{C^2h^{-1}}{\epsilon_L'}\|\boldsymbol{g}(t_m)\|_{0,\Gamma_N}^2 + \epsilon_L'\|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2 + C^2h^{-1}\|\boldsymbol{g}(0)\|_{0,\Gamma_N}^2.
\end{aligned}
$$

This completes the proof. □

With these preliminary results established we can now give the stability estimate. Note that the factor 'h^{-1}' is not observed in practical computations; its presence seems to be due only to a weakness in the proof.

Theorem 4.4 (discrete stability) *Assume that $\beta \geqslant (d-1)^{-1}$ along with $k \leqslant \hat{k}$ and $h \leqslant \hat{h}$. Then, for δ large enough, $\hat{k}$ and $\hat{h}$ small enough, and $m = 1, 2, \ldots, N$,*

$$\begin{aligned}
&\|\rho^{\frac{1}{2}}\boldsymbol{z}_m^h\|_0^2 + \|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}\,{}^*\boldsymbol{\sigma}_{im}^h\|_0^2 \\
&\quad + k\sum_{j=1}^{m} J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}\right) \\
&\quad + k\sum_{i=1}^{N_\varphi}\sum_{j=1}^{m}\left\|\boldsymbol{D}^{-\frac{1}{2}}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0^2 \\
&\leqslant C\|\rho^{\frac{1}{2}}\bar{\boldsymbol{z}}\|_0^2 + C\|\bar{\boldsymbol{u}}\|_2^2 + Ch^{-1}\|\boldsymbol{g}(0)\|_{0,\Gamma_N}^2 + Ch^{-1}\|\boldsymbol{g}(t_m)\|_{0,\Gamma_N}^2 \\
&\quad + Ch^{-1}k\sum_{j=1}^{m}\left\|\frac{\boldsymbol{g}(t_j) - \boldsymbol{g}(t_{j-1})}{k}\right\|_{0,\Gamma_N}^2 + Ck\sum_{j=1}^{m}\left\|\frac{\boldsymbol{f}(t_j) + \boldsymbol{f}(t_{j-1})}{2}\right\|_0^2,
\end{aligned}$$

where C represents a generic positive constant.

Proof: We choose

$$\boldsymbol{v} = (\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h)/k \in \mathbf{D}_r(\mathcal{E}_h)$$

in (35),

$$\boldsymbol{w}_i = \boldsymbol{D}^{-1}({}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h)/k \in \mathbf{L}_{r-1}(\mathcal{E}_h)$$

in (36),

$$\boldsymbol{v} = (\boldsymbol{z}_j^h - \boldsymbol{z}_{j-1}^h)/k \in \mathbf{D}_r(\mathcal{E}_h)$$

in (37), and we recall (38). Equation (37) then becomes,

$$\left(\rho\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \frac{\boldsymbol{z}_j^h - \boldsymbol{z}_{j-1}^h}{k}\right) = \left(\rho\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}, \frac{\boldsymbol{z}_j^h - \boldsymbol{z}_{j-1}^h}{k}\right)$$

and with this (35) becomes,

$$\left(\rho\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h-\boldsymbol{z}_{j-1}^h}{k}\right)+A\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2},\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right)$$
$$+J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right)$$
$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\gamma_i\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h+{}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right]$$
$$-\sum_{i=1}^{N_\varphi}\sum_E\int_E\gamma_i\frac{{}^*\boldsymbol{\sigma}_{ij}^h+{}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}:\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right)=L_j\left(\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right).$$

Also, (36) becomes,

$$\sum_{i=1}^{N_\varphi}\sum_E\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k},\boldsymbol{D}^{-1}\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)_E$$
$$+\sum_{i=1}^{N_\varphi}\sum_E\frac{1}{\tau_i}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h+{}^*\boldsymbol{\sigma}_{i,j-1}^h}{2},\boldsymbol{D}^{-1}\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)_E$$
$$=\sum_{i=1}^{N_\varphi}\sum_E\gamma_i\left(\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right),\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)_E$$
$$-\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right].$$

Adding the third of these to the second and noting that,

$$\left(\rho\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h-\boldsymbol{z}_{j-1}^h}{k}\right)+A\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2},\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right)$$
$$=\frac{1}{2k}\|\rho^{1/2}\boldsymbol{z}_j^h\|_0^2-\frac{1}{2k}\|\rho^{1/2}\boldsymbol{z}_{j-1}^h\|_0^2+\frac{1}{2k}\|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2-\frac{1}{2k}\|\boldsymbol{u}_{j-1}^h\|_{\mathcal{A}}^2$$
$$-\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right)\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right]$$
$$+\kappa\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right)\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right],$$

along with,

$$\sum_{i=1}^{N_\varphi}\sum_E \frac{1}{\tau_i}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}, \boldsymbol{D}^{-1}\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)$$
$$= \frac{1}{2k}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{ij}^h\|_0^2 - \frac{1}{2k}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{i,j-1}^h\|_0^2,$$

we obtain,

$$\frac{1}{2k}\|\rho^{1/2}\boldsymbol{z}_j^h\|_0^2 + \frac{1}{2k}\|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2 + J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}\right)$$
$$+\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{1}{2k\tau_i}\|\boldsymbol{D}^{-1/2*}\boldsymbol{\sigma}_{ij}^h\|_0^2$$
$$= \frac{1}{2k}\|\rho^{1/2}\boldsymbol{z}_{j-1}^h\|_0^2 + \frac{1}{2k}\|\boldsymbol{u}_{j-1}^h\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{2k\tau_i}\|\boldsymbol{D}^{-1/2*}\boldsymbol{\sigma}_{i,j-1}^h\|_0^2$$
$$+ L_j\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)$$
$$+\sum_{i=1}^{N_\varphi}\sum_E \gamma_i\left(\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right), \frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)_E$$
$$-\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right]$$
$$-\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\gamma_i\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right]$$
$$+\sum_{i=1}^{N_\varphi}\sum_E\int_E\gamma_i\frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2} : \boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)$$
$$+\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right)\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right]$$
$$-\kappa\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right].$$

Multiplying by $2k$, summing over $j = 1, \ldots, m$ and recalling that each ${}^*\boldsymbol{\sigma}_{i0}^h = \mathbf{0}$ then gives,

$$
\begin{aligned}
&\|\rho^{1/2}\boldsymbol{z}_m^h\|_0^2 + \|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2 + 2k\sum_{j=1}^m J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}\right) \\
&+ 2k\sum_{j=1}^m\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}{}^*\boldsymbol{\sigma}_{im}^h\|_0^2 \\
&\qquad = \|\rho^{1/2}\boldsymbol{z}_0^h\|_0^2 + \|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2 + 2k\sum_{j=1}^m L_j\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right) \\
&\qquad\qquad + T_1 + T_2 + T_3,
\end{aligned}
$$

where the T_i are given by,

$$
\begin{aligned}
T_1 = -2k\sum_{j=1}^m\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\Bigg(&\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right] \\
&+ \left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right]\Bigg), \\
T_2 = 2k\sum_{j=1}^m\sum_{i=1}^{N_\varphi}\sum_E\gamma_i\int_E\Bigg(&\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right) : \frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k} \\
&+ \boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right) : \frac{{}^*\boldsymbol{\sigma}_{ij}^h + {}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}\Bigg), \\
T_3 = 2k\sum_{j=1}^m\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\Bigg(&\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right)\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right] \\
&- \kappa\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k}\right)\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}\right]\Bigg).
\end{aligned}
$$

We take each of these T_i in turn. Firstly, using the summation identity,

$$
\sum_{j=1}^m\Big((a_j - a_{j-1})(b_j + b_{j-1}) + (a_j + a_{j-1})(b_j - b_{j-1})\Big) = 2a_m b_m - 2a_0 b_0,
$$

we get for T_2 that,

$$\begin{aligned} T_2 &= \sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\sum_{E}\gamma_i\int_E\biggl(\boldsymbol{\varepsilon}(\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h):({}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h) \\ &\qquad\qquad + \boldsymbol{\varepsilon}(\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h):({}^*\boldsymbol{\sigma}_{ij}^h+{}^*\boldsymbol{\sigma}_{i,j-1}^h)\biggr), \\ &= 2\sum_{i=1}^{N_\varphi}\sum_{E}\gamma_i\int_E\boldsymbol{\varepsilon}(\boldsymbol{u}_m^h):{}^*\boldsymbol{\sigma}_{im}^h, \end{aligned}$$

since ${}^*\boldsymbol{\sigma}_{i0}^h=\mathbf{0}$. Hence,

$$\begin{aligned} |T_2| &\leqslant 2\sum_{i=1}^{N_\varphi}\gamma_i\|\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\varepsilon}(\boldsymbol{u}_m^h)\|_0\,\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{im}^h\|_0, \\ &\leqslant \sum_{i=1}^{N_\varphi}\left(\frac{\bar{\epsilon}\gamma_i^2}{\varphi_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{im}^h\|_0^2+\frac{\varphi_i}{\bar{\epsilon}}\|\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\varepsilon}(\boldsymbol{u}_m^h)\|_0^2\right). \end{aligned}$$

But $\gamma_i^2/\varphi_i=1/\tau_i$, so,

$$|T_2|\leqslant\sum_{i=1}^{N_\varphi}\frac{\bar{\epsilon}}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{im}^h\|_0^2+\frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{u}_m^h|_{\mathcal{E}}^2,$$

because $\sum_{i=1}^{N_\varphi}\varphi_i=1-\varphi_0$.

