OPTIMAL LONG-TIME $L_p(0,T)$ STABILITY AND SEMIDISCRETE ERROR ESTIMATES FOR THE VOLTERRA FORMULATION OF THE LINEAR QUASISTATIC VISCOELASTICITY PROBLEM*

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Abstract. The purpose of this article is to show how the solution of the linear quasistatic (compressible) viscoelasticity problem, written in Volterra form with fading memory, may be sharply bounded in terms of the data if certain physically reasonable assumptions are satisfied. The bounds are derived by making precise assumptions on the memory term which then make it possible to avoid the Gronwall inequality, and use instead a comparison theorem which is more sensitive to the physics of the problem. Once the data-stability estimates are established we apply the technique also to deriving *a priori* error bounds for semidiscrete finite element approximations. Our bounds are derived for viscoelastic solids and fluids under the small strain assumption in terms of the eigenvalues of a certain matrix derived from the stress relaxation tensor. For isotropic materials we can be explicit about the form of these bounds, while for the general case we give a formula for their computation.

Key words. Viscoelasticity, fading memory, Volterra equation, renewal equation, comparison theorem, Gronwall, stability, finite element method, error estimates.

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1. Introduction. This paper is concerned with obtaining *a priori* data-stability and semidiscrete finite element error estimates for the solutions u of a certain class of second-kind Volterra integral equation with fading memory. Equations of this type arise when problems of viscoelastic deformation are modelled using history integral constitutive relations, and so our analysis is developed under assumptions which are physically reasonable for viscoelastic solids or fluids. We establish our results by using an abstract problem, so as not to obscure the main conclusions with detail, and then demonstrate later in § 5 how these abstract results can be applied to the physical problem.

So that there can be no misunderstanding, we mention here at the outset that our stability estimates will come as no surprise to workers in viscoelasticity, since their form can be demonstrated fairly easily by finding the exact solutions to illustrative model problems. Also, the quasistatic viscoelasticity problem we describe later in § 5 has received a great deal of attention in the literature, and we refer for example to [5, Chap. 4] for a summary of issues related to existence, uniqueness and well posedness. However, as far as we are aware data-stability estimates for the general problem have never been given in explicit form before, and indeed this is our purpose in this note.

The theoretical study of problems of viscoelasticity has moved on somewhat from quasistatic problems (where the inertia term in Newton's second law is neglected) to the full dynamic linear (and sometimes nonlinear) problem, and in this regard we mention Dafermos in [2], Dafermos and Nohel in [3], the collection in [23] and the monograph [14]. However, engineers still regard the quasistatic problem as an important model (see for example the beam problem described in [11]), and to solve practical design problems with complex geometries they of course need reliable adaptive software for numerical solution. It is our aim to provide such reliable numerical

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solution algorithms, built on rigorous *a posteriori* error estimates, and our first results are in [18, 19, 22, 21, 20].

Briefly, if U is the numerical approximation to u then these estimates have the form,

$$\|u - U\|_{L_p(0,T;H^1(\Omega))} \leqslant S(T)R(U),$$

where $R(U)$ is a residual computable in terms of the data and the approximation U, and $S(T)$ is a data-stability factor for an associated dual problem. If this estimate is put to practical use to drive an adaptive solution algorithm then using Gronwall's inequality to show that $S(T) = O(e^{cT})$ will just not do, instead one needs sharp, realistic estimates for $S(T)$ that contain quantitative information. It is the purpose of this paper to present a means of obtaining such estimates, and it is in this context that we present our results.

To illustrate the use of the estimates for numerical approximation we also derive *a priori* error estimates, with similarly "small" constants, for a semidiscrete finite element approximation to the problem. However, we leave the matter of *a posteriori* error estimation for another time.

Let $(\mathcal{V}, (\cdot,\cdot)_{\mathcal{V}})$ be a Hilbert space with (topological) dual $(\mathcal{V}', (\cdot,\cdot)_{\mathcal{V}'})$ and let $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ be a pivot Hilbert space such that

$$\langle f, v\rangle \equiv (f,v)_{\mathcal{H}} \qquad \forall f \in \mathcal{H}, v \in \mathcal{V},$$

where $\langle\cdot,\cdot\rangle : \mathcal{V}' \times \mathcal{V} \to \mathbb{R}$ denotes the duality pairing between $\mathcal{V}'$ and $\mathcal{V}$. We denote the natural norms induced by the inner products in the obvious way by $\|\cdot\|_{\mathcal{V}}$, $\|\cdot\|_{\mathcal{V}'}$ and $\|\cdot\|_{\mathcal{H}}$.

For a real number $T \in [0,\infty)$ we let $\mathcal{J} := [0,T]$ denote a time interval and consider the abstract problem of finding a map $u : \mathcal{J} \to \mathcal{V}$ such that,

$$Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds \qquad \text{a.e. in } \mathcal{J}. \tag{1}$$

Here $A\colon \mathcal{V} \to \mathcal{V}'$ is a self-adjoint and time-independent linear elliptic partial differential operator (acting in an open bounded domain $\Omega \subset \mathbb{R}^n$), $f \in \mathcal{V}'$ a.e. in $\mathcal{J}$ is a given load, and we assume that appropriate boundary data are specified. In addition $B(t,s)$ is a.e. in the triangle,

$$\mathcal{T} := \{s \in \mathcal{J} : 0 \leqslant s \leqslant t \in \mathcal{J}\},$$

a partial differential operator that is "similar" to A in a way that we make precise below (see Assumptions 1, part (ii)). For the moment it is sufficient to think of $B(t,s)$ in the form,

$$B(t,s) := \phi(t-s)A, \tag{2}$$

where $\phi\colon \mathcal{J} \to \mathbb{R}$ is a given function.

Equation (1) represents an abstract formulation of the linear quasistatic compressible viscoelasticity problem, as described for example in either of [16] or [17], and can be thought of as a Volterra integral equation of the second kind taking values in the Hilbert space $\mathcal{V}'$. In this paper we will demonstrate that by making physically reasonable assumptions of fading memory on the function ϕ, it is possible to use some relatively simple results from convolution Volterra theory to derive optimal $L_p(\mathcal{J};\cdot)$

data-stability estimates for u. Here, for a Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ we denote by $L_p(\mathcal{J};\mathcal{B})$ the Banach space of maps $w\colon \mathcal{J} \to \mathcal{B}$ with norm,

$$\|w\|_{L_p(\mathcal{J};\mathcal{B})} := \left(\int_0^T \|w\|_{\mathcal{B}}^p \, ds \right)^{\frac{1}{p}}.$$

The obvious modification is intended in the case $p = \infty$.

The estimates we derive are optimal in the sense that we do not use the usual Gronwall inequality and thus avoid the presence of an exponentially large factor in the bounding constants. Although similar in spirit, the stability results presented below for viscoelastic **solids** are more general than those already given in [16], and, as we mentioned above, have application in the adaptive numerical solution of viscoelasticity problems represented by (1). Our results in this paper are even more general than in [16] in that we also derive stability estimates for viscoelastic **fluids** as considered by Golden and Graham in [8].

We will outline the specific nature of the linear quasistatic viscoelasticity problem later in § 5, but for now we prefer to retain the abstract form (1) since we shall use several results from Volterra theory. Our first steps are to associate with A a symmetric bilinear form $a\colon \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ and with $B(t,s)$ a bilinear form $b\colon \mathcal{T}\times\mathcal{V}\times\mathcal{V} \to \mathbb{R}$. Also, we introduce a time dependent linear form $L\colon \mathcal{J} \times \mathcal{V} \to \mathbb{R}$ to represent the load f and any nonhomogeneous Neumann boundary data. Thus we cast (1) into the weak form: for $p \in [1,\infty]$ find $u \in L_p(\mathcal{J};\mathcal{V})$ such that a.e. in $\mathcal{J}$,

$$a(u(t), v) = L(t; v) + \int_0^t b(t, s; u(s), v)\, ds \qquad \forall v \in \mathcal{V}. \tag{3}$$

Our first aim is to derive stability estimates of the form,

$$\|u\|_{L_p(\mathcal{J};\mathcal{V})} \leqslant C$$

which can be usefully applied to the viscoelasticity problem even for large T, although it is only possible to allow $T \to \infty$ in the case of a viscoelastic solid. We then apply the proof technique to deriving the *a priori* error estimates.

The bounds given below are best possible in the sense that the form of the time dependence of the constants is captured exactly under the given physically reasonable assumptions on the data. Our results are based on the following general assumptions.

ASSUMPTIONS 1. *For some $p \in [1,\infty]$ we assume that the weak problem* (3) *has a unique solution $u \in L_p(\mathcal{J};\mathcal{V})$ and has also the following properties.*

(i) *The bilinear form $a(\cdot,\cdot)$ is continuous and coercive on $\mathcal{V}$ in the respective senses,*

$$|a(w,v)| \leqslant C\|w\|_{\mathcal{V}}\|v\|_{\mathcal{V}} \qquad \text{and} \qquad a(v,v) \geqslant c\|v\|_{\mathcal{V}}^2,$$

for all $w, v \in \mathcal{V}$ and where C and c are positive constants. Using these properties we define the energy norm,

$$\|v\|_H := \sqrt{a(v,v)}$$

for all $v \in \mathcal{V}$, which is clearly equivalent to the norm $\|\cdot\|_{\mathcal{V}}$. In this context the space $(H, (\cdot,\cdot)_H) := (\mathcal{V}, a(\cdot,\cdot))$ is in some sense the natural *Hilbert space for this problem.*

(ii) *The operator $B(t,s)$ is similar to A in the sense that,*

$$|b(t,s;w,v)| \leqslant \phi(t-s)\|w\|_H\|v\|_H,$$

a.e. in $\mathcal{T}$, for all $w,v \in \mathcal{V}$ and where $\phi \in L_1\big(\mathcal{J};[0,\infty)\big)$ is a known function.

