$L_p(0,T)$ error control using the derivative of the residual for a finite element approximation of a second-kind Volterra equation*

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Summary. We consider a piecewise constant finite element approximation to the convolution Volterra equation problem of the second kind: find $u \in L_p(0,T)$ such that $u = f + \phi * u$ in a time interval $[0,T]$. An *a posteriori* $L_p(0,T)$ error estimate involving the derivative of the residual weighted with the time steps is developed, and this can be used to construct an adaptive time stepping scheme. We assume that $\phi \in W_1^1(0,T)$ but need only assume that f is piecewise $W_p^1(0,T)$. The convolution kernel can be replaced with a more general Volterra kernel at the expense of additional technicalities.

1 Introduction

This is a sequel to the papers [2, 3] and is again concerned with adaptive error control for finite element discretizations of Volterra integral equations of the second kind. We take as a prototype the problem: find $u \in L_p(\mathcal{J})$ such that,

$$u(t) = f(t) + \int_0^t \phi(t-s)u(s)\,ds \qquad \text{a.e. in } \mathcal{J} := [0,T]. \tag{1}$$

Here $T > 0$ and f and ϕ are assumed smooth in a sense made precise below in Assumptions 1.1. Below we also use the notation,

$$(\phi * w)(t) \equiv \Lambda w(t) := \int_0^t \phi(t-s)w(s)\,ds.$$

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Equation (1) is a prototype for quasistatic viscoelasticity problems (see e.g. [4]) where smooth convolution kernels with a special form of exponentially fading memory are realistic. The material presented below can be extended to more general kernels, $\phi(t, s)$, at the price of extra technical difficulties. We flag these as they occur.

Our goal is to derive an *a posteriori* error estimate for a finite element discretization of (1) which can be used to drive a reliable adaptive time stepping algorithm delivering guaranteed control of the error in $L_p(\mathcal{J})$. This is an application of the so-called *Johnson paradigm* (e.g. [1]) wherein the strong (derivative) stability of an associated dual problem plays a crucial role. Although we are primarily concerned with the space-time viscoelasticity problem, the special difficulties that arise due to the lack of a strong stability estimate for (1) of the form $\|u'\|_{L_p(\mathcal{J})} \leqslant C\|f\|_{L_p(\mathcal{J})}$ suggests that a study of this pure-time scalar problem is relevant.

We will introduce and define terms more precisely below but for now, and to set the context, we recall the results in [2, 3] and describe the main result in this paper.

Let U be a piecewise constant finite element approximation to u, with respect to some partition of the time interval $\mathcal{J}$. In [2] we gave the *a priori* error estimate,

$$\|u - U\|_{L_p(\mathcal{J})} \leqslant C\|\kappa u'\|_{L_p(\mathcal{J})}, \tag{2}$$

where κ is the piecewise constant time step function. Introducing the residual,

$$r(t) := f(t) - U(t) + \Lambda U(t), \tag{3}$$

inside of each time interval we also have from [2] the *a posteriori* error estimate,

$$\|u - U\|_{L_p(\mathcal{J})} \leqslant C\|r\|_{L_p(\mathcal{J})}, \tag{4}$$

where C is a constant which, in principle, is computable. It is clear that although the right hand side of (4) is computable, it does not contain any explicit reference to the time steps. Hence, it cannot be used in a robust way to adaptively select these time steps such that,

$$\|u - U\|_{L_p(\mathcal{J})} \leqslant \mathrm{TOL},$$

where $\mathrm{TOL} > 0$ is a user-specified tolerance level.

To incorporate the time steps into the *a posteriori* error estimate we took a different approach in [3] (similar to that used by Süli and Houston in, for example, [6], but for a very different problem) and derived the negative norm estimate,

$$\|u - U\|_{W_p^{-1}(\mathcal{J})} \leqslant C\|\kappa r\|_{L_p(\mathcal{J})}. \tag{5}$$

Now that the time steps appear explicitly in the right hand side we can use the bound to adaptively select them and guarantee,

$$\|u - U\|_{W_p^{-1}(\mathcal{J})} \leqslant \mathrm{TOL}.$$

The shortcoming here is obvious. The negative norm is hard to interpret when u is the quantity of physical interest.