Now, using (38) and $|e|^\beta \leqslant Ch^{(d-1)\beta}$ we have for T_1 that,

$$
\begin{aligned}
|T_1| \leqslant\ & 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left|\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right]\right| \\
& + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left|\left(\frac{|e|^\beta}{\delta}\right)^{\frac12}\left\{\frac{{}^*\boldsymbol{\sigma}_{ij}^h+{}^*\boldsymbol{\sigma}_{i,j-1}^h}{2}\cdot\boldsymbol{n}_e\right\}\right. \\
& \qquad\left.\cdot\left(\frac{\delta}{|e|^\beta}\right)^{\frac12}\left[\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right]\right|, \\
\leqslant\ & 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\gamma_i C(\boldsymbol{D})h^{(d-1)\beta/2-1/2}\left\|\boldsymbol{D}^{-\frac12}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0 \\
& \qquad\times J_0^{1,\beta}\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2},\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right)^{\frac12} \\
& + 2C(\boldsymbol{D},\delta)h^{(d-1)\beta/2-1/2}k \\
& \qquad\times\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\gamma_i J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right)^{\frac12} \\
& \qquad\times\left(\|\boldsymbol{D}^{-\frac12}{}^*\boldsymbol{\sigma}_{ij}^h\|_0+\|\boldsymbol{D}^{-\frac12}{}^*\boldsymbol{\sigma}_{i,j-1}^h\|_0\right), \\
\leqslant\ & 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\hat\epsilon_i}{2}\left\|\boldsymbol{D}^{-\frac12}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0^2 \\
& + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{Ch^{(d-1)\beta-1}\gamma_i^2}{2\hat\epsilon_i}J_0^{1,\beta}\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2},\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right) \\
& + 2Ck\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left(\frac{\tilde\epsilon_i h^{(d-1)\beta-1}}{2}J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right)\right. \\
& \qquad\left.+\frac{\gamma_i^2}{4\tilde\epsilon_i}\left(\|\boldsymbol{D}^{-\frac12}{}^*\boldsymbol{\sigma}_{ij}^h\|_0^2+\|\boldsymbol{D}^{-\frac12}{}^*\boldsymbol{\sigma}_{i,j-1}^h\|_0^2\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|T_1| \leqslant\ & 2k \sum_{j=1}^{m} \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \boldsymbol{D}^{-\frac{1}{2}} \left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k} \right) \right\|_0^2 \\
& + Ck \sum_{j=1}^{m} \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 h^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta} \left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}, \frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2} \right) \\
& + 2h^{(d-1)\beta-1} k \check{\epsilon} (1 - \varphi_0) \sum_{j=1}^{m} J_0^{\delta,\beta} \left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2} \right) \\
& + \frac{2Ck}{\check{\epsilon}} \sum_{j=0}^{m} \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \| \boldsymbol{D}^{-\frac{1}{2}} {}^*\boldsymbol{\sigma}_{ij}^h \|_0^2.
\end{aligned}
$$

Observing that,

$$
J_0^{\delta,\beta} \left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2}, \frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2} \right) \leqslant \frac{1}{2} J_0^{\delta,\beta}(\boldsymbol{u}_j^h, \boldsymbol{u}_j^h) + \frac{1}{2} J_0^{\delta,\beta}(\boldsymbol{u}_{j-1}^h, \boldsymbol{u}_{j-1}^h),
$$

then yields,

$$
\begin{aligned}
|T_1| \leqslant\ & 2k \sum_{j=1}^{m} \sum_{i=1}^{N_\varphi} \frac{\hat{\epsilon}_i}{2} \left\| \boldsymbol{D}^{-\frac{1}{2}} \left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h - {}^*\boldsymbol{\sigma}_{i,j-1}^h}{k} \right) \right\|_0^2 \\
& + Ck \sum_{j=0}^{m} \sum_{i=1}^{N_\varphi} \frac{\gamma_i^2 h^{(d-1)\beta-1}}{\hat{\epsilon}_i} J_0^{1,\beta} (\boldsymbol{u}_j^h, \boldsymbol{u}_j^h) \\
& + 2h^{(d-1)\beta-1} k \check{\epsilon} (1 - \varphi_0) \sum_{j=1}^{m} J_0^{\delta,\beta} \left(\frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2}, \frac{\boldsymbol{z}_j^h + \boldsymbol{z}_{j-1}^h}{2} \right) \\
& + \frac{2Ck}{\check{\epsilon}} \sum_{j=0}^{m} \sum_{i=1}^{N_\varphi} \frac{1}{\tau_i} \| \boldsymbol{D}^{-\frac{1}{2}} {}^*\boldsymbol{\sigma}_{ij}^h \|_0^2,
\end{aligned}
$$

where $\tilde{\epsilon}_i = 2\varphi_i \check{\epsilon}/C$.

Now, for T_3, we note first that,

$$
\begin{aligned}
& -\kappa \int_e \left\{ \boldsymbol{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k} \right) \cdot \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2} \right] \\
= & -\frac{\kappa}{k} \int_e \{ \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_j^h) \cdot \boldsymbol{n}_e \} \cdot [\boldsymbol{u}_j^h] + \frac{\kappa}{k} \int_e \{ \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^h) \cdot \boldsymbol{n}_e \} \cdot [\boldsymbol{u}_{j-1}^h] \\
& + \kappa \int_e \left\{ \boldsymbol{D}\boldsymbol{\varepsilon} \left(\frac{\boldsymbol{u}_j^h + \boldsymbol{u}_{j-1}^h}{2} \right) \cdot \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{u}_j^h - \boldsymbol{u}_{j-1}^h}{k} \right].
\end{aligned}
$$

Hence,

$$T_3 = 2k\sum_{j=1}^{m}\sum_{e\in\Gamma_h\cup\Gamma_D}\left(\int_e (1+\kappa)\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right)\cdot\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h}{k}\right]\right.$$
$$\left.-\frac{\kappa}{k}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_j^h)\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_j^h]+\frac{\kappa}{k}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^h)\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_{j-1}^h]\right).$$

Therefore, since $|\kappa|\leqslant 1$ and $|1+\kappa|\leqslant 2$,

$$\begin{aligned}
|T_3| \leqslant\ & 2k\sum_{j=1}^{m}2Ch^{(d-1)\beta/2-1/2}\left\|\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right\|_{\mathcal{A}} \\
&\qquad\times J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right)^{\frac{1}{2}} \\
&+2\sum_{e\in\Gamma_h\cup\Gamma_D}\left|\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_m^h)\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_m^h]\right| \\
&+2\sum_{e\in\Gamma_h\cup\Gamma_D}\left|\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{u}_0^h)\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_0^h]\right|, \\
\leqslant\ & 2k\left(\sum_{j=1}^{m}\frac{2C^2}{\epsilon'}\left\|\frac{\boldsymbol{u}_j^h+\boldsymbol{u}_{j-1}^h}{2}\right\|_{\mathcal{A}}^2\right. \\
&\qquad\left.+\sum_{j=1}^{m}\frac{\epsilon' h^{(d-1)\beta-1}}{2}J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right)\right) \\
&+2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{u}_m^h\|_{\mathcal{A}}J_0^{1,\beta}(\boldsymbol{u}_m^h,\boldsymbol{u}_m^h)^{\frac{1}{2}} \\
&+2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{u}_0^h\|_{\mathcal{A}}J_0^{\delta,\beta}(\boldsymbol{u}_0^h,\boldsymbol{u}_0^h)^{\frac{1}{2}}, \\
\leqslant\ & \frac{\epsilon''}{2}\|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2+\frac{2Ck}{\epsilon'}\sum_{j=0}^{m}\|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2 \\
&+k\epsilon' h^{(d-1)\beta-1}\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right) \\
&+\frac{2C^2h^{(d-1)\beta-1}}{\epsilon''}J_0^{1,\beta}(\boldsymbol{u}_m^h,\boldsymbol{u}_m^h)+Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{u}_0^h\|_{\mathcal{A}}^2.
\end{aligned}$$

With these bounds on T_1, T_2 and T_3 we can return to our earlier inequality, use Lemma 4.3, and obtain,

$$
\begin{aligned}
&(1-\epsilon_L k)\|\rho^{\frac{1}{2}}\boldsymbol{z}_m^h\|_0^2 \\
&+2k\left(1-\left(\frac{\epsilon'}{2}+(1-\varphi_0)\check{\epsilon}\right)\hat{h}^{(d-1)\beta-1}\right)\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right) \\
&+\left(1-\epsilon_L'-\epsilon_L k-\frac{2Ck}{\epsilon'}-\frac{\epsilon''}{2}\right)\|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2 \\
&-\left(\frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{u}_m^h|_{\mathcal{E}}^2+\hat{h}^{(d-1)\beta-1}\left(\frac{2C^2}{\epsilon''}+Ck\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2}{\hat{\epsilon}_i}\right)J_0^{1,\beta}(\boldsymbol{u}_m^h,\boldsymbol{u}_m^h)\right) \\
&+2k\sum_{i=1}^{N_\varphi}\left(1-\frac{\hat{\epsilon}_i}{2}\right)\sum_{j=1}^{m}\left\|\boldsymbol{D}^{-\frac{1}{2}}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0^2 \\
&+\left(1-\bar{\epsilon}-\frac{2Ck}{\check{\epsilon}}\right)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{im}^h\|_0^2 \\
&\leqslant \|\rho^{\frac{1}{2}}\bar{\boldsymbol{z}}\|_0^2+C_{\mathcal{A}}^2(2+C\hat{h}^{(d-1)\beta/2-1/2})\|\bar{\boldsymbol{u}}\|_2^2 \\
&+C^2h^{-1}\|\boldsymbol{g}(0)\|_{0,\Gamma_N}^2+\frac{C^2h^{-1}}{\epsilon_L'}\|\boldsymbol{g}(t_m)\|_{0,\Gamma_N}^2 \\
&+\frac{C^2h^{-1}k}{\epsilon_L}\sum_{j=1}^{m}\left\|\frac{\boldsymbol{g}(t_j)-\boldsymbol{g}(t_{j-1})}{k}\right\|_{0,\Gamma_N}^2+\frac{k}{\rho\epsilon_L}\sum_{j=1}^{m}\left\|\frac{\boldsymbol{f}(t_j)+\boldsymbol{f}(t_{j-1})}{2}\right\|_0^2 \\
&+k\sum_{j=0}^{m-1}\left(\epsilon_L\|\boldsymbol{z}_j^h\|_0^2+\left(\epsilon_L+\frac{2C}{\epsilon'}+\sum_{i=1}^{N_\varphi}\frac{C^2\gamma_i^2\hat{h}^{(d-1)\beta-1}}{\delta\hat{\epsilon}_i}\right)\|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2\right. \\
&\left.+\frac{2C}{\check{\epsilon}}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{ij}^h\|_0^2\right).
\end{aligned}
$$

