Toward robust adaptive finite element methods for partial differential Volterra equation problems arising in viscoelasticity theory.

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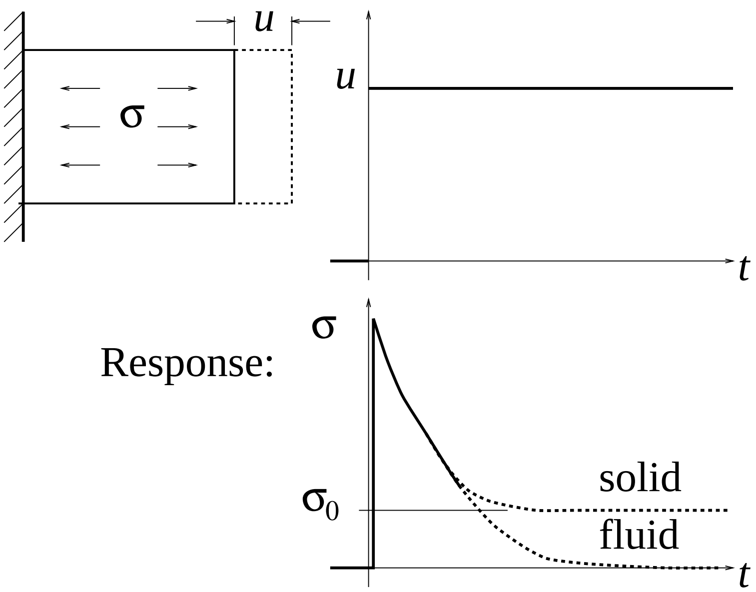
A brief introduction to how elliptic, parabolic and second-order hyperbolic partial differential equations with memory arise in modelling the response of linear viscoelastic materials is given. We focus on the elliptic-Volterra problem, and using the key assumption of fading memory—along with some comparison results from Volterra theory—we are able to by-pass the usual Gronwall lemma and give sharp data-stability estimates. These estimates also apply to a certain dual problem and this, coupled with the Galerkin orthogonality of the (space-time) finite element method, provides a route to a residual-based *a posteriori* Galerkin-error estimate. Such an estimate is derived and demonstrated numerically for a model problem in one space dimension.

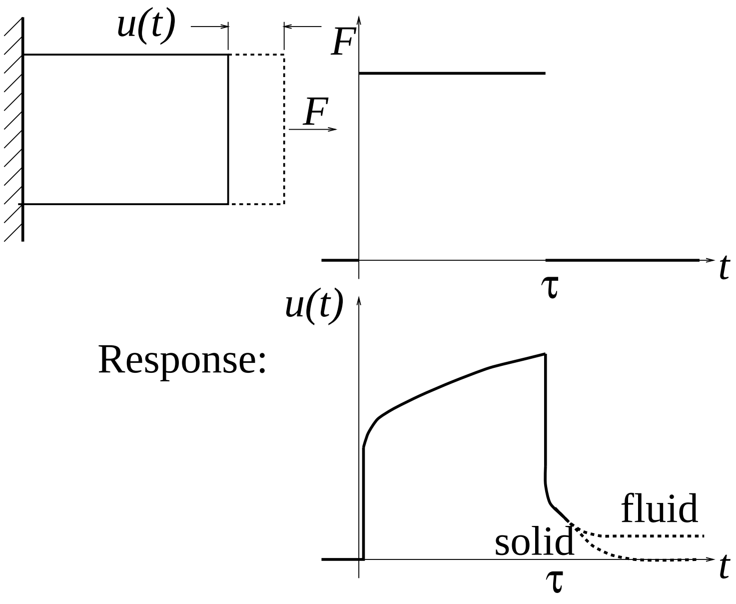
VISCOELASTICITY: STRESS RELAXATION & CREEP

Typical stress response of a 1-D viscoelastic bar subjected to an instantaneously applied, constant, elongation (strain) u:



stress relaxation.

Typical strain response of a 1-D viscoelastic bar subjected to an on-off step loading:



creep.