(iii) *The linear form $L(t;\cdot)$ is continuous on $\mathcal{V}$ in the sense that,*

$$|L(t;v)| \leqslant F(t)\|v\|_H,$$

a.e. in $\mathcal{J}$, for all $v \in \mathcal{V}$ and where $F \in L_p(\mathcal{J})$ is known.

Note the introduction of the energy norm $\|\cdot\|_H$ and its appearance in part (ii) of these assumptions. It is this similarity property of $b(t,s;\cdot,\cdot)$, along with the rather special properties of ϕ that can be assumed for viscoelasticity problems, that make it possible to derive sharp stability estimates.

REMARK 2. *To put these assumptions into context and to provide a concrete example of* (1) *we might think of the problem*: *find u such that,*

$$-\nabla^2 u(t) = f(t) - \int_0^t \phi(t-s)\nabla^2 u(s)\,ds,$$

in a domain $\Omega \times \mathcal{J}$ where Ω is an open bounded subset of $\mathbb{R}^n$, and where homogeneous Dirichlet data are specified on $\partial\Omega$. In this case we would set $\mathcal{V} := H_0^1(\Omega)$ with the usual $H^1(\Omega)$ inner product and norm, $\mathcal{V}' := H^{-1}(\Omega)$, $\mathcal{H} := L_2(\Omega)$, and $H = H_0^1(\Omega)$ with inner product $a(\,\cdot\,,\,\cdot\,) := (\nabla\,\cdot\,,\nabla\,\cdot\,)_{L_2(\Omega)}$. Also, we would require that $f \in L_p(\mathcal{J};H')$.

Following on from Golden and Graham in [8], we classify viscoelastic materials as either "solid" or "fluid" according to the long time behaviour of ϕ in Assumptions 1, part (ii). For viscoelastic solids it is reasonable to assume that there exists a constant, $C_\Psi > 0$, independent of $\mathcal{J}$ and such that,

$$\Psi(t) := \left(1 - \int_0^t \phi(s)\,ds\right)^{-1} \leqslant C_\Psi. \tag{4}$$

On the other hand to deal with viscoelastic fluids we find it necessary below to require that the domain of ϕ can be extended to the whole of $[0,\infty)$ in such a way that the function,

$$\Phi(t) := \frac{\phi(t)}{\displaystyle\int_t^\infty \phi(s)\,ds}, \tag{5}$$

is defined for every $t \geqslant 0$. This is physically reasonable as we show later in § 5, although the "power law" is excluded from our results on physical grounds.

On finite time intervals $\mathcal{J}$ it is possible to use the same assumption (4) to prove stability estimates for viscoelastic fluids also, but these will not usually be optimal. For example, in the simplest case ϕ is given by

$$\phi(t) := -\varphi'(t) \qquad \text{where} \qquad \varphi(t) := \varphi_0 + \varphi_1 e^{-\alpha t}, \tag{6}$$

and $\alpha, \varphi_0, \varphi_1 \geqslant 0$ such that $\varphi(0) = 1$. The function φ is an example of a stress relaxation function, see e.g. [8, 6], and when $\varphi_0 > 0$ models a viscoelastic solid. In this case,

$$\Psi(t) = \frac{1}{\varphi(t)} \leqslant \frac{1}{\varphi_0}, \tag{7}$$

which (see Theorem 9 below) results in the estimate $\|u\|_{L_p(\mathcal{J};\mathcal{V})} \leqslant C\|F\|_{L_p(\mathcal{J})}/\varphi_0$, where C is a coercivity constant such that $\|\cdot\|_{\mathcal{V}} \leqslant C\|\cdot\|_H$. This is the correct behaviour.

On the other hand the choice $\varphi_0 = 0$ corresponds to a viscoelastic fluid in which case,

$$\Psi(t) = \frac{e^{\alpha t}}{\varphi_1}. \tag{8}$$

This would yield the stability estimate $\|u\|_{L_p(\mathcal{J};\mathcal{V})} = O(e^{\alpha T})$ which is not optimal since Theorem 14 below gives the best possible estimate $\|u\|_{L_p(\mathcal{J};\mathcal{V})} = O(T)$.

Due to this necessary distinction we deal firstly in §§ 2 and 3 respectively with stability estimates for viscoelastic solids and fluids. These results are then followed up and applied to derive the *a priori* error estimates in § 4. In § 5 we then briefly outline our interpretation of the linear quasistatic viscoelasticity problem, and show how the estimates derived in §§ 2, 3 and 4 can be applied. We intend also that the material in this last section will go some way toward justifying the assumptions made on the bounding function ϕ in Assumptions 1, but for more detail we refer to other sources.

Finally in this section we recall some important results from convolution Volterra equation theory. Here, and throughout, we use $*$ to denote (Laplace-type) convolution so that,

$$(m * n)(t) := \int_0^t m(t-s)n(s)\,ds, \tag{9}$$

which is well defined for integrable functions m and n. In fact we have the following (well-known) result which we include for the sake of completeness.

LEMMA 3. *If $m \in L_1(\mathcal{J})$ and $n \in L_p(\mathcal{J})$ for some $p \in [1,\infty]$, then $m*n \in L_p(\mathcal{J})$ and $\|m*n\|_{L_p(0,t)} \leqslant \|m\|_{L_1(0,t)}\|n\|_{L_p(0,t)}$ for all $t \in \mathcal{J}$.*

Proof. The first part of the proof is a straightforward application of Hölder's inequality, with the obvious modifications intended if $p = 1$ or $p = \infty$. For a.e. $\tau \in \mathcal{J}$ we have,

$$\begin{aligned}
|(m*n)(\tau)|^p &= \left|\int_0^\tau \big(m(\tau-s)\big)^{\frac{p-1}{p}}\big(m(\tau-s)\big)^{\frac{1}{p}} n(s)\,ds\right|^p,\\
&\leqslant \left(\int_0^\tau |m(\tau-s)|\,ds\right)^{p-1}\int_0^\tau |m(\tau-s)|\,|n(s)|^p\,ds,\\
&= \|m\|_{L_1(0,\tau)}^{p-1}\int_0^\tau |m(\tau-s)|\,|n(s)|^p\,ds.
\end{aligned}$$

Integrating, using this inequality, and then interchanging the order of integration now gives for every $t \in \mathcal{J}$ that,

$$\begin{aligned}
\|m*n\|_{L_p(0,t)}^p &\leqslant \|m\|_{L_1(0,t)}^{p-1}\int_0^t\int_0^\tau |m(\tau-s)|\,|n(s)|^p\,ds d\tau,\\
&= \|m\|_{L_1(0,t)}^{p-1}\int_0^t\int_s^t |m(\tau-s)|\,d\tau\ |n(s)|^p\,ds,\\
&= \|m\|_{L_1(0,t)}^{p-1}\int_0^t \|m\|_{L_1(0,t-s)}|n(s)|^p\,ds,\\
&\leqslant \|m\|_{L_1(0,t)}^{p}\|n\|_{L_p(0,t)}^p,
\end{aligned}$$

and taking p^{th} roots completes the proof. □

For $n \in L_p(\mathcal{J})$ and $\phi \in L_1(\mathcal{J})$ recall that the prototype convolution Volterra equation problem of the second kind is to find $m \in L_p(\mathcal{J})$ such that,

$$m(t) = n(t) + (\phi * m)(t) \qquad \text{a.e. in } \mathcal{J}. \tag{10}$$

For such problems we have the following two important results.

LEMMA 4 (**variation of constants**). *If $\phi \in L_1(\mathcal{J})$ there exists a unique resolvent, $R \in L_1(\mathcal{J})$, such that $R = \phi + \phi * R$ and $R = \phi + R * \phi$ a.e. in $\mathcal{J}$. Moreover, for such ϕ, for any $p \in [1, \infty]$ and for every $n \in L_p(\mathcal{J})$, there is a unique solution $m \in L_p(\mathcal{J})$ to* (10) *given by the variation of constants formula*: $m = n + R * n$.

LEMMA 5 (**comparison theorem**). *For $p \in [1, \infty]$ and for $\phi \in L_1\big(\mathcal{J}; [0, \infty)\big)$ let m, $n \in L_p\big(\mathcal{J}; [0, \infty)\big)$ satisfy the Volterra inequality $m(t) \leqslant n(t) + (\phi * m)(t)$ a.e. in $\mathcal{J}$. Then $m(t) \leqslant M(t)$ a.e. in $\mathcal{J}$, where $M \in L_p\big(\mathcal{J}; [0, \infty)\big)$ solves the comparison equation $M = n + \phi * M$ a.e. in $\mathcal{J}$.*

For proofs of these lemmas we refer, respectively, to Theorems 3.1 and 3.5 in [9, Chap. 2], and to Proposition 8.1 and Lemma 8.2 in [9, Chap. 9]. Note that Lemma 5 may easily be used to prove Gronwall's inequality by taking $\phi = \text{constant} > 0$. Similar comparison theorems can be found in the papers [1] and [4].

2. Stability of viscoelastic solids. The basic defining characteristic of a viscoelastic solid is contained in the following definition.

DEFINITION 6 (**viscoelastic solid**). *We say that a function $\phi \in L_1\big(\mathcal{J}; [0, \infty)\big)$ characterizes a viscoelastic solid if there exists a constant $C_\Psi > 0$, and independent of $\mathcal{J}$, such that,*

$$0 \leqslant \Psi(t) \leqslant C_\Psi \qquad \textit{where} \qquad \Psi(t) := \left(1 - \int_0^t \phi(s)\, ds\right)^{-1}.$$

Note that we also have $\Psi(t) = \big(1 - \|\phi\|_{L_1(0,t)}\big)^{-1}$ since ϕ is a.e. non-negative in $\mathcal{J}$.

The basic result for this section is the following stability estimate for (10).