The negative norm appears in (5) in order that we can use an optimal interpolation-error estimate (yielding the time steps in the bound) together with a *pseudo strong* stability estimate of the form $\|u'\|_{L_p(\mathcal{J})} \leqslant C\|f'\|_{L_p(\mathcal{J})}$, for a related dual (backward) problem. The negative norm then arises by taking an appropriate supremum over all functions in $\mathring{W}^1_q(\mathcal{J})$.

In this article we modify the approach. We revert to using only weak stability (see (7) and (18) below) but, in order to introduce the time steps into the bound, we *differentiate* the residual. Our result (see Theorem 3.3 below) is now of the form,

$$\|u - U\|_{L_p(\mathcal{J})} \leqslant C\|\kappa r'\|_{L_p(\mathcal{J})}. \tag{6}$$

Error control in the L_p norm is now possible using adaptive time stepping, but we pay a price in that the data have to be smoother than is natural for the problem (1). In fact our result is a little more general than that given above in that we have only to assume that f is piecewise smooth.

The plan of the paper is as follows. In Section 2 we outline the discretization and the resulting numerical scheme, and give a discrete stability estimate. This is followed with the error analysis in Section 3, and we finish with some general remarks on the approach in Section 4. We finish this section by detailing our assumptions.

Assumption 1.1 *We assume that for some $p \in [1, \infty]$ the following hold.*

1. *Equation* (1) *has a unique solution $u \in L_p(\mathcal{J})$.*
2. $\phi \in W^1_1(\mathcal{J})$.
3. *There exists a finite set of intervals $\{K_i\}_{i=1}^I$ satisfying $K_i \cap K_j = \emptyset$, for $i \neq j$, and $\overline{\cup_i K_i} = [0, T]$ such that $f \in W^1_p(K_i)$ for each K_i.*
4. *There exists a* stability factor $S\colon \mathcal{J} \to [0, \infty)$ *such that,*

$$\|u\|_{L_p(0,t)} \leqslant S(t)\|f\|_{L_p(0,t)} \qquad \forall t \in \mathcal{J}. \tag{7}$$

See [5] *for a proof of this in the viscoelasticity context.*

Note that with our assumption on ϕ we have that $\Lambda\colon L_p(\mathcal{J}) \to L_p(\mathcal{J})$ is continous,

$$\|\Lambda w\|_{L_p(0,t)} \leqslant \|\phi\|_{L_1(0,t)}\|w\|_{L_p(0,t)} \qquad \forall w \in L_p(0, t).$$

This follows from Hölder's inequality applied to a convolution. It also follows from these asumptions that we can take $\phi \in C[0, T]$.

2 The finite element approximation

To obtain a piecewise constant finite element approximation of (1) we discretize $\mathcal{J}$ into time intervals $\{\mathcal{J}_i := (t_{i-1}, t_i)\}_{i=1}^N$ such that,

$$t_0 = 0, \quad t_{i-1} < t_i \text{ for } i \in \mathbb{N}(1, N), \quad \text{and} \quad t_N = T.$$

Here $\mathbb{N}(m,n) := \{m, m+1, \ldots, n-1, n\}$ for integers $n \geqslant m \geqslant 1$. We also define the time steps $k_i := t_i - t_{i-1}$ and the piecewise constant time step function $\kappa \in L_\infty(\mathcal{J})$ given by $\kappa|_{\mathcal{J}_i} := k_i$ for each interval $\mathcal{J}_i$.

Introducing the $L_2(\mathcal{J})$ inner product,

$$(w, v) := \int_0^T wv\, dt,$$

we write (1) in the "variational form": find $u \in L_p(\mathcal{J})$ such that,

$$(u - \Lambda u, v) = (f, v) \qquad \forall v \in L_q(\mathcal{J}), \tag{8}$$

where q is the conjugate Hölder index to p. Introducing the finite element space of piecewise constant functions,

$$V^h := \left\{ v \in L_\infty(\mathcal{J}) \colon v|_{\mathcal{J}_i} = \text{ constant } \forall i \in \mathbb{N}(1, N) \right\},$$

our finite element approximation to (8) is then: find $U \in V^h$ such that,

$$(U - \Lambda U, v) = (f, v) \qquad \forall v \in V^h. \tag{9}$$

The Galerkin "orthogonality" property then follows by subtracting (9) from (8),

$$(e - \Lambda e, v) = 0 \quad \forall v \in V^h \quad \Longrightarrow \quad (r, v) = 0 \quad \forall v \in V^h, \tag{10}$$

where $e := u - U$ is the error and r is the residual, defined earlier in (3).