To relate stress, $\boldsymbol{\sigma}$, to strain, $\boldsymbol{\varepsilon}$, for this behaviour requires **Hooke's law with memory**. If $\boldsymbol{\varepsilon}(t) = 0\ \forall t < 0$:

$$\boldsymbol{\sigma}(t) = \boldsymbol{D}(0)\boldsymbol{\varepsilon}(t) - \int_0^t \boldsymbol{D}_s(t-s)\boldsymbol{\varepsilon}(s)\,ds.$$

(See Shaw, Warby & Whiteman, Proc. MAFELAP 1993.) Here $\boldsymbol{D}$ is the viscoelastic matrix analogue to the linear elastic Hooke's tensor. Simple form for a viscoelastic solid:

$$\boldsymbol{D}(t) = \varphi(t)\boldsymbol{D} \quad \text{with} \quad \varphi(t) = \varphi_0 + \sum_{i=1}^{N_\varphi} \varphi_i \exp(-\alpha_i t),$$

where: φ_i, $\alpha_i > 0$, $\varphi(0) = 1$ and $\boldsymbol{D}$ is a constant matrix.

Note the **fading memory**:

$$1 - \int_0^t \varphi_s(t-s)\,ds = \varphi(t) \in [\varphi_0, 1] \qquad \forall t \in [0, \infty].$$

Suppose a viscoelastic body occupies the bounded domain $\Omega \subset \mathbb{R}^n$, and is subjected to a system of body forces $(f_i)_{i=1}^n$. Newton's second law gives the momentum balance:

$$\varrho \ddot{u}_i(t) - \sum_{j=1}^n \frac{\partial \sigma_{ij}}{\partial x_j}(t) = f_i(t),$$

for $i = 1, \ldots, n$ and for displacements $\boldsymbol{u} = (u_i)_{i=1}^n$.

PARTIAL DIFFERENTIAL VOLTERRA EQUATIONS IN VISCOELASTICITY I

Using the small strain assumption: $\varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2$, and substituting the hereditary constitutive relationship for $\boldsymbol{\sigma}$ into the momentum balance, we obtain an example of the generic **hyperbolic-Volterra** problem: find u such that

$$\ddot{u}(t) + Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds, \qquad \text{(HV)}$$

$+$ initial and boundary data. (See: Golden & Graham, *Boundary value problems in linear viscoelasticity*, Springer-Verlag, 1988.)

- **Recent numerical analysis of (HV):**

'91: Lin, Thomée & Wahlbin, SIAM JNA, 28. Detailed study of Ritz-Volterra projection. Semidiscrete *a priori* error estimates given.

'92: Pani, Thomée & Wahlbin, J. Integral Equations Appl., 4. *A priori* estimates for finite-element-in-space, finite-difference-and-sparse-quadrature-in-time discretization.

'96: Lubich, Sloan & Thomée, Math. Comp., 65. Integrate by parts in time to reformulate (HV) with $\dot{u}$ the "unknown".

PARTIAL DIFFERENTIAL VOLTERRA EQUATIONS IN VISCOELASTICITY II

If the motion of the body is quasistatic, that is $\ddot{\boldsymbol{u}} \approx 0$, then (HV) reduces to an example of the generic **elliptic-Volterra** problem: find u such that

$$Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds, \qquad \text{(EV)}$$

$+$ boundary data. Typically this leads to the numerical scheme:

$$\Big(\boldsymbol{A} - \varpi_{ii}\boldsymbol{B}(t_i,t_i)\Big)\boldsymbol{U}_i = \boldsymbol{F}(t_i) + \sum_{j=0}^{i-1} \varpi_{ij}\boldsymbol{B}(t_i,t_j)\boldsymbol{U}_j,$$

where the ϖ_{ij} are quadrature weights.

- **Recent numerical analysis of (EV):**

'94: Shaw, Warby, Whiteman, Dawson & Wheeler, Comp. Meth. Appl. Mech. Eng., 118. Finite-element-in-space, trapezoidal-rule-in-time scheme with *a priori* energy-error estimates.