LEMMA 7. *Suppose that $M \in L_p(\mathcal{J})$ is the unique solution to the Volterra problem $M = n + \phi * M$ a.e. in $\mathcal{J}$ where $n \in L_p(\mathcal{J})$ is given, and ϕ characterizes a viscoelastic solid. Then $\|M\|_{L_p(0,t)} \leqslant \Psi(t)\|n\|_{L_p(0,t)}$ for all $t \in \mathcal{J}$.*

Proof. Since $M = n + \phi * M$ we have from Lemma 4 that $M = n + R * n$ where $R \in L_1(\mathcal{J})$ is the resolvent of ϕ and satisfies $R = \phi + \phi * R$. Now, for any $t \in \mathcal{J}$ we have for this resolvent equation that,

$$\begin{aligned}
&\|R\|_{L_1(0,t)} \leqslant \|\phi\|_{L_1(0,t)} + \|\phi\|_{L_1(0,t)}\|R\|_{L_1(0,t)}, \\
\Longrightarrow \quad &\|R\|_{L_1(0,t)} \leqslant \Psi(t)\|\phi\|_{L_1(0,t)},
\end{aligned}$$

where we used Lemma 3. Taking $L_p(0,t)$ norms in the variation of constants formula $M = n + R * n$ now yields:

$$\begin{aligned}
\|M\|_{L_p(0,t)} &\leqslant \|n\|_{L_p(0,t)} + \|R\|_{L_1(0,t)}\|n\|_{L_p(0,t)}, \\
&\leqslant \big(1 + \Psi(t)\|\phi\|_{L_1(0,t)}\big)\, \|n\|_{L_p(0,t)}, \\
&= \Psi(t)\|n\|_{L_p(0,t)},
\end{aligned}$$

where we again used Lemma 3. This completes the proof. □

REMARK 8. *To see how this result is optimal for large times we use* ϕ *as given by* (6) *and find that* $\Psi(t) = 1/\varphi(t) \leqslant 1/\varphi_0$, *since* $\varphi(0) = 1$. *Now consider the problem:* $M = 1 + \phi * M$, *which has the solution*

$$M(t) = \frac{1 - \varphi_1 e^{-\alpha\varphi_0 t}}{\varphi_0},$$

and from which we obtain

$$\|M\|_{L_\infty(0,t)} = M(t) \leqslant \frac{\|1\|_{L_\infty(0,t)}}{\varphi_0}.$$

This agrees with the bound for $p = \infty$ *just given in Lemma* 7. *For* $p = 1$ *we have,*

$$\begin{aligned}\|M\|_{L_1(0,t)} &= \frac{t}{\varphi_0} - \frac{\varphi_1}{\alpha\varphi_0^2} + \frac{\varphi_1 e^{-\alpha\varphi_0 t}}{\alpha\varphi_0^2} = \left(\frac{1}{\varphi_0} - \frac{\varphi_1}{\alpha\varphi_0^2 t} + \frac{\varphi_1 e^{-\alpha\varphi_0 t}}{\alpha\varphi_0^2 t}\right)\|1\|_{L_1(0,t)},\\ &\leqslant \frac{\|1\|_{L_1(0,t)}}{\varphi_0},\end{aligned}$$

which also agrees with Lemma 7. *We conclude therefore that our bound is best possible for large times.*

The first result we state for (3) is an "energy-stability" estimate and for this we introduce the norm

$$\|u\|_{L_p(0,t;H)} := \left(\int_0^t \|u(s)\|_H^p\, ds\right)^{\frac{1}{p}},$$

which we call the "L_p-energy" norm.

THEOREM 9 (**energy stability of a solid**). *Let Assumptions* 1 *be satisfied in the particular case where* ϕ *characterizes a viscoelastic solid, then the solution* u *to* (3) *satisfies*

$$\|u\|_{L_p(0,t;H)} \leqslant \Psi(t)\|F\|_{L_p(0,t)}$$

for all $t \in \mathcal{J}$.

Proof. In equation (3) we take $v = u(t) \in \mathcal{V}$ for almost any $t \in \mathcal{J}$ and obtain

$$\|u(t)\|_H \leqslant F(t) + \int_0^t \phi(t-s)\|u(s)\|_H\, ds.$$

Setting $m(t) := \|u(t)\|_H$ and $n(t) = F(t)$ we have that $m \leqslant n + \phi * m$ a.e. in $\mathcal{J}$, and then from Lemma 5 that $m(t) \leqslant M(t)$ a.e., where M solves the equation $M = n + \phi * M$. From Lemma 7 we now find that $\|M\|_{L_p(0,t)} \leqslant \Psi(t)\|n\|_{L_p(0,t)}$ from which the desired result follows since $m \leqslant M$ implies $\|m\|_{L_p(0,t)} \leqslant \|M\|_{L_p(0,t)}$. ∎

The next result is not quite so general since it requires an extra assumption on $B(t,s)$ and also that $f \in L_p(\mathcal{J};\mathcal{H})$, whereas Assumption 1, part (iii) requires only that $f \in L_p(\mathcal{J};H')$.

THEOREM 10 (**operator stability of a solid**). *Suppose firstly that the solution to equation* (1) *also satisfies* $Au(t) \in \mathcal{H}$ *a.e. in* $\mathcal{J}$, *and suppose also that* $f \in L_p(\mathcal{J};\mathcal{H})$ *and* $B(t,s)$ *is similar to* A *in the sense that,*

$$\|B(t,s)u(s)\|_{\mathcal{H}} \leqslant \phi(t-s)\|Au(s)\|_{\mathcal{H}} \qquad \text{a.e. in } \mathcal{T},$$

where ϕ characterizes a viscoelastic solid. Then

$$\|Au\|_{L_p(0,t;\mathcal{H})} \leqslant \Psi(t)\|f\|_{L_p(0,t;\mathcal{H})}$$

for all $t \in \mathcal{J}$.

Proof. For almost any $t \in \mathcal{J}$ we take the $\mathcal{H}$ inner product of (1) with $Au(t)$ giving,

$$\begin{aligned} \|Au(t)\|_{\mathcal{H}}^2 &= \big(f(t), Au(t)\big)_{\mathcal{H}} + \int_0^t \big(B(t,s)u(s), Au(t)\big)_{\mathcal{H}}\, ds, \\ \Longrightarrow \quad \|Au(t)\|_{\mathcal{H}} &\leqslant \|f(t)\|_{\mathcal{H}} + \int_0^t \phi(t-s)\|Au(s)\|_{\mathcal{H}}\, ds. \end{aligned}$$

Setting $m(t) := \|Au(t)\|_{\mathcal{H}}$ and $n(t) := \|f(t)\|_{\mathcal{H}}$ we have firstly from Lemma 5 that $m(t) \leqslant M(t)$ a.e. in $\mathcal{J}$, and therefore that $\|m\|_{L_p(0,t)} \leqslant \|M\|_{L_p(0,t)}$ everywhere in $\mathcal{J}$ where M solves the comparison equation $M = n + \phi * M$. Lemma 7 again gives $\|M\|_{L_p(0,t)} \leqslant \Psi(t)\|n\|_{L_p(0,t)}$ from which the theorem follows immediately. □

This last result may be useful for problems posed on a spatial domain $\Omega \subset \mathbb{R}^n$ that has either a smooth or a convex polygonal/polyhedral boundary $\partial\Omega$. In such cases one could make use of elliptic regularity estimates of the form $\|D^2u\|_{L_2(\Omega)} \leqslant C\|Au\|_{L_2(\Omega)}$, which can be useful for deriving $L_2(\Omega)$ error bounds for numerical approximations.

Note the simplicity with which the stability estimates follow once the bounding function ϕ in Assumption 1, part (ii) has been precisely characterized, and the relevant Volterra theory established. Recalling the discussion surrounding equation (7) we can see that these stability estimates represent a significant improvement over the usual Gronwall-type bounds. We now move on and study the case where (1) models a (linear compressible quasistatic) viscoelastic fluid.

3. Stability of viscoelastic fluids. For viscoelastic fluids the function ϕ has similar properties to a probability density function and it turns out that the long-time behaviour of u is tied to the solution of the renewal equation (see e.g. [15, Chap. 3]). The asymptotic behaviour of this solution is usually established with a limit theorem (as $t \to \infty$) for the Laplace transform. This is useful in that it suggests a form to aim at for our problem, but it does not give the quantitative information at intermediate times necessary for an adaptive numerical solution. Accordingly, in this section we formulate a different assumption for ϕ and derive a bound for the renewal equation that holds for all times $t \in \mathcal{J}$.

We find that the situation regarding viscoelastic fluids is a little more delicate in that we have to strengthen our assumptions on the bounding function ϕ. In particular we assume that ϕ is defined (or can be so extended) on the whole positive real line and is continuous there such that it satisfies the following definition.

DEFINITION 11 (**viscoelastic fluid**). *In contrast to the case for a viscoelastic solid, we say that a function $\phi \in C\big([0,\infty);(0,\infty)\big)$ characterizes a viscoelastic fluid if,*

$$\int_0^\infty \phi(s)\, ds = 1,$$

and if the function,

$$\Phi(t) := \frac{\phi(t)}{\displaystyle\int_t^\infty \phi(s)\, ds},$$

is defined (by taking a limit if necessary) for every $t \geqslant 0$.

Note that it is the first condition on ϕ in this definition that renders the previous analysis for a solid almost useless for a fluid (recall the discussion earlier near equation (8)). In place of Lemma 7 we now have the following basic estimate.

LEMMA 12. *Suppose that* $M \in L_p(\mathcal{J})$ *is the unique solution to the Volterra problem* $M = n + \phi * M$ *a.e. in* $\mathcal{J}$, *where* $n \in L_p(\mathcal{J})$ *is given and* ϕ *characterizes a viscoelastic fluid. Then* $\|M\|_{L_p(0,t)} \leqslant \left(1 + t\|\Phi\|_{C(0,t)}\right)\|n\|_{L_p(0,t)}$ *for all* $t \in \mathcal{J}$.