We now choose,

$$\epsilon_L=\frac{1}{\varphi_0},\qquad \epsilon_L'=\frac{\varphi_0}{8-4\varphi_0},\qquad \bar{\epsilon}=1-\frac{\varphi_0}{2},\qquad \hat{\epsilon_i}=1,$$

$$\epsilon''=\frac{\varphi_0/4}{1-\varphi_0/2},\qquad \epsilon'=2\varphi_0,\qquad \check{\epsilon}=\frac{1}{2},$$

and insist that,

$$\delta\geqslant\frac{4C^2(2-\varphi_0)^2\hat{h}^{(d-1)\beta-1}}{(2-2\varphi_0)\varphi_0}.$$

Then, observing that,

$$
\begin{aligned}
-&\left(\frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{u}_m^h|_{\mathcal{E}}^2+\frac{2C^2\hat{h}^{(d-1)\beta-1}}{\epsilon''}J_0^{1,\beta}(\boldsymbol{u}_m^h,\boldsymbol{u}_m^h)\right)\\
&=-\left(\frac{2-2\varphi_0}{2-\varphi_0}|\boldsymbol{u}_m^h|_{\mathcal{E}}^2+\frac{4C^2(2-\varphi_0)\hat{h}^{(d-1)\beta-1}}{\delta\varphi_0}J_0^{\delta,\beta}(\boldsymbol{u}_m^h,\boldsymbol{u}_m^h)\right)\\
&\geqslant\frac{2\varphi_0-2}{2-\varphi_0}\|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2,
\end{aligned}
$$

our stability estimate can now be re-cast as,

$$
\begin{aligned}
&\left(1-\frac{\hat{k}}{\varphi_0}\right)\|\rho^{\frac{1}{2}}\boldsymbol{z}_m^h\|_0^2+\left(\frac{\varphi_0}{4-2\varphi_0}-C\hat{k}\right)\|\boldsymbol{u}_m^h\|_{\mathcal{A}}^2\\
&\quad+\left(\frac{\varphi_0}{2}-4C\hat{k}\right)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{im}^h\|_0^2\\
&\quad+(2-(1+\varphi_0)\hat{h}^{(d-1)\beta-1})k\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2},\frac{\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h}{2}\right)\\
&\quad+k\sum_{i=1}^{N_\varphi}\sum_{j=1}^{m}\left\|\boldsymbol{D}^{-\frac{1}{2}}\left(\frac{{}^*\boldsymbol{\sigma}_{ij}^h-{}^*\boldsymbol{\sigma}_{i,j-1}^h}{k}\right)\right\|_0^2\\
&\leqslant\|\rho^{\frac{1}{2}}\bar{\boldsymbol{z}}\|_0^2+C\|\bar{\boldsymbol{u}}\|_2^2+Ch^{-1}\|\boldsymbol{g}(0)\|_{0,\Gamma_N}^2+Ch^{-1}\|\boldsymbol{g}(t_m)\|_{0,\Gamma_N}^2\\
&\quad+Ch^{-1}k\sum_{j=1}^{m}\left\|\frac{\boldsymbol{g}(t_j)-\boldsymbol{g}(t_{j-1})}{k}\right\|_{0,\Gamma_N}^2+Ck\sum_{j=1}^{m}\left\|\frac{\boldsymbol{f}(t_j)+\boldsymbol{f}(t_{j-1})}{2}\right\|_0^2\\
&\quad+Ck\sum_{j=0}^{m-1}\left(\|\boldsymbol{z}_j^h\|_0^2+\|\boldsymbol{u}_j^h\|_{\mathcal{A}}^2+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-\frac{1}{2}}{}^*\boldsymbol{\sigma}_{ij}^h\|_0^2\right).
\end{aligned}
$$

Lastly, we choose $\hat{k}$ and $\hat{h}$ small enough, and then an application of the discrete Gronwall lemma completes the proof. □

Since uniqueness implies existence for linear finite dimensional problems, we can assert the existence and uniqueness of the discrete solution.

Theorem 4.5 (well-posedness) *Under the conditions of Theorem 4.4, the discrete solution exists and is unique.*

Our next goal is a fully discrete error estimate and the first step toward this is to derive an error equation. For this we set,

$$
\begin{aligned}
\boldsymbol{\chi}_j &:= \boldsymbol{u}_j^h-\breve{\boldsymbol{u}}(t_j), & \boldsymbol{\psi}_j &:= \boldsymbol{z}_j^h-\breve{\boldsymbol{z}}(t_j), & \boldsymbol{\eta}_{ij} &:= {}^*\boldsymbol{\sigma}_{ij}^h-{}^*\breve{\boldsymbol{\sigma}}_i(t_j),\\
\boldsymbol{\xi}_j &:= \boldsymbol{u}(t_j)-\breve{\boldsymbol{u}}(t_j), & \boldsymbol{\phi}_j &:= \boldsymbol{z}(t_j)-\breve{\boldsymbol{z}}(t_j), & \boldsymbol{\theta}_{ij} &:= {}^*\boldsymbol{\sigma}_i(t_j)-{}^*\breve{\boldsymbol{\sigma}}_i(t_j),
\end{aligned}
$$

where $\{\breve{\boldsymbol{u}}(t), {}^*\breve{\boldsymbol{\sigma}}_1(t), \ldots\} \subset \mathbf{D}_r(\mathcal{E}_h)$ for each t and with $\boldsymbol{z} := \boldsymbol{u}_t$ and $\breve{\boldsymbol{z}} := \breve{\boldsymbol{u}}_t$.

We choose $\breve{\boldsymbol{u}}(t) \in \mathbf{D}_r(\mathcal{E}_h)$ as the continuous interpolant of $\boldsymbol{u}(t)$ and ${}^*\breve{\boldsymbol{\sigma}}_i$ as the $\boldsymbol{L}_2(\Omega)$ projection of ${}^*\boldsymbol{\sigma}_i$ into $\mathbf{L}_{r-1}(\mathcal{E}_h)$. We then have, $\breve{\boldsymbol{z}}(t) = \breve{\boldsymbol{u}}_t(t) \in \mathbf{D}_r(\mathcal{E}_h)$ and if $\boldsymbol{u}(t_j),\ \boldsymbol{u}_t(t_j) \in C(\bar{\Omega})^d$ it follows that,

$$[\boldsymbol{u}(t_j)] = \mathbf{0}, \qquad [\breve{\boldsymbol{u}}(t_j)] = \mathbf{0}, \qquad [\boldsymbol{\xi}_j] = \mathbf{0}, \tag{43}$$

and

$$[\boldsymbol{u}_t(t_j)] = \mathbf{0}, \qquad [\breve{\boldsymbol{u}}_t(t_j)] = \mathbf{0}, \qquad [\boldsymbol{\phi}_j] = \mathbf{0}. \tag{44}$$

Moreover,

$$(\boldsymbol{\theta}_i, \boldsymbol{w}_i) = (\dot{\boldsymbol{\theta}}_i, \boldsymbol{w}_i) = (\ddot{\boldsymbol{\theta}}_i, \boldsymbol{w}_i) = \cdots = 0 \qquad \forall \boldsymbol{w}_i \in \mathbf{L}_{r-1}(\mathcal{E}_h), \tag{45}$$

and from standard arguments we also have,

$$\left\| \frac{\partial^n \breve{\boldsymbol{\sigma}}_i}{\partial t^n} \right\|_0 \leqslant \left\| \frac{\partial^n \boldsymbol{\sigma}}{\partial t^n} \right\|_0.$$

Furthermore, from (13) we have,

$$\|\boldsymbol{\theta}_{ij}\|_0 = \|{}^*\boldsymbol{\sigma}_i(t_j) - {}^*\breve{\boldsymbol{\sigma}}_i(t_j)\|_0 \leqslant Ch^r \|{}^*\boldsymbol{\sigma}_i(t_j)\|_r.$$

Now, motivated by the terms that arise below, define,

$$\Delta_j \boldsymbol{v} := \frac{\boldsymbol{v}_t(t_j) + \boldsymbol{v}_t(t_{j-1})}{2} - \frac{\boldsymbol{v}(t_j) - \boldsymbol{v}(t_{j-1})}{k}.$$

Then by standard estimates for the trapezoidal quadrature rule and the Cauchy-Schwarz inequality we have the following result.

Lemma 4.6 *We have,*

$$\Delta_j \boldsymbol{v} = \frac{1}{2k} \int_{t_{j-1}}^{t_j} \boldsymbol{v}_{ttt}(t)(t_j - t)(t - t_{j-1}).$$

Moreover, if $\boldsymbol{v}_{ttt} \in L_2((t_{j-1}, t_j); \boldsymbol{L}_2(\Omega))$*, then,*

$$\|\Delta_j \boldsymbol{v}\|_0^2 \leqslant \frac{k^3}{4} \int_{t_{j-1}}^{t_j} \|\boldsymbol{v}_{ttt}(t)\|_0^2.$$

We will also make use of the following ‘summation by parts’ identity,

$$\sum_{j=1}^{m} (\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}, \boldsymbol{p}_j) = (\boldsymbol{\psi}_m, \boldsymbol{p}_m) - (\boldsymbol{\psi}_0, \boldsymbol{p}_1) + \sum_{j=1}^{m-1} (\boldsymbol{\psi}_j, \boldsymbol{p}_j - \boldsymbol{p}_{j+1}). \tag{46}$$

We will also need the following estimate which is proven by using Taylor’s series with integral remainder.