'95: Shaw, Warby & Whiteman, SIAM JNA, to appear. As above but with non-exponentially large constants in the *a priori* bounds.

PARTIAL DIFFERENTIAL VOLTERRA EQUATIONS IN VISCOELASTICITY III

To model diffusion in polymers Fick's law is modified to

$$\boldsymbol{J} = -D\nabla u - E\nabla\left(\mu\int_0^t \exp\left(-\int_s^t \beta(\tau)\,d\tau\right)u(s)\,ds\right),$$

where u is the local flux. Using this in the mass conservation equation $\dot{u} = -\nabla\cdot\boldsymbol{J}$ we arrive at an example of the generic **parabolic-Volterra** problem: find u such that

$$\dot{u}(t) + Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds, \qquad \text{(PV)}$$

+ initial and boundary data. (Cohen *et al.*, SIAM JAM, 55.)

- **Recent numerical analysis of (PV):**

'86: Sloan & Thomée, SIAM JNA, 23. Linear problem. Special attention paid to deriving **sparse quadrature** rules.

'90: Cannon & Lin, SIAM JNA, 27. *A priori* estimates for a nonlinear problem. **Ritz-Volterra** projection generalizes the usual elliptic projection.

'94: Thomée & Wahlbin, Math. Comp., 62. Linear problem with fading memory Volterra kernel assumed. *A priori* estimates with non-exponentially large constants are given.

PARTIAL DIFFERENTIAL VOLTERRA EQUATIONS IN VISCOELASTICITY IV

In summary, the PDV equation problems we are concerned with take one of the generic forms. Find u solving either: the **elliptic Volterra** problem,

$$Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds, \qquad \text{(EV)}$$

or the **parabolic-Volterra** problem,

$$\dot{u}(t) + Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds, \qquad \text{(PV)}$$

or the **hyperbolic-Volterra** problem,

$$\ddot{u}(t) + Au(t) = f(t) + \int_0^t B(t,s)u(s)\,ds. \qquad \text{(HV)}$$

With boundary and initial data as appropriate, and where A and $B(t,s)$ are either spatial differential operators, with A elliptic, or, for prototype studies of pure-time problems, functions of time only.

NEED FOR ADAPTIVE PDV SOLVERS

The "N^2 problem". Typical discretizations of the Volterra term involve a summation over previous times. To march to time t_N requires about $1+2+\cdots+N = O(N^2)$ operations.

- Sloan & Thomée's sparse quadrature gives $O(N\sqrt{N})$ or smaller, but needs a very smooth integrand.
- Keep N small without trading accuracy $\Rightarrow$ **adaptivity!**
- Adaptive time steppers for pure-time (EV) problem:

  '85 Jones & McKee, Math. Comp., 44. Linear Multistep.

  '87 Blom & Brunner, SIAM JSSC, 8. Collocation.

The canonical elliptic, parabolic and wave equations each have a physically relevant Volterra analogue.
Very little work done on *a posteriori* error estimation and adaptivity for space-time problems. To solve physical problems efficiently and reliably requires adaptive computations predicated on *a posteriori* error estimates.

- **Goal:** adaptive algorithms for (EV), (PV) and (HV), with particular regard to viscoelasticity problems.
- **Departure point:** the finite element methodology of Eriksson, Johnson *et al.*

Consider the abstract problem: find u such that

$$Au = f \qquad \text{in } \Omega.$$

Let $(\cdot,\cdot)$ denote the $L_2(\Omega)$ inner product and define the weak formulation: find $u \in V$ such that

$$(Au, v) = (f, v) \quad \forall v \in V = \text{ a suitable function space,}$$

and the finite element scheme: find $U \in V^h \subset V$ such that

$$(AU, v) = (f, v) \quad \forall v \in V^h.$$

Let $e := u - U$ and $r := Ae = f - AU$ be respectively the Galerkin error and residual and note the Galerkin-orthogonality:

$$(r, v) = (Ae, v) = 0 \quad \forall v \in V^h.$$

Create the dual problem: find $\chi \in V$ such that

$$(v, A^*\chi) = (v, g) \quad \forall v \in V,$$

where A^* is defined by $(\cdot, A^*\cdot) = (A\cdot, \cdot)$, set $\|\cdot\|^2 := (\cdot,\cdot)$ and assume the data-stability estimate holds:

$$\|D^p\chi\| \leqslant C_s\|g\|, \qquad \text{for an integer } p \geqslant 0.$$

THE ERIKSSON-JOHNSON *et al.* ERROR ESTIMATE
(continued)