To obtain a numerical scheme from (9) take $v = 1$ in $\mathcal{J}_i$ and $v = 0$ elsewhere. Then $v \in V^h$, and defining $U_i := U|_{\mathcal{J}_i}$, for each i, we obtain the time stepping scheme,

$$U_i = \frac{\displaystyle\int_{t_{i-1}}^{t_i} f(t)\, dt + \sum_{j=1}^{i-1} U_j \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \phi(t-s)\, ds dt}{\displaystyle\int_{t_{i-1}}^{t_i} \left(1 - \int_{t_{i-1}}^{t} \phi(t-s)\, ds \right) dt}, \tag{11}$$

for $i = 1, 2, \ldots, N$.

Remark In general we need to impose a ϕ-dependent upper bound, k_ϕ, on the time steps in order that the denominator in (11) is bounded above zero. For example, in viscoelasticity we would typically have $\phi(t) = -\varphi'(t)$, where φ is a smooth, positive, monotone decreasing *stress relaxation function* satisfying $\varphi(0) = 1$. In this case,

$$\int_{t_{i-1}}^{t_i} 1 - \int_{t_{i-1}}^{t} \phi(t-s)\, ds\, dt = \int_{t_{i-1}}^{t_i} \varphi(t - t_{i-1})\, dt \geqslant c_\phi k_i,$$

where $c_\phi := \varphi(k_\phi)$. This is the basis of the assumption made in the discrete stability estimate given below. This estimate also makes use of the following L_p generalization of Gronwall's lemma, due to Willett [7, Lemma 2.2].

Lemma 2.1 (generalized Gronwall) *For $p \in [1,\infty)$ let $\alpha, \beta, \psi \in L_p(\mathcal{J})$ be non-negative a.e. on $\mathcal{J}$ and satisfy,*

$$\alpha(t) \leqslant \beta(t) + \psi(t)\|\alpha\|_{L_p(0,t)},$$

a.e. in $\mathcal{J}$. Then,

$$\|\alpha\|_{L_p(0,t)} \leqslant \frac{\|\beta\epsilon^{\frac{1}{p}}\|_{L_p(0,t)}}{1-(1-\epsilon(t))^{\frac{1}{p}}},$$

where $\epsilon(t) := \exp(-\|\psi\|^p_{L_p(0,t)})$.

Lemma 2.2 (discrete stability) *Assume for* (11) *that, for time steps bounded above by k_ϕ, the denominator is bounded below by $c_\phi k_i$, where c_ϕ is a positive constant independent of the time steps. Then for $f \in L_p(\mathcal{J})$ there is a positive constant C_S such that,*

$$\|U\|_{L_p(0,t_i)} \leqslant C_S\|f\|_{L_p(0,t_i)} \qquad \forall i \in \mathbb{N}(1,N). \tag{12}$$

The constant C_S depends upon c_ϕ^{-1}, ϕ, p, and an $\epsilon(t)$ similar to that in Lemma 2.1.

We break the proof into two cases: $p = \infty$ and $p \in [1,\infty)$.

Proof for the case $p = \infty$. Note that $\|U\|_{L_\infty(\mathcal{J}_i)} = |U_i|$ and also that from (11) we have,

$$\|U\|_{L_\infty(\mathcal{J}_i)} \leqslant c_\phi^{-1}\|f\|_{L_\infty(0,t_i)} + c_\phi^{-1}\|\phi\|_{L_\infty(\mathcal{J})}\sum_{j=1}^{i-1} k_j\|U\|_{L_\infty(\mathcal{J}_j)} \qquad \forall i \in \mathbb{N}(1,N).$$