Lemma 4.7 *We have,*

$$\Delta_j \boldsymbol{v} - \Delta_{j+1}\boldsymbol{v} = k\left(\frac{\boldsymbol{v}(t_{j+1}) - 2\boldsymbol{v}(t_j) + \boldsymbol{v}(t_{j-1})}{k^2} - \frac{\boldsymbol{v}_t(t_{j+1}) - \boldsymbol{v}_t(t_{j-1})}{2k}\right).$$

Moreover, if, a.e. in Ω*, we have* $\boldsymbol{v}_{tttt} \in L_2(t_{j-1}, t_{j+1})$*, then,*

$$|\Delta_j \boldsymbol{v} - \Delta_{j+1}\boldsymbol{v}|^2 \leqslant Ck^5 \int_{t_{j-1}}^{t_{j+1}} |\boldsymbol{v}_{tttt}(t)|^2.$$

Averaging (17), (18) and (19) between t_j and t_{j-1}, and subtracting the result from the fully discrete scheme given by (35), (36) and (37) then gives three error equations,

$$\begin{aligned}
&\left(\rho\frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k}, \boldsymbol{v}\right) + A\left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2}, \boldsymbol{v}\right) + J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2}, \boldsymbol{v}\right) \\
&\quad + \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \gamma_i \left\{\frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2}\boldsymbol{n}_e\right\} \cdot [\boldsymbol{v}] \\
&\quad - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \\
&= (\rho\Delta_j \boldsymbol{z}, \boldsymbol{v}) + \left(\rho\frac{\boldsymbol{\phi}_j - \boldsymbol{\phi}_{j-1}}{k}, \boldsymbol{v}\right) + A\left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}, \boldsymbol{v}\right) \\
&\quad + J_0^{\delta,\beta}\left(\frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2}, \boldsymbol{v}\right) + \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \gamma_i \left\{\frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2}\boldsymbol{n}_e\right\} \cdot [\boldsymbol{v}] \\
&\quad - \sum_{i=1}^{N_\varphi} \sum_E \int_E \gamma_i \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathbf{D}_r(\mathcal{E}_h),
\end{aligned} \tag{47}$$

and,

$$\sum_{i=1}^{N_\varphi}\sum_{E}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}+\frac{1}{\tau_i}\frac{\boldsymbol{\eta}_{ij}+\boldsymbol{\eta}_{i,j-1}}{2},\boldsymbol{w}_i\right)_E$$
$$-\sum_{i=1}^{N_\varphi}\sum_{E}\gamma_i\left(\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right),\boldsymbol{w}_i\right)_E$$
$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot\left[\frac{\boldsymbol{\chi}_j+\boldsymbol{\chi}_{j-1}}{2}\right]$$
$$=\sum_{i=1}^{N_\varphi}\sum_{E}(\Delta_j{}^*\boldsymbol{\sigma}_i,\boldsymbol{w}_i)_E+\sum_{i=1}^{N_\varphi}\sum_{E}\left(\frac{\boldsymbol{\theta}_{ij}-\boldsymbol{\theta}_{i,j-1}}{k}+\frac{1}{\tau_i}\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2},\boldsymbol{w}_i\right)_E$$
$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\{\boldsymbol{D}\boldsymbol{w}_i\boldsymbol{n}_e\}\cdot\left[\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right]$$
$$-\sum_{i=1}^{N_\varphi}\sum_{E}\gamma_i\left(\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right),\boldsymbol{w}_i\right)_E\qquad\forall\{\boldsymbol{w}_i\}\in\{\mathbf{L}_{r-1}(\mathcal{E}_h)\},\qquad(48)$$

and,

$$\left(\rho\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\boldsymbol{v}\right)_E-\left(\rho\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k},\boldsymbol{v}\right)_E=-(\rho\Delta_j\boldsymbol{u},\boldsymbol{v})_E$$
$$+\left(\rho\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2},\boldsymbol{v}\right)_E-\left(\rho\frac{\boldsymbol{\xi}_j-\boldsymbol{\xi}_{j-1}}{k},\boldsymbol{v}\right)_E\qquad\forall\boldsymbol{v}\in\mathbf{D}_r(\mathcal{E}_h).\qquad(49)$$

Now, choosing $\boldsymbol{v}=(\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1})/k$ in (47), $\boldsymbol{w}_i=\boldsymbol{D}^{-1}(\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1})/k$ in (48) and $\boldsymbol{v}=(\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1})/k$ in (49), adding the first two resulting equations together and noting from the third that,

$$\left(\rho\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)=\left(\rho\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right)$$
$$-\left(\rho\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right)+\left(\rho\frac{\boldsymbol{\xi}_j-\boldsymbol{\xi}_{j-1}}{k},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right)$$
$$+\left(\rho\Delta_j\boldsymbol{u},\frac{\boldsymbol{\psi}_j-\boldsymbol{\psi}_{j-1}}{k}\right),$$

we multiply by $2k$ and sum over $j = 1, \ldots, m$ to obtain,

$$
\begin{aligned}
&\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + 2k\sum_{j=1}^{m} J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right) \\
&+ 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2 \\
&= \|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + \|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{i0}\|_0^2 \\
&+ 2k\sum_{j=1}^{m} G_j\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right) + 2k\sum_{j=1}^{m} H_j\left(\boldsymbol{D}^{-1}\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right) \\
&+ T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \qquad (50)
\end{aligned}
$$

where,

$$\begin{aligned}
T_1 \quad &:= \quad -2k \sum_{j=1}^{m} \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \Bigg(\left\{ \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right] \\
&\qquad + \left\{ \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right] \Bigg), \\
T_2 \quad &:= \quad 2k \sum_{j=1}^{m} \sum_{i=1}^{N_\varphi} \sum_{E} \gamma_i \int_E \Bigg(\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right) : \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \\
&\qquad + \boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) : \frac{\boldsymbol{\eta}_{ij} + \boldsymbol{\eta}_{i,j-1}}{2} \Bigg), \\
T_3 \quad &:= \quad 2k \sum_{j=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \Bigg(\left\{ \boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right) \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right] \\
&\qquad - \kappa \left\{ \boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j + \boldsymbol{\chi}_{j-1}}{2} \right] \Bigg), \\
T_4 \quad &:= \quad -2k \sum_{j=1}^{m} J_0^{\delta,\beta} \left(\Delta_j \breve{\boldsymbol{u}}, \frac{\boldsymbol{\psi}_j + \boldsymbol{\psi}_{j-1}}{2} \right), \\
T_5 \quad &:= \quad -2k \sum_{j=1}^{m} \left(\rho \frac{\boldsymbol{\psi}_j - \boldsymbol{\psi}_{j-1}}{k}, \Delta_j \breve{\boldsymbol{u}} \right), \\
T_6 \quad &:= \quad 2k \sum_{j=1}^{m} \left(\rho \Delta_j \boldsymbol{z}, \frac{\boldsymbol{\chi}_j - \boldsymbol{\chi}_{j-1}}{k} \right) \\
&\qquad + 2k \sum_{j=1}^{m} \sum_{i=1}^{N_\varphi} \sum_{E} \left(\Delta_j{}^* \boldsymbol{\sigma}_i, \boldsymbol{D}^{-1} \frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k} \right)_E,
\end{aligned}$$

along with,

$$\begin{aligned}
G_j(\boldsymbol{v}) \quad &:= \quad J_0^{\delta,\beta} \left(\frac{\boldsymbol{\phi}_j + \boldsymbol{\phi}_{j-1}}{2}, \boldsymbol{v} \right) + \left(\rho \frac{\boldsymbol{\phi}_j - \boldsymbol{\phi}_{j-1}}{k}, \boldsymbol{v} \right) \\
&\qquad + A\left(\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}, \boldsymbol{v} \right) - \sum_{i=1}^{N_\varphi} \sum_{E} \int_E \gamma_i \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \\
&\qquad + \sum_{i=1}^{N_\varphi} \sum_{e \in \Gamma_h \cup \Gamma_D} \gamma_i \int_e \left\{ \frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2} \boldsymbol{n}_e \right\} \cdot [\boldsymbol{v}],
\end{aligned}$$

and,

$$
\begin{aligned}
H_j(\boldsymbol{v}) \quad := \quad & \sum_{i=1}^{N_\varphi} \sum_{e\in\Gamma_h\cup\Gamma_D} \gamma_i \int_e \{\boldsymbol{D}\boldsymbol{v}\boldsymbol{n}_e\} \cdot \left[\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right] \\
& - \sum_{i=1}^{N_\varphi} \sum_E \gamma_i \left(\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right), \boldsymbol{v}\right)_E \\
& + \sum_{i=1}^{N_\varphi} \sum_E \left(\frac{\boldsymbol{\theta}_{ij}-\boldsymbol{\theta}_{i,j-1}}{k} + \frac{1}{\tau_i}\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}, \boldsymbol{v}\right)_E .
\end{aligned}
$$

To get these we noted first that,

$$
\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2} - \frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k} = -\Delta_j \breve{\boldsymbol{u}}, \tag{51}
$$

because $(\boldsymbol{z}_j^h+\boldsymbol{z}_{j-1}^h)/2 = (\boldsymbol{u}_j^h-\boldsymbol{u}_{j-1}^h)/k$, and secondly that,

$$
\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2} - \frac{\boldsymbol{\xi}_j-\boldsymbol{\xi}_{j-1}}{k} = \Delta_j \boldsymbol{u} - \Delta_j \breve{\boldsymbol{u}}.
$$

We now start estimating. These lemmas are all subject to the assumptions made later in Theorem 4.13.