Also, assume that V^h has the approximation property:

$$\left\| h^{-p}(I-\pi)\chi \right\| \leqslant C_i \|D^p \chi\| \quad \forall \chi \in V,$$

where $\pi\chi \in V^h$ is an interpolant to χ.

Taking $v = g = e$ in the dual problem gives:

$$\begin{aligned}
(v,g) = \|e\|^2 = (e, A^*\chi) = (Ae,\chi) &= (r, \chi - \pi\chi) \\
&= (h^p r, h^{-p}(I-\pi)\chi) \\
&\leqslant \|h^p r\| \, \|h^{-p}(I-\pi)\chi\| \\
&\leqslant C_i \, \|h^p r\| \, \|D^p \chi\| \\
&\leqslant C_s C_i \, \|h^p r\| \, \|e\|
\end{aligned}$$

$$\Longrightarrow \quad \|u - U\| \leqslant C_s C_i \, \|h^p r\| \, .$$

This is the basis of a rigorous, non-asymptotic *a posteriori* Galerkin-error estimate which, if $p > 0$, may be used to rationally select the discretization function h so as to control the Galerkin-error $\|u - U\|$ on a user-specified tolerance level.

- Quadrature error is an issue.

(See: C. Johnson, Proc. MAFELAP 1993.)

A posteriori TEMPORAL ESTIMATES FOR (EV)

A temporally constant spatial finite element discretization of (EV) results in the Volterra system of the second kind:

$$\boldsymbol{A}\boldsymbol{U}(t) = \boldsymbol{F}(t) + \int_0^t \boldsymbol{B}(t,s)\boldsymbol{U}(s)\,ds.$$

Consider the prototype scalar problem: find $u \in L_2(0,T)$ such that

$$\begin{aligned} Lu(t) := u(t) - \Lambda u(t) = &f(t) \\ \text{where} \quad \Lambda u(t) := &\int_0^t \phi(t,s)u(s)\,ds. \end{aligned}$$

Set $V := L_2(0,T)$ and let V^h be a finite element space of piecewise constant functions with their discontinuities at $\{t_i\}_{i=1}^{N-1} \subset (0,T)$.

→ "weak" problem: find $u \in V$ such that

$$(u,v) - (\Lambda u, v) = (f,v)\ \forall v \in V.$$

→ F.E. problem: find $U \in V^h$ such that

$$(U,v) - (\Lambda U, v) = (f,v)\ \forall v \in V^h \subset V.$$

→ Galerkin orthogonality: with $r := f - U + \Lambda U$ as the residual we get $(Le,v) = (r,v) = 0\ \forall v \in V^h$.

→ Dual backward problem: find $\chi \in V$ such that

$$(v,\chi) - (v,\Lambda^*\chi) = (v,g)\ \forall v \in V,$$

$g \in V$ given and Λ^* defined by $(v,\Lambda^*\chi) = (\Lambda v,\chi)$.

A posteriori TEMPORAL ESTIMATES FOR (EV) *(continued)*

→ Weak data-stability: $\|\chi\| \leqslant C_s\|g\|$ for all $g \in V$.

→ Interpolation estimate: Let $\pi\chi \in V^h$ be the L_2 projection of χ, then $\|\chi - \pi\chi\| \leqslant C_i\|\chi\|$.

→ Take $v = g = e := u - U$ then:

$$\begin{aligned}\|e\|^2 = (e, g) = (e, \chi - \Lambda^*\chi) &= (r, \chi - \pi\chi) \\ \leqslant C_i\|r\|\|\chi\| &\leqslant C_iC_s\|r\|\|e\|.\end{aligned}$$

For $p \in [1, \infty]$ more subtle reasoning yields:

$$\|e\|_{L_p(0,t_i)} \leqslant C_{ip}C_{sp}\|r\|_{L_p(0,t_i)} \quad \text{for } i = 1, \ldots, N.$$

(See: Shaw & Whiteman, Numer. Math., to appear.)