Proof. From Lemma 4 we find that $M = n + R * n$ where the resolvent R satisfies $R = \phi + \phi * R$. Now, since ϕ is positive and continuous on $\mathcal{J}$, R is also and so for any $\tau \in \mathcal{J}$ we can find $t' \in [0, \tau]$ and $\epsilon > 0$ for which,

$$\|R\|_{C(0,\tau)} = R(t') \leqslant \frac{1}{\epsilon}.$$

Now we require a lower bound for ϵ:

$$\begin{aligned}
&\frac{1}{\epsilon} = R(t') = \phi(t') + \int_0^{t'} \phi(t' - s)R(s)\,ds \leqslant \phi(t') + \frac{1}{\epsilon}\int_0^{t'} \phi(s)\,ds, \\
\Longrightarrow\quad & 1 - \int_0^{t'} \phi(s)\,ds \leqslant \epsilon\phi(t'), \\
\Longrightarrow\quad & \epsilon \geqslant \frac{1}{\phi(t')}\int_{t'}^{\infty} \phi(s)\,ds \quad\Longrightarrow\quad \frac{\phi(t')}{\displaystyle\int_{t'}^{\infty} \phi(s)\,ds} \geqslant \frac{1}{\epsilon} \geqslant R(t').
\end{aligned}$$

Since τ is arbitrary we therefore have $\|R\|_{C(0,t)} \leqslant \|\Phi\|_{C(0,t)}$ for all $t \in \mathcal{J}$. From the resolvent equation for M we now easily get

$$\begin{aligned}
\|M\|_{L_p(0,t)} &\leqslant \|n\|_{L_p(0,t)} + \|R\|_{L_1(0,t)}\|n\|_{L_p(0,t)}, \\
&\leqslant \|n\|_{L_p(0,t)}\left(1 + t\|\Phi\|_{C(0,t)}\right),
\end{aligned}$$

and this completes the proof. □

REMARK 13. *For* $p = \infty$ *this result is best possible under the assumptions given and to see this consider the problem* $M = 1 + \phi * M$ *where* $\phi(t) := \alpha e^{-\alpha t}$ *for* $\alpha > 0$. *The solution to this problem is* $M(t) = 1 + \alpha t$. *Now, it is easy to see that this function* ϕ *characterizes a fluid according to Definition* 11 *and that* $\Phi(t) = \alpha$. *Hence Lemma* 12 *gives the bound,*

$$\|M\|_{L_\infty(0,t)} \leqslant (1 + \alpha t)\|1\|_{L_\infty(0,t)},$$

which is clearly best possible. On the other hand, for $p = 1$ *the lemma over-estimates by a factor of about two since it now gives,*

$$\|M\|_{L_1(0,t)} \leqslant (1 + \alpha t)\|1\|_{L_1(0,t)} = t(1 + \alpha t),$$

whereas a direct calculation yields $\|M\|_{L_1(0,t)} = t(1 + \alpha t/2)$.

With this result the proofs of energy and operator stability for fluids can be established in exactly the same way as for solids.

THEOREM 14 (**energy stability of a fluid**). *Let Assumptions* 1 *be satisfied in the particular case where* ϕ *characterizes a viscoelastic fluid, then the solution* u *to* (3) *satisfies*

$$\|u\|_{L_p(0,t;H)} \leqslant \left(1 + t\|\Phi\|_{C(0,t)}\right)\|F\|_{L_p(0,t)}$$

for all $t \in \mathcal{J}$.

Proof. In equation (3) we take $v = u(t) \in \mathcal{V}$ for almost any $t \in \mathcal{J}$ and obtain

$$\|u(t)\|_H \leqslant F(t) + \int_0^t \phi(t-s)\|u(s)\|_H\, ds.$$

Setting $m(t) := \|u(t)\|_H$ and $n(t) = F(t)$ we have that $m \leqslant n + \phi * m$ a.e. in $\mathcal{J}$, and then from Lemma 5 that $m(t) \leqslant M(t)$ a.e., where M solves the equation $M = n + \phi * M$. From Lemma 12 we now find that $\|M\|_{L_p(0,t)} \leqslant (1 + t\|\Phi\|_{C(0,t)})\|n\|_{L_p(0,t)}$ from which the desired result follows since $\|m\|_{L_p(0,t)} \leqslant \|M\|_{L_p(0,t)}$. ∎

This theorem is the "fluid" analogue to Theorem 9, and we also have the following companion to Theorem 10.

THEOREM 15 (**operator stability of a fluid**). *Suppose that the solution to equation* (1) *also satisfies* $Au(t) \in \mathcal{H}$ *a.e. in* $\mathcal{J}$, *and also that* $f \in L_p(\mathcal{J};\mathcal{H})$ *and* $B(t,s)$ *is similar to* A *in the sense that*

$$\|B(t,s)u(s)\|_{\mathcal{H}} \leqslant \phi(t-s)\|Au(s)\|_{\mathcal{H}} \qquad \text{a.e. in } \mathcal{T},$$

where ϕ *characterizes a viscoelastic fluid. Then*

$$\|Au\|_{L_p(0,t;\mathcal{H})} \leqslant \left(1 + t\|\Phi\|_{C(0,t)}\right)\|f\|_{L_p(0,t;\mathcal{H})}$$

for all $t \in \mathcal{J}$.

Proof. For almost any $t \in \mathcal{J}$ we take the $\mathcal{H}$ inner product of (1) with $Au(t)$ giving

$$\|Au(t)\|_{\mathcal{H}}^2 = \left(f(t), Au(t)\right)_{\mathcal{H}} + \int_0^t \left(B(t,s)u(s), Au(t)\right)_{\mathcal{H}} ds,$$

$$\implies \quad \|Au(t)\|_{\mathcal{H}} \leqslant \|f(t)\|_{\mathcal{H}} + \int_0^t \phi(t-s)\|Au(s)\|_{\mathcal{H}}\, ds.$$

Setting $m(t) := \|Au(t)\|_{\mathcal{H}}$ and $n(t) := \|f(t)\|_{\mathcal{H}}$ we have firstly from Lemma 5 that $m(t) \leqslant M(t)$ a.e. in $\mathcal{J}$, and therefore that $\|m\|_{L_p(0,t)} \leqslant \|M\|_{L_p(0,t)}$ everywhere in $\mathcal{J}$ where M solves the comparison equation $M = n + \phi * M$. Lemma 12 again gives $\|M\|_{L_p(0,t)} \leqslant (1 + t\|\Phi\|_{C(0,t)})\|n\|_{L_p(0,t)}$ from which the theorem now follows. ∎

4. Semidiscrete error estimates. The objective of this section is to apply the stability estimates derived above to obtain *a priori* error estimates for semidiscrete finite element approximations to (3). These estimates will contain sharper constants than would be possible using the usual Gronwall inequality. We first describe the discretization and then, by making a standard assumption on the approximation property of the finite element space in terms of the Ritz projection, we then derive an error estimate in the L_p-energy norm, $\|\cdot\|_{L_p(0,t;H)}$. Under the additional assumption of elliptic regularity we also provide a "lift" via an auxilliary dual backward problem to give an optimal $L_p(0,t;\mathcal{H})$ error estimate.

Let $\{H_h; a(\cdot\,,\cdot)\}_{h>0}$ be a family of finite element subspaces of H indexed by the parameter h. We have in mind here of course spaces of piecewise polynomial functions of degree $r > 0$ defined relative to a quasiuniform mesh of mesh-width h. Each H_h suggests a semidiscrete finite element approximation to (3) as the problem: find $U \in L_p(\mathcal{J}; H_h)$ such that,

$$a(U(t), v) = L(t; v) + \int_0^t b(t,s; U(s), v)\, ds \qquad \forall v \in H_h, \text{ a.e. in } \mathcal{J}. \tag{11}$$

Note that H_h does not change with time; such adaptive approximations are considered elsewhere.

Subtracting (11) from (3) gives for all $v \in H_h$ that,

$$a(u(t) - U(t), v) = \int_0^t b(t, s; u(s) - U(s), v)\, ds,$$

which is the Galerkin "orthogonality" property of the finite element approximation. We write this in the alternative form,

$$a(Ru(t) - U(t), v) = \int_0^t b(t, s; u(s) - Ru(s), v)\, ds + \int_0^t b(t, s; Ru(s) - U(s), v)\, ds,$$

where Ru is the Ritz (or elliptic) projection of u on H_h defined by,

$$a(u(t) - Ru(t), v) = 0 \qquad \forall v \in H_h, \text{ a.e. in } \mathcal{J}.$$

The idea of using this projection was first introduced by Wheeler in [24], and we are using it here to obtain estimates for a Ritz-Volterra projection in the sense of [12]. Setting $\eta := u - Ru$ and $\zeta := Ru - U$ gives,

$$a(\zeta(t), v) = \ell(t; v) + \int_0^t b(t, s; \zeta(s), v)\, ds \qquad \forall v \in H_h,$$

and where we have set,

$$\ell(t; v) := \int_0^t b(t, s; \eta(s), v)\, ds.$$

Observe that by replacing ζ, ℓ and H_h in this equation by u, L and H we arrive at (3). The goal now is to apply the stability estimates for (3) in this new context to estimate ζ in terms of η. To do this we first need to check that Assumptions 1 are satisfied.