Lemma 4.8 *We have,*

$$
\begin{aligned}
|T_4+T_5+T_6| \leqslant 2k \sum_{j=1}^{m}\sum_{i=1}^{N_\varphi} \frac{\epsilon_6'}{2} \left\| \boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 \\
+ \frac{k}{\epsilon_6}\sum_{j=0}^{m} \|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 + \epsilon_5 \|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + k\sum_{j=1}^{m-1}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 \\
+ Ck^3\int_0^{t_1}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{Ck^3}{\epsilon_5}\int_{t_{m-1}}^{t_m}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + Ck^4\int_0^{t_m}\|\breve{\boldsymbol{u}}_{tttt}(t)\|_0^2 \\
+ Ck^4\int_0^{t_m}\left((\epsilon_6+1)\|\boldsymbol{z}_{ttt}(t)\|_0^2 + \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon_6'}\sum_{i=1}^{N_\varphi}\|{}^*\ddot{\boldsymbol{\sigma}}_i(t)\|_0^2\right).
\end{aligned}
$$

Proof. Firstly, $T_4 = 0$ by (43) and (44). Now, using (46), we can write T_5 as,

$$
T_5 = -2(\rho\boldsymbol{\psi}_m, \Delta_m\breve{\boldsymbol{u}}) + 2(\rho\boldsymbol{\psi}_0, \Delta_1\breve{\boldsymbol{u}}) - 2\sum_{j=1}^{m-1}(\rho\boldsymbol{\psi}_j, \Delta_j\breve{\boldsymbol{u}} - \Delta_{j+1}\breve{\boldsymbol{u}}).
$$

From this we estimate with the Cauchy-Schwarz and Young's inequalities as follows,

$$|T_5| \leqslant \epsilon_5 \|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + k\sum_{j=1}^{m-1} \|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2$$
$$+ \frac{C(\rho)}{\epsilon_5}\|\Delta_m \breve{\boldsymbol{u}}\|_0^2 + C(\rho)\|\Delta_1 \breve{\boldsymbol{u}}\|_0^2 + \frac{C(\rho)}{k}\sum_{j=1}^{m-1} \|\Delta_j \breve{\boldsymbol{u}} - \Delta_{j+1}\breve{\boldsymbol{u}}\|_0^2.$$

Using Lemma 4.6 results in,

$$\frac{C(\rho)}{\epsilon_5}\|\Delta_m \breve{\boldsymbol{u}}\|_0^2 \leqslant \frac{Ck^3}{\epsilon_5}\int_{t_{m-1}}^{t_m} \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2,$$

and

$$C(\rho)\|\Delta_1 \breve{\boldsymbol{u}}\|_0^2 \leqslant Ck^3 \int_0^{t_1} \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2,$$

while Lemma 4.7 leads to,

$$\frac{C(\rho)}{k}\|\Delta_j \breve{\boldsymbol{u}} - \Delta_{j+1}\breve{\boldsymbol{u}}\|_0^2 \leqslant Ck^4 \int_{t_{j-1}}^{t_{j+1}} \|\breve{\boldsymbol{u}}_{tttt}(t)\|_0^2.$$

For T_6 we have by the Cauchy-Schwarz and Young's inequalities, Lemma 4.6 and the triangle inequality with (51), that,

$$|T_6| \leqslant \frac{k}{\epsilon_6}\sum_{j=0}^{m} \|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi} \frac{\epsilon_6'}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij} - \boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2$$
$$+ Ck^4 \int_0^{t_m}\left((\epsilon_6 + 1)\|\boldsymbol{z}_{ttt}(t)\|_0^2 + \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon_6'}\sum_{i=1}^{N_\varphi} \|{}^*\ddot{\boldsymbol{\sigma}}_i(t)\|_0^2\right).$$

These estimates complete the proof. □

The terms T_1, T_2 and T_3 can be handled in much the same way as in the proof of Theorem 4.4, although some modifications are necessary.

Lemma 4.9 *We have*

$$
\begin{aligned}
|T_1+T_2+T_3| &\leqslant 2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 \\
&+ 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\hat{\epsilon}_i}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 \\
&+ \frac{2Ck}{\check{\epsilon}}\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2 \\
&+ \sum_{i=1}^{N_\varphi}\frac{\bar{\epsilon}}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2 + \frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2 + \frac{2C^2h^{(d-1)\beta-1}}{\epsilon''}J_0^{1,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m) \\
&+ Ck\sum_{j=0}^{m}\frac{1}{\epsilon'}\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + \frac{\epsilon''}{2}\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + Ck\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2h^{(d-1)\beta-1}}{\hat{\epsilon}_i}J_0^{1,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j) \\
&+ 2h^{(d-1)\beta-1}k(2\check{\epsilon}(1-\varphi_0)+\epsilon')\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right).
\end{aligned}
$$

Proof. First, using (51), (43) and (44), we have,

$$
J_0^{\delta,\beta}\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right) = J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right).
$$

With this we refer back to the proof of Theorem 4.4 and obtain,

$$
\begin{aligned}
|T_1| &\leqslant 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\hat{\epsilon}_i}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 \\
&+ Ck\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2h^{(d-1)\beta-1}}{\hat{\epsilon}_i}J_0^{1,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j) \\
&+ \frac{2Ck}{\check{\epsilon}}\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2 \\
&+ 2h^{(d-1)\beta-1}k\check{\epsilon}(1-\varphi_0)\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right).
\end{aligned}
$$

Similarly, for T_3 we obtain,

$$|T_3| \leqslant \frac{2Ck}{\epsilon'}\sum_{j=0}^{m}\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + \frac{\epsilon''}{2}\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + \frac{2C^2h^{(d-1)\beta-1}}{\epsilon''}J_0^{1,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m)$$
$$+ 2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2$$
$$+ 2k\epsilon' h^{(d-1)\beta-1}\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right).$$

Finally,

$$|T_2| \leqslant \sum_{i=1}^{N_\varphi}\frac{\bar{\epsilon}}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2 + \frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2,$$

and this completes the proof. □

Lemma 4.10 *Assuming that* $h \leqslant \hat{h}$, $\beta \geqslant (d-1)^{-1}$, $|e| \leqslant Ch^{d-1}$ *and* $\|\boldsymbol{v}\|_{0,e} \leqslant Ch^{-1/2}\|\boldsymbol{v}\|_{0,E}$ *if* e *is an edge of* E, *we have,*

$$\left|2k\sum_{j=1}^{m}H_j\left(\boldsymbol{D}^{-1}\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right|$$
$$\leqslant 2k\sum_{j=1}^{m}\left(\sum_{i=1}^{N_\varphi}\epsilon'_H\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 + \frac{C}{\epsilon'_H}\left|\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right|_{\mathcal{E}}^2\right).$$

Proof. Using (45) we have,

$$H_j\left(\boldsymbol{D}^{-1/2}\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right) = T_1 + T_2$$

where,

$$T_1 := \sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left\{\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right],$$

and

$$T_2 := -\sum_{i=1}^{N_\varphi}\sum_{E}\gamma_i\left(\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right),\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)_E.$$

Now, $T_1 = 0$ due to (43), and for T_2 we have,

$$|T_2| \leqslant \sum_{i=1}^{N_\varphi}\gamma_i\left\|\boldsymbol{D}^{1/2}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right)\right\|_0\left\|\boldsymbol{D}^{-1/2}\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right\|_0,$$
$$\leqslant \sum_{i=1}^{N_\varphi}\epsilon'_H\left\|\boldsymbol{D}^{-1/2}\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right\|_0^2 + \sum_{i=1}^{N_\varphi}\frac{\gamma_i^2}{4\epsilon'_H}\left|\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right|_{\mathcal{E}}^2.$$

This completes the proof. □

Lemma 4.11 *We have,*

$$\left|2k\sum_{j=1}^{m}G_j\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)\right|\leqslant 2k\sum_{j=1}^{m}\frac{1}{2}\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0^2$$

$$+2k\sum_{j=1}^{m}\frac{1+\epsilon_G''}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2+2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}C\epsilon_G'\left\|\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}\right\|_0^2$$

$$+\frac{k}{\epsilon_G'}\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)+2k\sum_{j=0}^{m}\frac{1}{2\epsilon_G''}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2$$

$$+Ch^{2r}\big((1+\epsilon_G''')\|\boldsymbol{u}\|^2_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}+\|\boldsymbol{u}_t\|^2_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}\big)$$

$$+\frac{1}{2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2+\frac{1}{2\epsilon_G'''}\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2+\frac{k}{2}\sum_{j=1}^{m-1}\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2,$$

for all positive ϵ_G', ϵ_G'' *and* ϵ_G'''.