The estimate then follows from the standard Gronwall lemma. $\square$

Proof for the case $p \in [1,\infty)$. In this case $\|U\|_{L_p(\mathcal{J}_i)} = k_i^{1/p}|U_i|$ and, hence, using Hölder's inequality for sums we have for the term on the right in (11),

$$\left|\sum_{j=1}^{i-1} U_j \int_{t_{i-1}}^{t_i}\int_{t_{j-1}}^{t_j} \phi(t-s)\,dsdt\right| \leqslant k_i\|\phi\|_{L_\infty(\mathcal{J})} t_{i-1}^{1/q}\|U\|_{L_p(0,t_{i-1})},$$

where $q = p/(p-1)$ is the conjugate Hölder index to p. Using this in (11) we then obtain,

$$|U_i| \leqslant c_\phi^{-1}k_i^{-1}\|f\|_{L_1(\mathcal{J}_i)} + c_\phi^{-1}\|\phi\|_{L_\infty(\mathcal{J})} t_{i-1}^{1/q}\|U\|_{L_p(0,t_{i-1})}.$$

To use Lemma 2.1 we define, piecewise, the following non-negative L_p functions:

$$\begin{aligned}
\alpha(t)|_{\mathcal{J}_i} &:= |U_i|,\\
\beta(t)|_{\mathcal{J}_i} &:= c_\phi^{-1}k_i^{-1}\|f\|_{L_1(\mathcal{J}_i)},\\
\psi(t)|_{\mathcal{J}_i} &:= c_\phi^{-1}\|\phi\|_{L_\infty(\mathcal{J})} t_{i-1}^{1/q},
\end{aligned}$$

and then, noting that $\|U\|_{L_p(0,t)} = \|\alpha\|_{L_p(0,t)}$ and $\|U\|_{L_p(0,t_{i-1})} \leqslant \|U\|_{L_p(0,t)}$ for $t \in \mathcal{J}_i$, we have,

$$\|U\|_{L_p(0,t)} \leqslant \frac{\|\beta\epsilon^{1/p}\|_{L_p(0,t)}}{1-(1-\epsilon(t))^{1/p}},$$

where, for each $i \in \mathbb{N}(1,N)$,

$$\epsilon(t)|_{\mathcal{J}_i} := \exp\left(-c_\phi^{-p}\|\phi\|^p_{L_\infty(\mathcal{J})} t_{i-1}^{p/q} t\right).$$

Clearly,

$$0 < \exp\left(-c_\phi^{-p}\|\phi\|^p_{L_\infty(\mathcal{J})} t_i^p\right) \leqslant \epsilon(t)|_{\mathcal{J}_i} \leqslant 1, \qquad \forall i \in \mathbb{N}(1,N),$$

giving $1-(1-\epsilon(t))^{1/p} > 0$ and $\|\beta\epsilon^{1/p}\|_{L_p(0,t)} \leqslant \|\beta\|_{L_p(0,t)}$, and from the definition of β it follows that,

$$\|\beta\|^p_{L_p(0,t_i)} = \sum_{j=1}^{i} c_\phi^{-p} k_j^{1-p} \|f\|^p_{L_1(\mathcal{J}_j)} \leqslant c_\phi^{-p}\|f\|^p_{L_p(0,t_i)}.$$

This completes the proof. □

We now move on to the error analysis.

3 *A posteriori* error analysis

We introduce the projection $\pi\colon L_1(\mathcal{J}) \to V^h$ defined piecewise through local averaging as follows,

$$\pi w|_{\mathcal{J}_i} := \frac{1}{k_i}\int_{t_{i-1}}^{t_i} w(t)\,dt \qquad \forall w \in L_1(\mathcal{J}) \text{ and } \forall i \in \mathbb{N}(1,N). \tag{13}$$

The following results are straightforward but important.

Lemma 3.1 *For all $w \in L_1(\mathcal{J})$, for all $i \in \mathbb{N}(1,N)$ and for all $q \in [1,\infty]$:*

$$\int_{t_{i-1}}^{t_i} w - \pi w\,dt = 0, \tag{14}$$

$$\|\pi w\|_{L_q(\mathcal{J}_i)} \leqslant \|w\|_{L_q(\mathcal{J}_i)}, \tag{15}$$

$$\|w - \pi w\|_{L_q(\mathcal{J}_i)} \leqslant C_q\|w\|_{L_q(\mathcal{J}_i)}. \tag{16}$$

The $C_q \leqslant 2$ are positive constants and the value $C_1 = 2$ is best possible.