Clearly Parts (i) and (ii) hold in this new context since $H_h \subset H$. To meet Part (iii) we have that,

$$|\ell(t, v)| \leqslant \int_0^t \phi(t - s)\|\eta(s)\|_H\, ds\ \|v\|_H$$

for all $v \in H \supset H_h$, and so we may set $F(t) := (\phi * \|\eta\|_H)(t)$. Noting now that $\zeta : \mathcal{J} \to H_h$ a direct application of either Theorem 9 or Theorem 14 gives,

$$\|\zeta\|_{L_p(0,t;H)} \leqslant S(t)\, \|\phi * \|\eta\|_H\|_{L_p(0,t)}\, ,$$

where we use $S(t)$ to mean either $\Psi(t)$ or $(1 + t\|\Phi\|_{C(0,t)})$ depending on whether we are dealing with a solid or a fluid. We now have,

$$\begin{aligned}\|u - U\|_{L_p(0,t;H)} &\leqslant \|\zeta\|_{L_p(0,t;H)} + \|\eta\|_{L_p(0,t;H)}, \\ &\leqslant (1 + S(t)\|\phi\|_{L_1(0,t)})\|\eta\|_{L_p(0,t;H)}.\end{aligned}$$

Now, recalling the definition of $S(t) = \Psi(t)$ in the case of a solid we see that,

$$1 + S(t)\|\phi\|_{L_1(0,t)} = 1 + \frac{\|\phi\|_{L_1(0,t)}}{1 - \|\phi\|_{L_1(0,t)}} = \frac{1}{1 - \|\phi\|_{L_1(0,t)}} = S(t).$$

On the other hand, for a fluid we get,

$$1 + S(t)\|\phi\|_{L_1(0,t)} = 1 + (1 + t\|\Phi\|_{C(0,t)})\|\phi\|_{L_1(0,t)} \leqslant (1 + \|\phi\|_{L_1(0,t)})S(t).$$

For a solid the result above is the best we could hope for since it compares directly to the stability estimate. However, for a fluid we have an additional time dependence appearing in front of the stability factor $S(t)$. The effect is small though since,

$$(1 + \|\phi\|_{L_1(0,t)})S(t) \leqslant 2S(t),$$

where we recalled that $\|\phi\|_{L_1(0,\infty)} = 1$ by assumption (see Definition 11). We can now state our first error estimate, the proof of which follows immediately from the foregoing.

THEOREM 16 (**semidiscrete energy-error estimate**). *The semidiscrete finite element error satisfies,*

$$\|u - U\|_{L_p(0,t;H)} \leqslant \Big(1 + S(t)\|\phi\|_{L_1(0,t)}\Big)\|u - Ru\|_{L_p(0,t;H)},$$

where $R : H \to H_h$ is the Ritz map defined above. Furthermore, if the spaces $\{H_h\}$ are such that Ru satisfies the estimate $\|u(t) - Ru(t)\|_H \leqslant C_R h^r |||u(t)|||$ a.e. in $\mathcal{J}$ for some $C_R > 0$, and where $|||\cdot|||$ is an appropriately strong norm (or seminorm) measuring the spatial derivatives of u, then,

$$\|u - U\|_{L_p(0,t;H)} \leqslant C_R \mathcal{S}(t) h^r |||u|||_{L_p(0,t)},$$

where we use $\mathcal{S}(t) := \Psi(t)$ for a solid and $\mathcal{S}(t) := \big(1 + \|\phi\|_{L_1(0,t)}\big)\big(1 + t\|\Phi\|_{C(0,t)}\big)$ for a fluid.

To derive optimal error estimates in the weaker $L_p(0,t;\mathcal{H})$ norm we proceed as follows. For any $\tau \in \mathcal{J}$ and q the conjugate Hölder index to p we introduce the dual backward problem: find $w \in L_q(0,\tau;H)$ such that a.e. in $[0,\tau]$,

$$a(v, w(t)) = (v, g(t))_{\mathcal{H}} + \int_t^\tau b(s,t;v,w(s))\,ds \qquad \forall v \in H.$$

Here $g \in L_q(0,\tau;\mathcal{H})$ is an arbitrary load. Changing the variables to $\xi := \tau - t$ and $\eta := \tau - s$ and defining,

$$\bar{w}(\,\cdot\,) := w(\tau - \,\cdot\,), \qquad \bar{g}(\xi) := g(\tau - \xi) \qquad \text{and} \qquad \bar{b}(\xi,\eta;\,\cdot\,,v) := b(\tau-\eta,\tau-\xi;v,\,\cdot\,),$$

it is easy to see that this backward problem can be written in forward form as: find $\bar{w} \in L_q(0,\tau;H)$ such that for a.e. $\xi \in [0,\tau]$,

$$a(\bar{w}(\xi), v) = (\bar{g}(\xi), v)_{\mathcal{H}} + \int_0^\xi \bar{b}(\xi,\eta;\bar{w}(\eta),v)\,d\eta \qquad \forall v \in H.$$

This is now essentially identical to (3) and we would like to apply all of the results obtained so far to this dual problem. This is possible once we know that Assumptions 1 are satisfied. Part (i) is clearly satisfied, and so is Part (iii) if we notice that $\bar{g}$ generates a linear form $L_g(\xi) \in H'$ via,

$$L_g(\xi; v) := (\bar{g}(\xi), v)_{\mathcal{H}}.$$

We then have $F(\xi) \sim \|L_g(\xi)\|_{\mathcal{H}'} \in L_q(0,\tau)$. To satisfy Part (ii) in this new context we use it in its original context to see that,

$$\begin{aligned}|\bar{b}(\xi,\eta;\bar{w}(\eta),v)| &= |b(\tau-\eta,\tau-\xi;v,\bar{w}(\eta))| \leqslant \phi\Big((\tau-\eta)-(\tau-\xi)\Big)\|v\|_H\|\bar{w}(\eta)\|_H,\\ &= \phi(\xi-\eta)\|v\|_H\|\bar{w}(\eta)\|_H.\end{aligned}$$

So, with an appropriate change of notation in all the previous sections we may freely use the stability and error estimates obtained thus far for the dual problem. In particular, the operator stability estimate reads,

$$\|Aw\|_{L_q(0,\tau;\mathcal{H})} \leqslant S(\tau)\|g\|_{L_q(0,\tau;\mathcal{H})},$$

with $S(t)$ as defined above. Also, if W is the semidiscrete finite element approximation to w then Theorem 16 gives,

$$\|w-W\|_{L_q(0,\tau;H)} \leqslant \Big(1+S(t)\|\phi\|_{L_1(0,\tau)}\Big)\|w-Rw\|_{L_q(0,\tau;H)}.$$

Now, allowing a time dependent test function $v \in L_p(0,\tau;H)$ in the dual problem and integrating over $[0,\tau]$ gives,

$$\begin{aligned}\int_0^\tau (v(t),g(t))_{\mathcal{H}}\,dt &= \int_0^\tau a(v(t),w(t)) - \int_t^\tau b(s,t;v(t),w(s))\,ds\ dt,\\ &= \int_0^\tau a(v(t),w(t)) - \int_0^t b(t,s;v(s),w(t))\,ds\ dt,\end{aligned}$$

by interchanging the order of integration in the Volterra term. Choosing $v = e := u-U$ and using Galerkin "orthogonality" then gives,

$$\begin{aligned}\int_0^\tau (e(t),g(t))_{\mathcal{H}}\,dt &= \int_0^\tau a(e(t),w(t)-W(t)) - \int_0^t b(t,s;e(s),w(t)-W(t))\,ds\ dt,\\ &\leqslant \int_0^\tau \Big(\|e(t)\|_H + \int_0^t \phi(t-s)\|e(s)\|_H\,ds\Big)\,\|w(t)-W(t)\|_H\,dt,\\ &\leqslant (1+\|\phi\|_{L_1(0,\tau)})\|e\|_{L_p(0,\tau;H)}\|w-W\|_{L_q(0,\tau;H)}.\end{aligned}$$

Recalling that the choice for $\tau \in \mathcal{J}$ was arbitrary, and using an appropriate elliptic regularity assumption, results in the following.

THEOREM 17 (**semidiscrete lifted error estimate**). *Assume that the L_q-energy error estimate for the dual problem can be written in the non-optimal (unless $r=1$) form,*

$$\|w-W\|_{L_q(0,\tau;H)} \leqslant C_l\mathcal{S}(\tau)h\|Aw\|_{L_q(0,\tau;\mathcal{H})},$$

and let Theorem 10 or 15 (as appropriate) hold also. Then,

$$\|u-U\|_{L_p(0,t;\mathcal{H})} \leqslant C_RC_l(1+\|\phi\|_{L_1(0,t)})\mathcal{S}^2(t)S(t)h^{r+1}|||u|||_{L_p(0,t)},$$

where $S(t)$ and $\mathcal{S}(t)$ are as defined previously. Furthermore, if the estimate in Theorem 16 in the case $r=1$ can be written,

$$\|u-U\|_{L_p(0,t;H)} \leqslant C_e\mathcal{S}(t)h\|Au\|_{L_p(0,t;\mathcal{H})},$$

then we also have,

$$\|u-U\|_{L_p(0,t;\mathcal{H})} \leqslant C_e C_l(1+\|\phi\|_{L_1(0,t)})\mathcal{S}^2(t)S^2(t)h^2\|f\|_{L_p(0,t;\mathcal{H})},$$

provided that Theorem 10 *or* 15 *holds as appropriate.*

Proof. For $z \in L_q(0,\tau)$ we set $g(t) := z(t)e(t)\|e(t)\|_{\mathcal{H}}^{-1}$ and observe that,

$$\|g(t)\|_{\mathcal{H}} = |z(t)| \quad \Longrightarrow \quad \|g\|_{L_q(0,\tau;\mathcal{H})} = \|z\|_{L_q(0,\tau)}.$$