Proof. Using (45) we have,

$$G_j\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)=J_0^{\delta,\beta}\left(\frac{\boldsymbol{\phi}_j+\boldsymbol{\phi}_{j-1}}{2},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)$$

$$+\left(\rho\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)$$

$$+\sum_{i=1}^{N_\varphi}\sum_{e\in\Gamma_h\cup\Gamma_D}\gamma_i\int_e\left\{\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right]$$

$$+A\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2},\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right)$$

$$=T_1+T_2+T_3+T_4,$$

but, firstly, $T_1=0$ by (44). Secondly, for T_2,

$$|T_2|\leqslant\left|\left(\rho\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)\right|+\left|\left(\rho\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k},\Delta_j\breve{\boldsymbol{u}}\right)\right|,$$

$$\leqslant\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0\left(\left\|\rho^{1/2}\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right\|_0+\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0\right),$$

$$\leqslant\frac{\epsilon_G''+1}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2+\frac{1}{2}\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0^2$$

$$+\frac{1}{4\epsilon_G''}\left(\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2+\|\rho^{1/2}\boldsymbol{\psi}_{j-1}\|_0^2\right).$$

Thirdly, for T_3 we use (51) and (43) and get,

$$
\begin{aligned}
|T_3| \leqslant \sum_{i=1}^{N_\varphi} \gamma_i \sum_{e\in\Gamma_h\cup\Gamma_D} \left\| \left\{ \frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2} \right\} \right\|_{0,e} \left\| \left[\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2} \right] \right\|_{0,e}, \\
\leqslant \sum_{i=1}^{N_\varphi} \gamma_i \left(\sum_e \left(\frac{|e|^\beta}{\delta} \left\| \left\{ \frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2} \right\} \right\|_{0,e}^2 \right) \right)^{1/2} \\
\times \left(\sum_e \left(\frac{\delta}{|e|^\beta} \left\| \left[\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2} \right] \right\|_{0,e}^2 \right) \right)^{1/2}, \\
\leqslant \sum_{i=1}^{N_\varphi} C\gamma_i h^{(d-1)\beta/2-1/2} \left\| \frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2} \right\|_0 \\
\times J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2} \right)^{1/2}, \\
\leqslant \sum_{i=1}^{N_\varphi} C\epsilon_G' \left\| \frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2} \right\|_0^2 + \frac{1}{2\epsilon_G'} J_0^{\delta,\beta} \left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}, \frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2} \right).
\end{aligned}
$$

Turning to T_4 and noting (43) we have,

$$
\begin{aligned}
A\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}, \frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k} \right) = \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2} \right) : \boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k} \right) \\
- \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \left\{ \boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2} \right) \boldsymbol{n}_e \right\} \cdot \left[\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k} \right].
\end{aligned}
$$

Taking the sum over j as needed by the lemma we use a variant of (46) and get for the first term that,

$$
\begin{aligned}
2k\sum_{j=1}^{m} \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2} \right) : \boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k} \right) \\
= -\sum_{j=1}^{m-1} \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_{j+1}-\boldsymbol{\xi}_{j-1}) : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_j) \\
+\sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m+\boldsymbol{\xi}_{m-1}) : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_m) - \sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_0) : \boldsymbol{\varepsilon}(\boldsymbol{\chi}_0).
\end{aligned}
$$

Hence,

$$\left|\sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1})\colon \boldsymbol{\varepsilon}(\boldsymbol{\chi}_m)\right|$$
$$\leqslant \epsilon_G''' C\|\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}\|_{1,\Omega}^2 + \frac{1}{2\epsilon_G'''}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2,$$
$$\leqslant \epsilon_G''' Ch^{2r}\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2 + \frac{1}{2\epsilon_G'''}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2,$$

because,

$$\|\boldsymbol{\xi}_m + \boldsymbol{\xi}_{m-1}\|_{1,\Omega} \leqslant Ch^r\|\boldsymbol{u}(t_m) + \boldsymbol{u}(t_{m-1})\|_{r+1}.$$

Similarly,

$$\left|\sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_0)\colon \boldsymbol{\varepsilon}(\boldsymbol{\chi}_0)\right| \leqslant Ch^{2r}\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2 + \frac{1}{2}|\boldsymbol{\chi}_0|_{\mathcal{E}}^2.$$

And,

$$\left|\sum_{j=1}^{m-1}\sum_E \int_E \boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1})\colon \boldsymbol{\varepsilon}(\boldsymbol{\chi}_j)\right| \leqslant Ck\sum_{j=1}^{m-1}\left\|\frac{\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}}{k}\right\|_1 |\boldsymbol{\chi}_j|_{\mathcal{E}},$$
$$\leqslant Ch^{2r}\|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2 + \frac{k}{2}\sum_{j=1}^{m-1}|\boldsymbol{\chi}_j|_{\mathcal{E}}^2,$$

where we used,

$$\begin{aligned}\left\|\frac{\boldsymbol{\xi}_{j+1} - \boldsymbol{\xi}_{j-1}}{k}\right\|_1 &= \left\|\frac{\boldsymbol{u}(t_{j+1}) - \boldsymbol{u}(t_{j-1})}{k} - \frac{\breve{\boldsymbol{u}}(t_{j+1}) - \breve{\boldsymbol{u}}(t_{j-1})}{k}\right\|_1, \\ &\leqslant Ch^r\left\|\frac{\boldsymbol{u}(t_{j+1}) - \boldsymbol{u}(t_{j-1})}{k}\right\|_{r+1},\end{aligned}$$

and (by the fundamental theorem of calculus),

$$k\sum_{j=1}^{m-1}\left\|\frac{\boldsymbol{u}(t_{j+1}) - \boldsymbol{u}(t_{j-1})}{k}\right\|_{r+1} \leqslant C\|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}.$$

For the second term in T_4 we proceed similarly:

$$
\begin{aligned}
-2k\sum_{j=1}^{m}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\left\{\boldsymbol{D}\boldsymbol{\varepsilon}\left(\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right)\boldsymbol{n}_e\right\}\cdot\left[\frac{\boldsymbol{\chi}_j-\boldsymbol{\chi}_{j-1}}{k}\right] \\
=-\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m+\boldsymbol{\xi}_{m-1})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_m] \\
+\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_0)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_0] \\
+\sum_{j=1}^{m-1}\sum_{e\in\Gamma_h\cup\Gamma_D}\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_{j+1}-\boldsymbol{\xi}_{j-1})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_j].
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left|\sum_e\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m+\boldsymbol{\xi}_{m-1})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_m]\right| \\
&\leqslant\frac{\epsilon_G'''Ch^{(d-1)\beta}}{2\delta}\sum_e\|\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m+\boldsymbol{\xi}_{m-1})\boldsymbol{n}_e\|_{0,e}^2+\frac{1}{2\epsilon_G'''}J_0^{\delta,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m), \\
&\leqslant\frac{\epsilon_G'''Ch^{2r}}{2\delta}\|\boldsymbol{u}(t_m)+\boldsymbol{u}(t_{m-1})\|_{r+1}^2+\frac{1}{2\epsilon_G'''}J_0^{\delta,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m),
\end{aligned}
$$

since $(d-1)\beta-1\geqslant 0$ and,

$$
\|\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_m+\boldsymbol{\xi}_{m-1})\boldsymbol{n}_e\|_{0,e}\leqslant Ch^{r-1/2}\|\boldsymbol{u}(t_m)+\boldsymbol{u}(t_{m-1})\|_{r+1}.
$$

Similarly,

$$
\begin{aligned}
&\left|\sum_e\int_e\{\boldsymbol{D}\boldsymbol{\varepsilon}(\boldsymbol{\xi}_1+\boldsymbol{\xi}_0)\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_0]\right| \\
&\leqslant\frac{Ch^{2r}}{2\delta}\|\boldsymbol{u}(t_1)+\boldsymbol{u}(t_0)\|_{r+1}^2+\frac{1}{2}J_0^{\delta,\beta}(\boldsymbol{\chi}_0,\boldsymbol{\chi}_0).
\end{aligned}
$$

We also have,

$$\left|\sum_{j=1}^{m-1}\sum_{e}\int_{e}\{\boldsymbol{D\varepsilon}(\boldsymbol{\xi}_{j+1}-\boldsymbol{\xi}_{j-1})\boldsymbol{n}_e\}\cdot[\boldsymbol{\chi}_j]\right|$$
$$\leqslant k\sum_{j=1}^{m-1}\sum_{e}\left\|\boldsymbol{D\varepsilon}\left(\frac{\boldsymbol{u}(t_{j+1})-\boldsymbol{u}(t_{j-1})}{k}-\frac{\breve{\boldsymbol{u}}(t_{j+1})-\breve{\boldsymbol{u}}(t_{j-1})}{k}\right)\boldsymbol{n}_e\right\|_{0,e}$$
$$\times\|[\boldsymbol{\chi}_j]\|_{0,e},$$
$$\leqslant k\sum_{j=1}^{m-1}\frac{Ch^{2r-1+(d-1)\beta}}{2\delta}\left\|\frac{\boldsymbol{u}(t_{j+1})-\boldsymbol{u}(t_{j-1})}{k}\right\|_{r+1}^2+\frac{k}{2}\sum_{j=1}^{m-1}J_0^{\delta,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j),$$
$$\leqslant\frac{Ch^{2r}}{\delta}\|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2+\frac{k}{2}\sum_{j=1}^{m-1}J_0^{\delta,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j),$$

where we used,

$$\left\|\boldsymbol{D\varepsilon}\left(\frac{\boldsymbol{u}(t_{j+1})-\boldsymbol{u}(t_{j-1})}{k}-\frac{\breve{\boldsymbol{u}}(t_{j+1})-\breve{\boldsymbol{u}}(t_{j-1})}{k}\right)\boldsymbol{n}_e\right\|_{0,e}$$
$$\leqslant Ch^{r-1/2}\left\|\frac{\boldsymbol{u}(t_{j+1})-\boldsymbol{u}(t_{j-1})}{k}\right\|_{r+1,E}.$$

Assembling these estimates then yields,

$$\left\|2k\sum_{j=1}^{m}T_4\right\|\leqslant Ch^{2r}\big((1+\epsilon_G''')\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2+\|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2\big)$$
$$+\frac{1}{2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2+\frac{1}{2\epsilon_G'''}\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2+\frac{k}{2}\sum_{j=1}^{m-1}\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2,$$

which completes the proof. □

Before stating the error estimate we need one more estimate—connected with a term in the previous lemma.