Proof. (14) follows from the definition of π and (15) follows from Hölder's inequality. (16) is then immediate from the triangle inequality which gives $C_q \leqslant 2$. To show that $C_1 = 2$ is best possible choose $w = 1$ on (t_{i-1}, a) and $w = 0$ on (a, t_i) for some $a \in \mathcal{J}_i$. Then $\|w - \pi w\|_{L_1(\mathcal{J}_i)} \to 2\|w\|_{L_1(\mathcal{J}_i)}$ as $a \to t_{i-1}$. □

We now introduce the dual backward problem: find $\chi \in L_q(\mathcal{J})$ such that,

$$(v, \chi - \Lambda^*\chi) = (v, g) \qquad \forall v \in L_p(\mathcal{J}), \tag{17}$$

where $g \in L_q(\mathcal{J})$ is arbitrary and Λ^* is dual (adjoint) to Λ in the sense that $(v, \Lambda^*\chi) = (\Lambda v, \chi)$. Analogously to (7) we have the following weak stability estimate. For all $\tau \in \mathcal{J}$,

$$\|\chi\|_{L_q(0,\tau)} \leqslant S(\tau)\|g\|_{L_q(0,\tau)} \qquad \forall g \in L_q(\mathcal{J}) \text{ such that } g|_{(\tau,T)} = 0. \tag{18}$$

The stability factor S is actually the same as that appearing in (7), as can be seen by introducing new variables $\xi := T - t$ and $\eta := T - s$ into the backward problem (17). This converts it to a forward problem of exactly the same form as (1) (because of the convolution kernel). Consequently (7) applies.

Our *a posteriori* error estimate will involve the derivative of the residual, so we first need to establish that it "makes sense". This is another occasion where the convolution kernel simplifies the argument.

Lemma 3.2 *Let Assumptions* 1.1 *hold, and choose the time steps so that* $f \in W_p^1(\mathcal{J}_i)$ *for each* $\mathcal{J}_i$*. Then,*

$$\|r'\|_{L_p(\mathcal{J}_i)} \leqslant C_{r,i} := \|f'\|_{L_p(\mathcal{J}_i)} + C_S\Big(|\phi(0)| + \|\phi'\|_{L_1(0,t_i)}\Big)\|f\|_{L_p(0,t_i)}, \tag{19}$$

for each $i \in \mathbb{N}(1, N)$*, and where* C_S *is from Lemma* 2.2.

Proof. For $t \in \mathcal{J}_i$ differentiate (3) to get,

$$r'(t) = f'(t) + \phi(0)U(t) + \int_0^t \phi_t(t-s)U(s)\,ds. \tag{20}$$

Taking $L_p(\mathcal{J}_i)$ norms and using Hölder's inequality for convolutions gives,

$$\|r'\|_{L_p(\mathcal{J}_i)} \leqslant \|f'\|_{L_p(\mathcal{J}_i)} + |\phi(0)|\,\|U\|_{L_p(\mathcal{J}_i)} + \|\phi'\|_{L_1(0,t_i)}\|U\|_{L_p(0,t_i)},$$

and the proof is completed by using Lemma 2.2. $\square$

Notice that due to our assumed piecewise smoothness of f the time steps can always be chosen in the way required by Lemma 3.2. In relation to this it is convenient to introduce "broken norms": for each $i \in \mathbb{N}(1, N)$ set,

$$\begin{aligned} &|\!|\!| w |\!|\!|_{L_p(0,t_i)} := \left(\sum_{j=1}^{i} \|w\|_{L_p(\mathcal{J}_j)}^p \right)^{\frac{1}{p}} \\ &\qquad \forall w \quad \text{such that} \quad w|_{\mathcal{J}_j} \in L_p(\mathcal{J}_j)\ \forall j \in \mathbb{N}(1, N), \end{aligned} \tag{21}$$

and with the obvious modification if $p = \infty$. We can now give the *a posteriori* error estimate.

Theorem 3.3 (a posteriori error estimate) *Let Assumptions* 1.1 *hold and assume that the time steps are chosen as in Lemma* 3.2. *Then,*

$$\|u - U\|_{L_p(0,t_j)} \leqslant C_q S(t_j) \|\kappa r'\|_{L_p(0,t_j)} \qquad \forall j \in \mathbb{N}(1, N). \tag{22}$$

Here C_q is the constant in (16), *S is the stability factor in* (18), *κ is the piecewise constant time step function defined earlier in Section* 2 *and r' is the time derivative of the residual, given by* (20).