With this we easily find that $(e(t),g(t))_{\mathcal{H}} = z(t)\|e(t)\|_{\mathcal{H}}$ and this, with our result above, the operator stability estimate for the dual problem and the assumed error estimate for W gives,

$$\begin{aligned}\int_0^\tau z(t)\|e(t)\|_{\mathcal{H}}\,dt &\leqslant C_l(1+\|\phi\|_{L_1(0,\tau)})\mathcal{S}(\tau)h\|e\|_{L_p(0,\tau;H)}\|Aw\|_{L_q(0,\tau;\mathcal{H})},\\ &\leqslant C_l(1+\|\phi\|_{L_1(0,\tau)})\mathcal{S}(\tau)S(\tau)h\|e\|_{L_p(0,\tau;H)}\|g\|_{L_q(0,\tau;\mathcal{H})}.\end{aligned}$$

For $p>1$ we get,

$$\begin{aligned}\|e\|_{L_p(0,\tau;\mathcal{H})} &= \sup_z\left\{\left|\int_0^\tau z(t)\|e(t)\|_{\mathcal{H}}\,dt\right| : \|z\|_{L_q(0,\tau)}=1\right\},\\ &\leqslant C_l(1+\|\phi\|_{L_1(0,\tau)})\mathcal{S}(\tau)S(\tau)h\|e\|_{L_p(0,\tau;H)},\end{aligned}$$

and the proof is completed by using the L_p-energy estimate from Theorem 16. For the case $p=1$ we simply choose $z(t)=1$. The final estimate in the Theorem, for the case $r=1$, follows immediately from this and either Theorem 10 or 15. □

We give concrete forms of these error estimates in the context of the physical problem in the next section. In closing here we remark that estimates for u_t, u_{tt}, etc, as well as error estimates resulting from incorrect data can also be derived in the L_p-energy norm thus generalizing Theorems 3.1 and 3.2 in [16].

5. Application to linear quasistatic viscoelasticity problems. For every time $t \in \mathcal{J}$ we assume that the interior of a linear compressible viscoelastic body $\mathcal{G}$ occupies the open bounded domain $\Omega \subset \mathbb{R}^n$, and is subject to a system of body forces $\boldsymbol{f} = \big(f_i(\boldsymbol{x},t)\big)_{i=1}^n$, for $\boldsymbol{x} \in \Omega$, acting throughout its volume. Also, we assume that the surface of $\mathcal{G}$ coincides with $\partial\Omega$ and that there exists a time independent closed set $\Gamma_D \subseteq \partial\Omega$ of positive measure on which the body is rigidly fixed in space. On the open (and possibly empty) set $\Gamma_N := \partial\Omega \setminus \Gamma_D$ there is prescribed a system of surface tractions $\boldsymbol{g} = \big(g_i(\boldsymbol{x},t)\big)_{i=1}^n$, for $\boldsymbol{x} \in \Gamma_N$, and below we shall denote the unit outward directed normal vector to Γ_N as $\widehat{\boldsymbol{n}} = \big(\widehat{n}_i(\boldsymbol{x})\big)_{i=1}^n$.

Let $\boldsymbol{u} = \big(u_i(\boldsymbol{x},t)\big)_{i=1}^n : \Omega \times \mathcal{J} \to \mathbb{R}^n$ denote the displacement at time t of the point $\boldsymbol{x} \in \Omega \cup \partial\Omega$ from the equilibrium configuration that results from the action of the loads $\boldsymbol{f}$ and $\boldsymbol{g}$. In the linear theory the strain tensor $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) = \big(\varepsilon_{ij}(\boldsymbol{u})\big)_{i,j=1}^n$ is given in terms of the displacement as

$$\varepsilon_{ij}(\boldsymbol{u}) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{12}$$

and the constitutive relation connecting the strain to the symmetric stress tensor $\underline{\boldsymbol{\sigma}} = (\sigma_{ij})_{i,j=1}^n$ is given by the linear functional

$$\sigma_{ij}(\boldsymbol{x},t) = D_{ijkl}(\boldsymbol{x},0)\varepsilon_{kl}(\boldsymbol{u}(\boldsymbol{x},t)) - \int_0^t \frac{\partial D_{ijkl}(\boldsymbol{x},t-s)}{\partial s}\varepsilon_{kl}(\boldsymbol{u}(\boldsymbol{x},s))\,ds. \tag{13}$$

Here $\underline{\boldsymbol{D}} = (D_{ijkl})_{i,j,k,l=1}^{n}$ is a fourth order tensor of stress relaxation functions satisfying the following symmetries:

$$D_{ijkl} = D_{jikl}, \qquad D_{ijkl} = D_{ijlk}, \qquad \text{and} \qquad D_{ijkl} = D_{klij}.$$

The first two of these are implied by the symmetry of $\underline{\boldsymbol{\sigma}}$ and $\underline{\varepsilon}$, while the last follows from energy considerations. The situation is analogous to that in linear elasticity theory since for $\underline{\boldsymbol{D}}$ time-independent this constitutive relation is no more than Hooke's law. Note that in viscoelasticity the third of these symmetries will hold for isotropic materials but in the general case only for $t = 0$ and $t = \infty$ (see [13, Equations (1.10) and (2.62)]). These are thermodynamic requirements and details are given in, for example, [5, Chap. 3]. In view of this we assume that $\underline{\boldsymbol{D}}(\boldsymbol{x}, 0)$ is positive definite (a.e. in Ω) in the sense that $\gamma_{ij}\gamma_{kl}D_{ijkl}(\boldsymbol{x}, 0) > 0$ for all non-zero symmetric second order tensors $\underline{\gamma}$. Usually below we will not indicate $\boldsymbol{x}$ dependence.

It is useful and convenient below to introduce the matrix-vector representation of (13). Collecting the stress and strain components into the vectors,

$$\boldsymbol{\sigma} := (\sigma_{11}\ \sigma_{22}\ \ldots \sigma_{12}\ \ldots)^T \qquad \text{and} \qquad \boldsymbol{\varepsilon} := (\varepsilon_{11}\ \varepsilon_{22}\ \ldots 2\varepsilon_{12}\ \ldots)^T$$

we then have,

$$\boldsymbol{\sigma}(t) = \boldsymbol{D}(0)\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \int_0^t \boldsymbol{D}_s(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\, ds,$$

where $\boldsymbol{D}(t)$ is the matrix representation of the tensor $\underline{\boldsymbol{D}}$ and we note that $\boldsymbol{D}(0)$ is symmetric and positive-definite.

In the quasistatic linear theory we assume that the inertia of $\mathcal{G}$ is negligible, and so obtain a mathematical model of the problem by merging (12) and (13) with the following equilibrium equations. For each $i = 1, \ldots, n$,

$$-\sigma_{ij,j} = f_i \qquad \text{in } \Omega \times \mathcal{J}, \tag{14}$$

$$u_i = 0 \qquad \text{in } \Gamma_D \times \mathcal{J}, \tag{15}$$

$$\sigma_{ij}\widehat{n}_j = g_i \qquad \text{in } \Gamma_N \times \mathcal{J}. \tag{16}$$

With appropriately defined operators A and $B(t,s)$ we then arrive at an example of (1) (i.e. A is the usual linear elasticity operator). As an aside, when the inertia is not negligible one will arrive at a system of hyperbolic-Volterra equations which describe the dynamic viscoelasticity problem.

To give a weak formulation of this problem we first define the product Hilbert spaces

$$\boldsymbol{H}^s(\Omega) := H^s(\Omega) \times \cdots \times H^s(\Omega) \qquad (n \text{ times}),$$

for $s = 0, 1, 2, \ldots$, and the accompanying inner products

$$(\boldsymbol{w}, \boldsymbol{v})_s := \sum_{i=1}^{n} (w_i, v_i)_{H^s(\Omega)},$$

for $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{H}^s(\Omega)$, and where $(\cdot\,,\cdot)_{H^s(\Omega)}$ is the usual inner product in $H^s(\Omega)$. These spaces have the natural norms $\|\cdot\|_s := \sqrt{(\cdot\,,\cdot)_s}$, and we note that $\boldsymbol{L}_2(\Omega) \equiv \boldsymbol{H}^0(\Omega)$. Defining now the test space,

$$V := \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D\}$$

with inner product $(\cdot,\cdot)_1$, we arrive at a concrete example of $(\mathcal{V},(\cdot,\cdot)_{\mathcal{V}})$. Taking the Euclidean scalar product of (14) with a test function $\boldsymbol{v} \in V$ and integrating by parts over Ω using the traction boundary condition (16), we get:

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,d\Omega &= -\int_\Omega \sigma_{ij,j}v_i\,d\Omega = \int_\Omega \sigma_{ij}v_{i,j}\,d\Omega - \oint_{\Gamma_N}\sigma_{ij}\widehat{n}_j v_i\,d\Gamma,\\ &= \int_\Omega \sigma_{ij}\varepsilon_{ij}(\boldsymbol{v})\,d\Omega - \oint_{\Gamma_N}\boldsymbol{g}\cdot\boldsymbol{v}\,d\Gamma, \qquad \forall \boldsymbol{v}\in V.\end{aligned}$$

Here we observed that $\sigma_{ij}v_{i,j} = (\sigma_{ij}v_{i,j}+\sigma_{ij}v_{j,i})/2 = \sigma_{ij}\varepsilon_{ij}(\boldsymbol{v})$. Finally we substitute for $\boldsymbol{\sigma}$ using the constitutive relations (13) and arrive at the weak problem: for some $p \in [1,\infty]$, find $\boldsymbol{u} \in L_p(\mathcal{J};V)$ such that

$$a(\boldsymbol{u}(t),v) = L(t;\boldsymbol{v}) + \int_0^t b(t,s;\boldsymbol{u}(s),\boldsymbol{v})\,ds \qquad \forall \boldsymbol{v}\in V, \text{ a.e. in } \mathcal{J}.$$

Here $a(\cdot,\cdot): V\times V \to \mathbb{R}$ and $b(t,s;\cdot,\cdot): \mathcal{T}\times V\times V\to\mathbb{R}$ are bilinear forms given by

$$\begin{aligned}a(\boldsymbol{w},\boldsymbol{v}) &:= \int_\Omega D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{w})\varepsilon_{ij}(\boldsymbol{v})\,d\Omega \equiv \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v})\cdot\boldsymbol{D}(0)\boldsymbol{\varepsilon}(\boldsymbol{w})\,d\Omega,\\ b(t,s;\boldsymbol{w},\boldsymbol{v}) &:= \int_\Omega \frac{\partial D_{ijkl}(t-s)}{\partial s}\varepsilon_{kl}(\boldsymbol{w})\varepsilon_{ij}(\boldsymbol{v})\,d\Omega \equiv \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v})\cdot\boldsymbol{D}_s(t-s)\boldsymbol{\varepsilon}(\boldsymbol{w})\,d\Omega,\end{aligned}$$

for all $\boldsymbol{w},\boldsymbol{v}\in V$ and where the dot denotes the Euclidean scalar product. Also, $L:\mathcal{J}\times V\to\mathbb{R}$ is a linear form given by,

$$L(t;\boldsymbol{v}) := \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,d\Omega + \oint_{\Gamma_N}\boldsymbol{g}(t)\cdot\boldsymbol{v}\,d\Gamma. \tag{17}$$

It is easy to see that this weak problem is an example of the abstract second-kind Volterra equation problem (3), and to apply our stability estimates we must now ensure that Assumptions 1 are satisfied. For more detail on the formulation of this problem see, for example, [17] or [16].