Lemma 4.12 *We have,*

$$2k\sum_{j=1}^{m}\frac{1+\epsilon_G''}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2$$
$$\leqslant Ch^{2r}\|\boldsymbol{u}_t\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2+C(1+\epsilon_G'')k^4\|\boldsymbol{u}_{tttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}^2.$$

Proof. By the triangle and Young's inequalities we have,

$$
\begin{aligned}
2k\sum_{j=1}^{m}\frac{1+\epsilon_G''}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2 &\leqslant 2k\sum_{j=1}^{m}(1+\epsilon_G'')\|\rho^{1/2}\boldsymbol{\phi}_t(t_{j-1/2})\|_0^2\\
&+2k\sum_{j=1}^{m}(1+\epsilon_G'')\left\|\rho^{1/2}\left(\boldsymbol{\phi}_t(t_{j-1/2})-\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right)\right\|_0^2,\\
&\leqslant Ct_m h^{2r}\|\boldsymbol{u}_t\|^2_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}+Ck^4(1+\epsilon_G'')\|\boldsymbol{\phi}_{ttt}\|^2_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}.
\end{aligned}
$$

Now use $\|\boldsymbol{\phi}_{ttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}\leqslant C\|\boldsymbol{u}_{tttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}$. □

Now we can give the error estimate.

Theorem 4.13 (fully discrete 'energy' error estimate) *Assume that we have $h\leqslant\hat{h}$, $k\leqslant\hat{k}$, $\beta\geqslant(d-1)^{-1}$, $\bar{\boldsymbol{u}}\in\boldsymbol{H}^{r+1}(\Omega)$, $\bar{\boldsymbol{z}}\in\boldsymbol{H}^{r}(\Omega)$ and $\boldsymbol{u}\in H^4(\boldsymbol{L}_2)\cap H^2(\boldsymbol{H}^1)\cap W^1_\infty(\boldsymbol{H}^{r+1})\cap C^1(C(\bar{\Omega})^d)$. Then, for $\hat{h}$ and $\hat{k}$ small enough, and δ large enough,*

$$
\begin{aligned}
&\|\rho^{1/2}(\boldsymbol{u}_t(t_m)-\boldsymbol{z}^h_m)\|_0+\|\boldsymbol{u}(t_m)-\boldsymbol{u}^h_m\|_{\mathcal{A}}\\
&\qquad+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}({}^*\boldsymbol{\sigma}_i(t_m)-{}^*\boldsymbol{\sigma}^h_{im})\|_0\leqslant C(h^r+k^2),
\end{aligned}
$$

where C is a 'Gronwall' constant independent of h and k.

Proof. We start with (50) and invoke Lemmas 4.8, 4.9, 4.10, 4.11 and 4.12.

These results collect up as follows:.

$$
\begin{aligned}
&\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2+\|\boldsymbol{\chi}_m\|^2_{\mathcal{A}}+2k\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)\\
&+2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2\\
&\qquad\leqslant\|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2+\|\boldsymbol{\chi}_0\|^2_{\mathcal{A}}+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{i0}\|_0^2
\end{aligned}
$$

$(T_1 + T_2 + T_3)$

$$
\begin{aligned}
&+ 2Ch^{(d-1)\beta/2-1/2}\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 \\
&+ 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\hat{\epsilon}_i}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 \\
&+ \frac{2Ck}{\check{\epsilon}}\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2 \\
&+ \sum_{i=1}^{N_\varphi}\frac{\bar{\epsilon}}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2 + \frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{\chi}_m|_{\mathcal{E}}^2 + \frac{2C^2h^{(d-1)\beta-1}}{\epsilon''}J_0^{1,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m) \\
&+ Ck\sum_{j=0}^{m}\frac{1}{\epsilon'}\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2 + \frac{\epsilon''}{2}\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2 + Ck\sum_{j=0}^{m}\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2\hat{h}^{(d-1)\beta-1}}{\hat{\epsilon}_i}J_0^{1,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j) \\
&+ 2h^{(d-1)\beta-1}k(2\check{\epsilon}(1-\varphi_0)+\epsilon')\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)
\end{aligned}
$$

$(T_4 + T_5 + T_6)$

$$
\begin{aligned}
&+ \frac{k}{\epsilon_6}\sum_{j=0}^{m}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\frac{\epsilon_6'}{2}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 \\
&+ \epsilon_5\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2 + \|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + k\sum_{j=1}^{m-1}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2 \\
&+ Ck^3\int_0^{t_1}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2\,dt + \frac{Ck^3}{\epsilon_5}\int_{t_{m-1}}^{t_m}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2\,dt + Ck^4\int_0^{t_m}\|\breve{\boldsymbol{u}}_{tttt}(t)\|_0^2\,dt \\
&+ Ck^4\int_0^{t_m}(1+\epsilon_6)\|\boldsymbol{z}_{ttt}(t)\|_0^2 + \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon_6'}\sum_{i=1}^{N_\varphi}\|{}^{*}\ddot{\boldsymbol{\sigma}}_i(t)\|_0^2\,dt
\end{aligned}
$$

(G_j)

$$+ 2k\sum_{j=1}^{m}\frac{1+\epsilon''_G}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j-\boldsymbol{\phi}_{j-1}}{k}\right\|_0^2 + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}C\epsilon'_G\left\|\frac{\boldsymbol{\theta}_{ij}+\boldsymbol{\theta}_{i,j-1}}{2}\right\|_0^2$$
$$+ \frac{k}{\epsilon'_G}\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right) + 2k\sum_{j=0}^{m}\frac{1}{2\epsilon''_G}\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2$$
$$+ Ch^{2r}\big((1+\epsilon'''_G)\|\boldsymbol{u}\|^2_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))} + \|\boldsymbol{u}_t\|^2_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}\big)$$
$$+\frac{1}{2}\|\boldsymbol{\chi}_0\|^2_{\mathcal{A}} + \frac{1}{2\epsilon'''_G}\|\boldsymbol{\chi}_m\|^2_{\mathcal{A}} + \frac{k}{2}\sum_{j=1}^{m-1}\|\boldsymbol{\chi}_j\|^2_{\mathcal{A}} + 2k\sum_{j=1}^{m}\frac{1}{2}\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0^2$$

(H_j)

$$+ 2k\sum_{j=1}^{m}\left(\sum_{i=1}^{N_\varphi}\epsilon'_H\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2 + \frac{C}{\epsilon'_H}\left|\frac{\boldsymbol{\xi}_j+\boldsymbol{\xi}_{j-1}}{2}\right|_{\mathcal{E}}^2\right).$$

Rearranging this gives,

$$\left(1-\epsilon_5-\frac{k}{\epsilon''_G}-\frac{k}{\epsilon_6}\right)\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2$$
$$+\left(1-\frac{\epsilon''}{2}-\frac{1}{2\epsilon'''_G}-\frac{Ck}{\epsilon'}\right)\|\boldsymbol{\chi}_m\|^2_{\mathcal{A}}$$
$$+ 2k\left(1-\frac{1}{2\epsilon'_G}-\hat{h}^{(d-1)\beta-1}(2\check{\epsilon}(1-\varphi_0)+\epsilon')\right)$$
$$\times\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)$$
$$+ 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left(1-\frac{\hat{\epsilon}_i}{2}-\epsilon'_H-\frac{\epsilon'_6}{2}\right)\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2$$
$$+\left(1-\bar{\epsilon}-\frac{2Ck}{\check{\epsilon}}\right)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2$$
$$-\left(\frac{1-\varphi_0}{\bar{\epsilon}}|\boldsymbol{\chi}_m|^2_{\mathcal{E}} + \hat{h}^{(d-1)\beta-1}\left(\frac{2C^2}{\epsilon''}+Ck\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2}{\hat{\epsilon}_i}\right)J_0^{1,\beta}(\boldsymbol{\chi}_m,\boldsymbol{\chi}_m)\right)$$
$$\leqslant T_1+T_2+T_3+T_4+T_5,$$

where,

$$\begin{aligned}
T_1 \quad &:= \quad 2\|\rho^{1/2}\boldsymbol{\psi}_0\|_0^2 + (\tfrac{3}{2} + 2C\hat{h}^{\frac{(d-1)\beta-1}{2}})\|\boldsymbol{\chi}_0\|_{\mathcal{A}}^2 + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{i0}\|_0^2,\\
T_2 \quad &:= \quad Ck^4\int_0^{t_m}\|\breve{\boldsymbol{u}}_{tttt}(t)\|_0^2 + 2k\sum_{j=1}^{m}\frac{1}{2}\|\rho^{1/2}\Delta_j\breve{\boldsymbol{u}}\|_0^2,\\
&\qquad + Ck^3\int_0^{t_1}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{Ck^3}{\epsilon_5}\int_{t_{m-1}}^{t_m}\|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2\\
&\qquad + Ck^4\int_0^{t_m}\left((1+\epsilon_6)\|\boldsymbol{z}_{ttt}(t)\|_0^2 + \|\breve{\boldsymbol{u}}_{ttt}(t)\|_0^2 + \frac{1}{\epsilon_6'}\sum_{i=1}^{N_\varphi}\|{}^*\dddot{\boldsymbol{\sigma}}_i(t)\|_0^2\right)\\
T_3 \quad &:= \quad 2k\sum_{j=1}^{m}\frac{1+\epsilon_G''}{2}\left\|\rho^{1/2}\frac{\boldsymbol{\phi}_j - \boldsymbol{\phi}_{j-1}}{k}\right\|_0^2,\\
T_4 \quad &:= \quad 2k\sum_{j=1}^{m}\frac{C}{\epsilon_H'}\left|\frac{\boldsymbol{\xi}_j + \boldsymbol{\xi}_{j-1}}{2}\right|_{\mathcal{E}}^2 + 2k\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}C\epsilon_G'\left\|\frac{\boldsymbol{\theta}_{ij} + \boldsymbol{\theta}_{i,j-1}}{2}\right\|_0^2\\
&\qquad + Ch^{2r}\big((1+\epsilon_G''')\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2 + \|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2\big),\\
T_5 \quad &:= \quad \frac{2Ck}{\check{\epsilon}}\sum_{j=0}^{m-1}\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2 + Ck\sum_{j=0}^{m-1}\left(\frac{1}{2} + \frac{1}{\epsilon'}\right)\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2\\
&\qquad + Ck\sum_{j=0}^{m-1}\sum_{i=1}^{N_\varphi}\frac{\gamma_i^2\hat{h}^{(d-1)\beta-1}}{\delta\hat{\epsilon}_i}J_0^{\delta,\beta}(\boldsymbol{\chi}_j,\boldsymbol{\chi}_j)\\
&\qquad + k\sum_{j=0}^{m-1}\left(1 + \frac{1}{\epsilon_G''} + \frac{1}{\epsilon_6}\right)\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2.
\end{aligned}$$