Proof. In (17) take $v = e := u - U \in L_p(\mathcal{J})$ and set $g = 0$ in $[t_j, T]$. Then $\chi = 0$ in $[t_j, T]$ also and, since $\pi\chi \in V^h$, we use (10) to get,

$$(e, g) = (e, \chi - \Lambda^*\chi) = (e - \Lambda e, \chi) = \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} (\chi - \pi\chi) r \, dt.$$

Integrating by parts on a typical subinterval gives,

$$\int_{t_{i-1}}^{t_i} (\chi - \pi\chi) r \, dt = r(t) \int_{t_{i-1}}^{t} \chi - \pi\chi \, ds \Bigg|_{t_{i-1}}^{t_i} - \int_{t_{i-1}}^{t_i} \left(\int_{t_{i-1}}^{t} \chi - \pi\chi \, ds \right) r'(t) \, dt,$$

and the boundary term vanishes identically at the lower limit and also at the upper limit due to (14). Hence,

$$\left| \int_{t_{i-1}}^{t_i} (\chi - \pi\chi) r \, dt \right| \leqslant \int_{t_{i-1}}^{t_i} |r'| \, dt \int_{t_{i-1}}^{t_i} |\chi - \pi\chi| \, dt \leqslant k_i \|r'\|_{L_p(\mathcal{J}_i)} \, \|\chi - \pi\chi\|_{L_q(\mathcal{J}_i)}.$$

Using (16) we therefore have,

$$|(e, g)| \leqslant C_q \sum_{i=1}^{j} \|k_i r'\|_{L_p(\mathcal{J}_i)} \, \|\chi\|_{L_q(\mathcal{J}_i)} \leqslant C_q \|\kappa r'\|_{L_p(0,t_j)} \, \|\chi\|_{L_q(0,t_j)}.$$

Using the dual stability estimate (18) in this we arrive at,

$$|(e, g)| \leqslant C_q S(t_j) \|\kappa r'\|_{L_p(0,t_j)} \, \|g\|_{L_q(0,t_j)},$$

and hence obtain (22) for $p > 1$,

$$\|e\|_{L_p(0,t_j)} = \sup \left\{ |(e, g)| : \|g\|_{L_q(0,t_j)} = 1 \right\} \leqslant C_q S(t_j) \|\kappa r'\|_{L_p(0,t_j)},$$

since $g|_{(t_j,T)} = 0$. For $p = 1$ we set (on $(0, t_j)$ only) $g := e \, |e|^{-1}$ if $e \neq 0$ and $g := 1$ otherwise. Then $\|g\|_{L_\infty(0,t_j)} = 1$ and,

$$\|e\|_{L_1(0,t_j)} = (e, g) \leqslant C_\infty S(t_j) \|\kappa r'\|_{L_1(0,t_j)}.$$

This completes the proof. $\square$

4 Conclusions

This short report has sought to demonstrate how the derivative of the residual can be used in an *a posteriori* error bound for a model second-kind Volterra problem. This bound explicitly contains the time step and we can conclude that it is sharp. For this we simply combine the estimates (19) with (22), and automatically obtain an optimal *a priori* error bound.

The *a posteriori* error bound can be used to construct an adaptive time stepping scheme as follows. To guarantee $\|u - U\|_{L_p(\mathcal{J})} \leqslant \mathrm{TOL}$ we require (for $p \in [1, \infty)$) that,

$$k_j^p \|r'\|_{L_p(\mathcal{J}_j)}^p = \frac{k_j}{T}\left(\frac{\mathrm{TOL}}{C_q S(T)}\right)^p \qquad \forall j \in \mathbb{N}(1, N).$$

This leads to the time step selector,

$$k_j^{\mathrm{new}} = \min\left\{k_j^{\mathrm{old}}, \frac{\mathrm{TOL}}{C_q S(T)\|r'\|_{L_p(\mathcal{J}_j)}}\left(\frac{k_j^{\mathrm{old}}}{T}\right)^{1/p}\right\}$$

on each time level.

We close by identifying some outstanding issues:

- Piecewise linear (discontinuous or continuous) approximations can also be brought into this framework. The main extra effort would appear to be the deriving of a bound on the derivative of the discrete solution for the analogue to Lemma 3.2.

- Quadrature error could also be considered but, as this is a non-Galerkin error, several additional steps in the error analysis would be required.

- Smooth nonlinearities can be included by introducing a "linearised" dual problem. See [1] for the general approach.

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