To satisfy the continuity requirement of part (i) it is necessary only to insist that the spatial regularity of the components of $\underline{\boldsymbol{D}}$ is of type $L_\infty(\Omega)$. The coercivity requirement can be verified by using Korn's inequality, see e.g. [10, 7]:

$$\int_\Omega \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v})\,d\Omega \geqslant c\|\boldsymbol{v}\|_1^2 \qquad \forall\boldsymbol{v}\in V.$$

This is a non-trivial inequality to establish in general, but from a physical point of view we can argue that it will be satisfied if all non-trivial rigid body motions (i.e. translation + rotation) are excluded from V. For our problem this is guaranteed by insisting upon the positive measure of Γ_D. Note that we can now define the energy norm

$$\|\boldsymbol{v}\|_H := \sqrt{a(\boldsymbol{v},\boldsymbol{v})},$$

which is equivalent to $\|\cdot\|_1$, and note that $H := V$ with inner product $(\cdot,\cdot)_H := a(\cdot,\cdot)$ is a Hilbert space with dual H'.

To satisfy part (ii) of Assumptions 1 we introduce the following simple lemma.

LEMMA 18. *For all vectors $\boldsymbol{x}$, $\boldsymbol{y}$ we have,*

$$\big| -\boldsymbol{x}\cdot\boldsymbol{D}'(t)\boldsymbol{y}\big| \leqslant \sqrt{\varrho(\boldsymbol{B}(t)^T\boldsymbol{B}(t))\,|\boldsymbol{x}\cdot\boldsymbol{D}(0)\boldsymbol{x}|\,|\boldsymbol{y}\cdot\boldsymbol{D}(0)\boldsymbol{y}|} \qquad \text{a.e. in } \Omega,$$

where $\boldsymbol{B}(t) := -\boldsymbol{D}^{-\frac{1}{2}}(0)\boldsymbol{D}'(t)\boldsymbol{D}^{-\frac{1}{2}}(0)$, and $\varrho(\cdot)$ denotes the spectral radius.

Proof. Since $\boldsymbol{D}(0)$ is symmetric positive definite we have,

$$\begin{aligned}\big| -\boldsymbol{x}\cdot\boldsymbol{D}'(t)\boldsymbol{y}\big| &\leqslant \Big|(\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{x})\cdot(-\boldsymbol{D}^{-\frac{1}{2}}(0)\boldsymbol{D}'(t)\boldsymbol{D}^{-\frac{1}{2}}(0))(\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{y})\Big|,\\ &\leqslant \|-\boldsymbol{D}^{-\frac{1}{2}}(0)\boldsymbol{D}'(t)\boldsymbol{D}^{-\frac{1}{2}}(0)\|_2\,\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{x}\|_2\,\|\boldsymbol{D}^{\frac{1}{2}}(0)\boldsymbol{y}\|_2.\end{aligned}$$

Here $\|\cdot\|_2$ denotes the Euclidean norm. The result follows by characterising the induced matrix norm in terms of spectral radius. □

We now observe that

$$\begin{aligned}|b(t,s;\boldsymbol{u}(s),\boldsymbol{v})| &= \left|\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v})\cdot\boldsymbol{D}_s(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,d\Omega\right|,\\ &\leqslant \int_\Omega \varrho(\boldsymbol{B}(t-s)^T\boldsymbol{B}(t-s))^{\frac{1}{2}}\,|\boldsymbol{\varepsilon}(\boldsymbol{v})\cdot\boldsymbol{D}(0)\boldsymbol{\varepsilon}(\boldsymbol{v})|^{\frac{1}{2}}\,|\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\cdot\boldsymbol{D}(0)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))|^{\frac{1}{2}}\,d\Omega,\\ &\leqslant \big\|\varrho(\boldsymbol{B}(t-s)^T\boldsymbol{B}(t-s))^{\frac{1}{2}}\big\|_{L_\infty(\Omega)}\|\boldsymbol{u}(s)\|_H\|\boldsymbol{v}\|_H,\end{aligned}$$

and this leads to the definition,

$$\phi(t) := \big\|\varrho(\boldsymbol{B}(t)^T\boldsymbol{B}(t))^{\frac{1}{2}}\big\|_{L_\infty(\Omega)}.$$

Part (ii) of Assumptions 1 is now satisfied if we assume the integrability of these eigenvalues $\{\lambda_i\}$ in the way demonstrated below.

Finally, part (iii) of the assumptions will be satisfied by weakening (17) and requiring only that $\boldsymbol{f}$ and $\boldsymbol{g}$ are of class $L_p(\mathcal{J})$, for some $p \in [1,\infty]$, and valued in appropriate dual spaces such that,

$$\|L\|_{L_p(\mathcal{J};H')} := \big\|\|L(t)\|_{H'}\big\|_{L_p(\mathcal{J})} < \infty.$$

We then set $F(t) := \|L(t)\|_{H'}$.

We are now in a position to apply our stability estimates to this linear viscoelasticity problem and to match the notation we set

$$(\mathcal{H};(\cdot,\cdot)_{\mathcal{H}}) := (\boldsymbol{L}_2(\Omega);(\cdot,\cdot)_0), \qquad (\mathcal{V};(\cdot,\cdot)_{\mathcal{V}}) := (V;(\cdot,\cdot)_1),$$

and $\mathcal{V}' := V'$. As already touched on above, the "natural space" H for this problem is V but with inner product given by $a(\cdot,\cdot)$. The L_p-energy norm is then defined by,

$$\|\boldsymbol{v}\|_{L_p(0,t;H)} := \left(\int_0^t \|\boldsymbol{v}(s)\|_H^p\,ds\right)^{\frac{1}{p}}.$$

To illustrate the application of our earlier results we proceed in increasing order of complexity of the viscoelastic law.

5.1. Synchronous materials. the simplest possible model of a viscoelastic material would appear to be the so-called *synchronous* approximation. In these models the Poisson ratio of the material is assumed constant in time which leads to the assumption that each component of $\underline{\boldsymbol{D}}$ shares the same time dependence. In this case we can write,

$$\underline{\boldsymbol{D}}(\boldsymbol{x},t) := \varphi(t)\underline{\boldsymbol{D}}_0(\boldsymbol{x}),$$

for a scalar-valued stress relaxation function φ which, from the fading memory hypothesis of viscoelasticity theory, we assume is a positive, monotone decreasing function of t that is smooth enough for its first time derivative to belong to $L_1(\mathcal{J})$. The thermodynamic symmetry requirements at $t = 0$ and $t = \infty$ imply that $\underline{\boldsymbol{D}}_0$ is symmetric and accordingly the corresponding matrices $\boldsymbol{D}(t)$ and $\boldsymbol{D}_0$ are symmetric with the latter positive definite.

In particular we have $\boldsymbol{D}'(t) = \varphi'(t)\boldsymbol{D}_0$ and so in Lemma 18 we have,

$$\boldsymbol{B}(t) = -\frac{\varphi'(t)\boldsymbol{D}_0^{-\frac{1}{2}}\boldsymbol{D}_0\boldsymbol{D}_0^{-\frac{1}{2}}}{\varphi(0)} = -\frac{\varphi'(t)}{\varphi(0)}\boldsymbol{I}.$$

It is natural to normalize so that $\varphi(0) = 1$ and in fact the prototype form for φ is the Dirichlet-Prony series,

$$\varphi(t) = \varphi_0 + \sum_{i=1}^{N} \varphi_i e^{-\alpha_i t}, \tag{18}$$

for $\alpha_1, \ldots, \alpha_M, \varphi_0, \varphi_1, \ldots, \varphi_N \geqslant 0$ such that $\varphi(0) = 1$ and where the case $\varphi_0 = 0$ corresponds to a viscoelastic fluid. It is easy to check that this function is compatible with either of Definitions 6 and 11. The distinction between a viscoelastic solid and fluid can be illustrated by considering the behaviour of φ for large t. In particular, if $\lim \varphi(t) = c$ then the case $c > 0$ corresponds to the material being a solid while $c = 0$ implies that the material is a fluid. The case $c < 0$ is non-physical and is therefore disallowed. To model a solid using (18) one would take $\varphi_0 > 0$, whereas $\varphi_0 = 0$ gives a fluid. For more detail on this theory see for example [8, 6, 5, 14].