We now choose,

$$\epsilon'' = \frac{\varphi_0/4}{1-\varphi_0/2}, \qquad \epsilon' = \varphi_0, \qquad \hat{\epsilon}_i = 1, \qquad \check{\epsilon} = \frac{1}{4}, \qquad \epsilon_G''' = \frac{2-\varphi_0}{\varphi_0},$$

$$\bar{\epsilon} = 1 - \frac{\varphi_0}{2}, \qquad \epsilon_5 = \frac{1}{2}, \qquad \epsilon_G'' = \epsilon_6 = \frac{2}{C} \quad \text{for some } C > 0,$$

$$\epsilon_G' = \frac{1}{1-\varphi_0}, \qquad \epsilon_H' = \frac{1}{6}, \qquad \epsilon_6' = \frac{1}{3},$$

and insist that,

$$\delta \geqslant \frac{4C^2(2-\varphi_0)^2\hat{h}^{(d-1)\beta-1}}{(2-2\varphi_0)\varphi_0}. \tag{52}$$

These lead to,

$$\begin{aligned}
&\left(\frac{1}{2}-C\hat{k}\right)\|\rho^{1/2}\boldsymbol{\psi}_m\|_0^2+\left(\frac{\varphi_0}{8-4\varphi_0}-C\hat{k}\right)\|\boldsymbol{\chi}_m\|_{\mathcal{A}}^2\\
&+k(1+\varphi_0)(1-\hat{h}^{(d-1)\beta-1})\sum_{j=1}^{m}J_0^{\delta,\beta}\left(\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2},\frac{\boldsymbol{\psi}_j+\boldsymbol{\psi}_{j-1}}{2}\right)\\
&+\frac{k}{3}\sum_{j=1}^{m}\sum_{i=1}^{N_\varphi}\left\|\boldsymbol{D}^{-1/2}\left(\frac{\boldsymbol{\eta}_{ij}-\boldsymbol{\eta}_{i,j-1}}{k}\right)\right\|_0^2\\
&+\left(\frac{\varphi_0}{2}-C\hat{k}\right)\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0^2\\
&\leqslant T_1+T_2+T_3+T_4+T_5.
\end{aligned}$$

Now, for T_2, using Lemmas 2.1, 4.7 and stability properties of the interpolant,

$$\begin{aligned}
|T_2| \leqslant\ & Ck^4\Big(\|\boldsymbol{u}_{ttt}\|_{L_\infty(0,t_m;\boldsymbol{L}_2(\Omega))}+\|\boldsymbol{u}\|_{H^2(0,t_m;\boldsymbol{H}^1(\Omega))}\\
&+\|\boldsymbol{u}_{ttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}+\|\boldsymbol{u}_{tttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}\Big)^2.
\end{aligned}$$

For T_3, using Lemma 4.12,

$$|T_3|\leqslant Ch^{2r}\|\boldsymbol{u}_t\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}^2+Ck^4\|\boldsymbol{u}_{tttt}\|_{L_2(0,t_m;\boldsymbol{L}_2(\Omega))}^2.$$

For T_4,

$$\begin{aligned}
|T_4| \leqslant\ & Ch^{2r}\Big(\|\boldsymbol{u}\|_{L_\infty(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}+\|\boldsymbol{u}_t\|_{L_2(0,t_m;\boldsymbol{H}^{r+1}(\Omega))}\\
&+\max_{1\leqslant i\leqslant N_\varphi}\|{}^*\boldsymbol{\sigma}_i\|_{L_\infty(0,t_m;\boldsymbol{H}^r(\Omega))}\Big)^2,
\end{aligned}$$

and use Lemma 2.2, and, for T_5,

$$|T_5|\leqslant Ck\sum_{j=0}^{m-1}\left(\|\rho^{1/2}\boldsymbol{\psi}_j\|_0^2+\|\boldsymbol{\chi}_j\|_{\mathcal{A}}^2+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{ij}\|_0^2\right).$$

Now select $\hat{h}$ and $\hat{k}$ small enough, use the initial conditions ${}^*\boldsymbol{\sigma}_i(0)=\boldsymbol{0}$ and apply the discrete Gronwall lemma to get,

$$\begin{aligned}
&\|\rho^{1/2}\boldsymbol{\psi}_m\|_0+\|\boldsymbol{\chi}_m\|_{\mathcal{A}}+\sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0\\
&\leqslant C(h^r+k^2)+C\|\rho^{1/2}\boldsymbol{\psi}_0\|_0+C\|\boldsymbol{\chi}_0\|_{\mathcal{A}}.
\end{aligned}$$

Now, by (40) and (13), we have,

$$\begin{aligned}\|\rho^{\frac{1}{2}}\boldsymbol{\psi}_0\|_0 &= \|\rho^{\frac{1}{2}}(\boldsymbol{z}_0^h - \breve{\boldsymbol{z}}(0))\|_0,\\ &\leqslant \|\rho^{\frac{1}{2}}(\boldsymbol{z}_0^h - \bar{\boldsymbol{z}})\|_0 + \|\rho^{\frac{1}{2}}(\breve{\boldsymbol{z}}(0) - \bar{\boldsymbol{z}})\|_0,\\ &\leqslant 2\|\rho^{\frac{1}{2}}(\breve{\boldsymbol{z}}(0) - \bar{\boldsymbol{z}})\|_0 \leqslant Ch^r\|\bar{\boldsymbol{z}}\|_r.\end{aligned}$$

Also, using standard results for the elliptic projection (see e.g. [12]) we have,

$$\begin{aligned}\|\boldsymbol{\chi}_0\|_{\mathcal{A}} &= \|\boldsymbol{u}_0^h - \breve{\boldsymbol{u}}(0)\|_{\mathcal{A}},\\ &\leqslant \|\bar{\boldsymbol{u}} - \boldsymbol{u}_0^h\|_{\mathcal{A}} + \|\bar{\boldsymbol{u}} - \breve{\boldsymbol{u}}(0)\|_{\mathcal{A}},\\ &\leqslant \|\bar{\boldsymbol{u}} - \boldsymbol{u}_0^h\|_{\mathcal{A}} + |\bar{\boldsymbol{u}} - \breve{\boldsymbol{u}}(0)|_{\mathcal{E}} \leqslant Ch^r\|\bar{\boldsymbol{u}}\|_{r+1}.\end{aligned}$$

Using the triangle inequality we now obtain,

$$\begin{aligned}&\|\rho^{1/2}(\boldsymbol{u}_t(t_m) - \boldsymbol{z}_m^h)\|_0 + \|\boldsymbol{u}(t_m) - \boldsymbol{u}_m^h\|_{\mathcal{A}}\\ &\quad + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}({}^*\boldsymbol{\sigma}_i(t_m) - {}^*\boldsymbol{\sigma}_{im}^h)\|_0\\ &\qquad\leqslant \|\rho^{1/2}\boldsymbol{\xi}_t(t_m)\|_0 + \|\boldsymbol{\xi}(t_m)\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\theta}_i(t_m)\|_0\\ &\qquad\quad + \|\rho^{1/2}\boldsymbol{\psi}_m\|_0 + \|\boldsymbol{\chi}_m\|_{\mathcal{A}} + \sum_{i=1}^{N_\varphi}\frac{1}{\tau_i}\|\boldsymbol{D}^{-1/2}\boldsymbol{\eta}_{im}\|_0,\\ &\qquad\leqslant Ch^r\|\boldsymbol{u}_t(t_m)\|_r + Ch^r\|\boldsymbol{u}(t_m)\|_{r+1}\\ &\qquad\quad + Ch^r\max_{1\leqslant i\leqslant N_\varphi}\|{}^*\boldsymbol{\sigma}_i(t_m)\|_r + C(h^r + k^2).\end{aligned}$$

To get this we noted that, $\|\boldsymbol{\xi}(t_j)\|_{\mathcal{A}} = |\boldsymbol{\xi}(t_j)|_{\mathcal{E}} \leqslant Ch^r\|\boldsymbol{u}(t_j)\|_{r+1}$ for $j = 0, 1, \ldots, m$. Finally, using Lemma 2.2 completes the proof. □

5 Conclusion

In this article we have extended the application of the DG FEM to dynamic linear viscoelasticity problems. This builds upon the algorithm and estimates in [12] in that we have now included the inertia term. It also varies the approach in [12] in that here we have chosen to represent the viscoelastic history through evolution equations for internal stress tensors, rather than augment the momentum equation with a Volterra (hereditary) integral.

This article with [12] represent the extension of DG FEM to elliptic and (second-order) hyperbolic problems with viscoelastic memory. The analogous parabolic problem is currently under study in [11]. Since code development for these type of problems is non-trivial, we do not present numerical results here. Instead, numerics for all three problems will be presented elsewhere at a later date when all the numerical issues have been identified.

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