- Lack of **<u>strong data stability</u>** in dual problem means "h" does not explicitly appear in the bound. Nonetheless, valuable feedback on the L_p-error is available.
- For viscoelasticity a sharp *a priori* bound on C_s is known.
- An *a priori* error estimate is also given in the reference.

A posteriori TEMPORAL ESTIMATES FOR (PV)

A temporally constant spatial finite element discretization of (PV) results in the ODE system with memory:

$$\dot{\boldsymbol{U}}(t) + \boldsymbol{A}\boldsymbol{U}(t) = \boldsymbol{F}(t) + \int_0^t \boldsymbol{B}(t,s)\boldsymbol{U}(s)\,ds. \qquad (\dagger)$$

Consider the prototype scalar problem: find $u \in W^1_\infty(0,T)$ such that

$$Lu := \dot{u}(t) + \sigma u(t) - \int_0^t \phi(t,s)u(s)\,ds = f(t),$$

with $u(0) = u_0$. A finite element discretization in $(0,T)$ using piecewise constant trial and test functions yields a Galerkin variant of the backward Euler method. For this the *a posteriori* Galerkin-error estimate holds:

$$|e(t_i)| \leqslant C_s^1 \max_{1\leqslant j\leqslant i} \left\{k_j\|r\|_{L_\infty(t_{j-1},t_j)} + |U_j - U_{j-1}|\right\}.$$

$U_j \approx u(t_j)$ is the finite element approximation, $e := u - U$ the error, $k_j := t_j - t_{j-1}$ and r "=" Le the residual.

Also, C_s^1 is a **strong** data stability constant such that $\|\dot{\chi}\|_{L_1(0,t_i)} \leqslant C_s^1|e_i|$ where χ solves the dual backward problem $L^*\chi = 0$ in $(0,t_i)$ subject to $\chi(t_i) := e_i$.

A posteriori TEMPORAL ESTIMATES FOR (PV) *(continued)*

Using continuous piecewise linear trial functions and retaining the test space used above results in a Galerkin variant of the Crank-Nicolson method. In this case we have

$$|e(t_i)| \leqslant C_s^1 \max_{1 \leqslant j \leqslant i} \left\{ k_j \|r\|_{L_\infty(t_{j-1},t_j)} \right\}.$$

(See: Shaw & Whiteman, Math. Comp., submitted.)

- Numerical experiments show these bounds to be sharp, reliable and useful.
- These bounds explicitly contain the time steps $\{k_j\}$ on the right which implies that one can guarantee $|e(t_i)| \leqslant$ TOL by choosing k_i small enough.
- An *a priori* error estimate is also given in the reference.
- A complete analysis of quadrature errors is given in the reference.
- Semidiscrete (HV) can be written as a system of type (†).
- For fading memory problems a sharp weak stability bound is given in Thomée & Wahlbin, Math. Comp., 62.

A MODEL PROBLEM OF TYPE (EV)

Find u such that

$$Au(t) = f(t) + \Lambda u(t) \text{ in } Q := \Omega \times \mathcal{J}, \qquad \text{(MEV)}$$

where: $\Omega := (0,1)$; $\mathcal{J} := [0,T]$;

$$\begin{aligned} f &\in L_\infty(\mathcal{J}; L_2); \\ Au(t) &:= -Pu''(x,t); \\ B(t,s)u(x,s) &:= -P_B(t,s)u''(x,s); \\ \Lambda u(t) &:= \int_0^t B(t,s)u(x,s)\,ds; \end{aligned}$$

$0 \leqslant s \leqslant t \in \mathcal{J}$; $x \in \Omega$; $P > 0$; and, the prime denotes x-differentiation.