With this form we get for Assumptions 1, part (ii) that $\phi(t) = -\varphi'(t)$ and hence:

$$\Psi(t) = \frac{1}{\varphi(t)} \qquad \text{and} \qquad \Phi(t) = -\frac{\varphi'(t)}{\varphi(t)},$$

since for a fluid $\varphi \to 0$ as $t \to \infty$. For a viscoelastic solid we have therefore that,

$$\|\boldsymbol{u}\|_{L_p(0,t;H)} \leqslant \frac{\|L\|_{L_p(0,t;H')}}{\varphi(t)},$$

and

$$\|A\boldsymbol{u}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))} \leqslant \frac{\|\boldsymbol{f}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))}}{\varphi(t)},$$

from Theorems 9 and 10, while for a fluid we get

$$\|\boldsymbol{u}\|_{L_p(0,t;H)} \leqslant \left(1 + t\left\|\frac{\varphi'}{\varphi}\right\|_{C(0,t)}\right)\|L\|_{L_p(0,t;H')},$$

and

$$\|A\boldsymbol{u}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))} \leqslant \left(1 + t\left\|\frac{\varphi'}{\varphi}\right\|_{C(0,t)}\right)\|\boldsymbol{f}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))},$$

from Theorems 14 and 15.

The estimates for the solid are self-explanatory and are based on the result that $\lim_{t\to\infty}\varphi(t) = \varphi_0 > 0$, but the estimates for fluids deserve a little more attention.

With φ given by the Dirichlet-Prony series we have from the mean value theorem that,

$$\left\|\frac{\varphi'}{\varphi}\right\|_{C(0,t)} = \left\|\frac{\sum_{i=1}^N \alpha_i\varphi_i e^{-\alpha_i t}}{\sum_{i=1}^N \varphi_i e^{-\alpha_i t}}\right\|_{C(0,t)} = \left\|\frac{\tilde{\alpha}(t)\sum_{i=1}^N \varphi_i e^{-\alpha_i t}}{\sum_{i=1}^N \varphi_i e^{-\alpha_i t}}\right\|_{C(0,t)} \leqslant \max_i\{\alpha_i\},$$

and so our estimates give a bounding constant of order $O(1+t)$.

Another possible choice for the relaxation function is the "corrected power law" where

$$\varphi(t) := \varphi_0 + (t+\varphi_1)^{-\alpha},$$

for $\alpha \in (0,1)$, $\varphi_0 \geqslant 0$, $\varphi_1 > 0$ and such that $\varphi(0) = 1$. The case $\varphi_0 > 0$ corresponds to a viscoelastic solid while $\varphi_0 = 0$ gives a fluid. Also we require $\varphi_1 > 0$ in order to ensure that the material has a finite stiffness at $t = 0$. This time we again have $\Psi(t) = 1/\varphi(t) \leqslant 1/\varphi_0$ for a solid, and for a fluid $u(t) \sim 1 + \alpha t/\varphi_1$ since

$$\left\|\frac{\varphi'}{\varphi}\right\|_{C(0,t)} \leqslant \left\|\frac{\alpha}{t+\varphi_1}\right\|_{C(0,t)} = \frac{\alpha}{\varphi_1}.$$

Note that on physical grounds we disallow the "classical" power law

$$\varphi(t) := \varphi_0 t^{-\alpha}$$

(for $\varphi_0 > 0$ and $\alpha \in (0,1)$), because this implies that the material has infinite instantaneous stiffness. This is non-physical since there could then be no instantaneous deformation possible in response to an instantaneously applied finite load.

5.2. Isotropic materials. To model an isotropic material (for $\Omega \subset \mathbb{R}^2$) it is usual to take,

$$\boldsymbol{D}(t) := \begin{pmatrix} \lambda(t) + 2\mu(t) & \lambda(t) & 0 \\ \lambda(t) & \lambda(t) + 2\mu(t) & 0 \\ 0 & 0 & \mu(t) \end{pmatrix},$$

where λ and μ are now stress relaxation functions with $\mu'(t) < 0$ and which vary independently in time. It is always the case that $\mu > 0$. The independence of λ and μ models a time dependent Poisson ratio (such materials are sometimes termed *assynchronous*) and it is typical in applications to model each of λ and μ by a Prony series. The eigenvalues of $\boldsymbol{D}$ are μ, 2μ and $2(\lambda+\mu)$ and for $\boldsymbol{D}$ to be positive definite it is necessary only that $\lambda > -\mu$ which means λ can be negative. The orthogonal eigenvectors of $\boldsymbol{D}$ are

$$(0\ 0\ 1)^T, \qquad (-1\ 1\ 0)^T \qquad \text{and} \qquad (1\ 1\ 0)^T,$$

and are clearly independent of time—this leads to a simple result. Since the eigenvalues of $\boldsymbol{D}'(t)$ are obviously $\mu'(t)$, $2\mu'(t)$ and $2(\lambda'(t)+\mu'(t))$, while the eigenvectors are the same as for $\boldsymbol{D}$, we find that the eigenvalues of $\boldsymbol{B}(t)$ have the pleasingly simple form,

$$\omega_1 = -\frac{\lambda'(t)+\mu'(t)}{\lambda(0)+\mu(0)} \qquad \text{and} \qquad \omega_2 = \omega_3 = -\frac{\mu'(t)}{\mu(0)},$$

and so, using exactly the same reasoning as before we can then satisfy Assumptions 1, part (ii) by setting,

$$\phi(t) := \Big\| \max\big\{\omega_1(t), \omega_2(t)\big\} \Big\|_{L_\infty(\Omega)}$$

and then form either Ψ or Φ and arrive at the stability estimates in the same way as for the synchronous model.

Of course, in a specific application where we know more about λ and μ it may be possible to simplify the expression for ϕ and obtain a more explicit bound. To see how this would work we consider as an example the case where λ and μ are spatially constant, we have then that,

$$\phi(t) := \max\big\{\omega_1(t), \omega_2(t)\big\}.$$

To go further we assume, again for the sake of example, that $\omega_2(t) \geqslant \omega_1(t)$ for all $t \geqslant 0$. Then, if $\mu(t) \to \text{constant} > 0$ as $t \to \infty$ we have,

$$\Psi(t) = \left(1 + \int_0^t \frac{\mu'(s)}{\mu(0)}\, ds\right)^{-1} = \frac{\mu(0)}{\mu(t)} < \infty,$$

which corresponds to a solid. On the other hand, if $\mu \to 0$ as $t \to \infty$ we have a fluid and this result is useless. However,

$$\Phi(t) = \frac{-\mu'(t)/\mu(0)}{-\displaystyle\int_t^\infty \frac{\mu'(s)}{\mu(0)}\, ds} = -\frac{\mu'(t)}{\mu(t)}.$$

In the case where μ is modelled by either a Prony series or a corrected power law this can then be dealt with in the same way as before for the synchronous material.

For three dimensional problems we have the eigenvalues,

$$\omega_1 = -\frac{3\lambda'(t)+2\mu'(t)}{3\lambda(0)+2\mu(0)} \qquad \text{and} \qquad \omega_2 = \cdots = \omega_6 = -\frac{\mu'(t)}{\mu(0)},$$

we can then proceed exactly as above.

5.3. The general case. In the general case it would be necessary to use the bound in Lemma 18 directly since it hardly seems possible to give useful explicit forms for ϕ. This can be done in the numerical software during the adaptive solution, or *a priori* as soon as the material data are available (perhaps using a computer algebra package).

5.4. Semidiscrete error estimates. To apply the L_p-energy error estimate of Theorem 16 to this problem we need only match the notation again and define the strong norm $|||\cdot|||$. For approximation by piecewise polynomials of degree r it is appropriate to set $|||\cdot||| := |\cdot|_{\boldsymbol{H}^{r+1}(\Omega)}$ (the $\boldsymbol{H}^{r+1}(\Omega)$ seminorm), then we get,

$$\|\boldsymbol{u}-\boldsymbol{U}\|_{L_p(0,t;H)} \leqslant C_R \mathcal{S}(t) h^r |\boldsymbol{u}|_{L_p(0,t;\boldsymbol{H}^{r+1}(\Omega))},$$

with $\mathcal{S}(t)$ as defined in Theorem 16. Assuming Ω and the problem data such that the elliptic regularity estimate,

$$|\cdot|_{\boldsymbol{H}^2(\Omega)} \leqslant C_e \|A\cdot\|_{\boldsymbol{L}_2(\Omega)},$$

holds, then from Theorem 17 we also have the lifted estimate,

$$\|\boldsymbol{u}-\boldsymbol{U}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))} \leqslant C_R C_l (1+\|\phi\|_{L_1(0,t)}) \mathcal{S}^2(t) S(t) h^{r+1} |\boldsymbol{u}|_{L_p(0,t;\boldsymbol{H}^{r+1}(\Omega))}.$$

Finally, if $\boldsymbol{f} \in L_p(\mathcal{J};\boldsymbol{L}_2(\Omega))$, then for $r=1$,

$$\|\boldsymbol{u}-\boldsymbol{U}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))} \leqslant C_e C_R C_l (1+\|\phi\|_{L_1(0,t)}) \mathcal{S}^2(t) S^2(t) h^2 \|\boldsymbol{f}\|_{L_p(0,t;\boldsymbol{L}_2(\Omega))},$$

where $S(t)$ is as defined in Section 4.

6. Conclusion. In closing let us first remark that if we replace u by W in equation (1) and add the term $\varrho W''(t)$ to the left hand side then we arrive at the dynamic problem (here ϱ is the mass density of the viscoelastic body). One can then form an equation of type (1) for the error $W-u$ in the quasistatic approximation with the forcing function $f(t) := -\varrho W''(t)$. If a useful computable bound on W'' could be established then the stability estimates given above could be used to test the validity of the quasistatic computation.

Lastly, although the material in Section 5 focuses on a specific problem, it is likely that the abstract framework developed in the earlier sections could equally well be applied to quasistatic problems in viscoelastic beam, plate or shell theory.

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