Towards robust adaptive numerical methods for linear and nonlinear viscoelastic solid deformation

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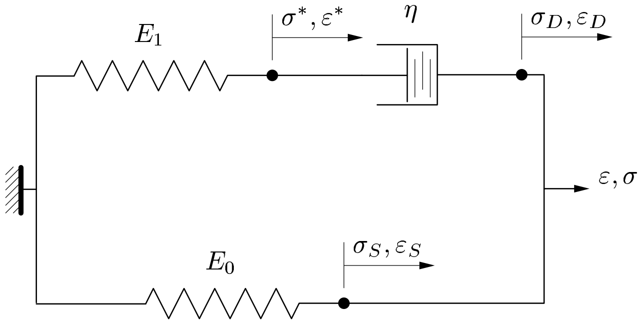
Quasistatic deformation of viscoelastic solid materials can be modelled mathematically using: equilibrium equations; the definition of strains in terms of derivatives of displacements; and, constitutive relations of history integral type. The form of the constitutive relationship is motivated by considering various **spring and dashpot** models which represent the particular solid- and fluid-like behaviour of viscoelastic materials.

THE MAXWELL SOLID



For the case of small strain (quasistatic) viscoelasticity the strain tensor $\underline{\varepsilon} = (\varepsilon_{ij})_{i,j=1}^3$ is defined componentwise by $\varepsilon_{ij}(\boldsymbol{u}) := (u_{i,j} + u_{j,i})/2$, where $\boldsymbol{u} = (u_i)_{i=1}^3$ is the displacement vector. The **Maxwell Solid** model shown in the figure leads to a constitutive relationship between stress $\underline{\boldsymbol{\sigma}} = (\sigma_{ij})_{i,j=1}^3$ and strain as

$$\begin{aligned}\sigma_{ij}(t) &= D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{u}(t)) \\ &\quad - \int_0^t \frac{\partial D_{ijkl}}{\partial s}(t-s)\varepsilon_{kl}(\boldsymbol{u}(s))\,ds,\end{aligned}$$

where $(D_{ijkl}(t))_{i,j,k,l=1}^3$ is a tensor of **stress relaxation functions**.

Use of this constitutive equation in the equilibrium equations leads to a deformation model of the type: find the displacement $\boldsymbol{u}$ such that

$$A\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{f}(\boldsymbol{x},t) + \int_0^t B(t-s)\boldsymbol{u}(\boldsymbol{x},s)\,ds,$$

where $\boldsymbol{x}$ is a point in the body, $\boldsymbol{f}$ is a given load, A and B are elliptic partial differential operators and t represents time.

One route to producing a numerical solution of this linear viscoelasticity problem is firstly to produce a spatially weak formulation by multiplying the differential equation by a (spatial) test function, and then integrating by parts over the domain of the body. One then produces a finite element semidiscretization in the usual way. Secondly, full discretization may be effected by applying a numerical quadrature rule (e.g. the trapezoidal rule) to the history integral. Such a scheme is proposed and analyzed in [4] where *a priori* error estimates are given. These estimates were subsequently improved (in terms of the constant) in [3], and a natural variant of the scheme was applied successfully to the small-strain, low-stress creep modelling of a certain Nylon 6.6 compound.

For larger strains and higher stresses this linear constitutive model is not adequate and one must look to nonlinear models. A scheme based on **reduced time** is employed in [2] where the long-time creep response of Nylon 6.6 is successfully modelled over a range of applied stresses. The reduced time variable responds to the time dependent changes in material properties due to dilatation effects caused by changes in **free volume**.

A major feature of modern computational methods is the inclusion of a **calculable** (i.e. *a posteriori*) error estimator which allows for **adaptive** numerical solution. Until recently this type of analysis has been sadly lacking in the viscoelasticity context but some early results are now available in [5]. This reference contains an *a posteriori* error estimate for a space-time finite element discretization of a model viscoelasticity problem. The method is based on the work and methods of Eriksson *et al.* described for example in [1]. This work is currently in its infancy but our intention is to extend it to model multidimensional reduced-time problems.

References

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