We supply this differential-Volterra problem with the mixed boundary conditions:

$$\begin{aligned} &u(0,t) = 0, \\ &Pu'(1,t) - \int_0^t P_B(t,s)u'(1,s)\,ds = f_\Gamma(t) \in L_\infty(\mathcal{J}). \end{aligned}$$

for all $t \in \mathcal{J}$.

SEMI-WEAK FORMULATION OF (EV)

Usual numerical schemes for (MEV) are based on finite element spatial discretization. Define the trial/test space:

$$\mathcal{H} := \{ v \in H^1(\Omega) : v(0) = 0 \},$$

then the "spatially weak" formulation of (MEV) is:
find $u \in L_\infty(\mathcal{J}; \mathcal{H})$ such that

$$A(u(t), v) = (f(t), v) + v(1) f_\Gamma(t) + \int_0^t B(t, s; u(s), v)\, ds$$

for all $v \in \mathcal{H}$, a.e. in $\mathcal{J}$.

Here:

$$A(v, w) := \int_0^1 P v' w' \, dx;$$
$$B(t, s; v, w) := \int_0^1 P_B(t, s) v' w' \, dx.$$

$A(\cdot, \cdot)$ is a continuous and $\mathcal{H}$-coercive bilinear form and the "Energy" norm,

$$\| \cdot \|_{\mathcal{H}} := \sqrt{A(\cdot, \cdot)}$$

is equivalent to $\| \cdot \|_{H^1(\Omega)}$.

Seeking u in $L_\infty(\mathcal{J};\mathcal{H})$ seems natural because there are no time derivatives in (MEV). Such functions do not have unique pointwise values—the "fully weak" problem is:
find $u \in L_\infty(\mathcal{J};\mathcal{H})$ such that

$$a(u,v) = L(v) \quad \forall v \in L_1(\mathcal{J};\mathcal{H}).$$

Here:

$$a(u,v) := \int_0^T \Big(A\big(u(t),v(t)\big) - \int_0^t B\big(t,s;u(s),v(t)\big)\,ds \Big)\,dt,$$
$$L(v) := \int_0^T \Big(\big(f(t),v(t)\big) - v(1,t) f_\Gamma(t) \Big) dt.$$

Dual problem: find $\chi \in L_1(\mathcal{J};\mathcal{H})$ such that

$$a^*(\chi,v) = L^*(v) \quad \forall v \in L_\infty(\mathcal{J};\mathcal{H}).$$

Here, with $g \in L_1\big(\mathcal{J};L_2(\Omega)\big)$ and $g_\Gamma \in L_1(\mathcal{J})$ given:

$$a^*(\chi,v) := \int_0^T A\big(v(t),\chi(t)\big) - \int_t^T B\big(s,t;v(t),\chi(s)\big)\,ds\;dt,$$
$$L^*(v) := \int_0^T \big(v(t),g(t)\big) - g_\Gamma(t)v(1,t)\,dt,$$

$$\boxed{a^*(w,v) = a(v,w) \; \forall v \in L_\infty(\mathcal{J};\mathcal{H}),\, \forall w \in L_1(\mathcal{J};\mathcal{H}).}$$

The prototype convolution Volterra problem of the second kind is: find m such that

$$m(t) = n(t) + (\phi * m)(t) \qquad \text{for } t \in \mathcal{J}, \qquad (\dagger)$$
$$\text{where} \quad (\phi * m)(t) := \int_0^t \phi(t-s)m(s)\,ds.$$

For this problem we have the following two propositions. (For proofs see Prop. **9**.8.1, Lemma **9**.8.2 and Theorems **2**.3.1, **2**.3.5 in: Gripenberg, Londen & Staffans, Volterra integral and functional equations, CUP, 1990.)

Existence. *If* $\phi \in L_1(\mathcal{J})$, $\exists!\, R \in L_1(\mathcal{J})$ *such that* $R = \phi + \phi * R$ *and* $R = \phi + R * \phi$ a.e. *in* $\mathcal{J}$. *For any* $p \in [1, \infty]$ *and* $n \in L_p(\mathcal{J})$, $\exists!\, m \in L_p(\mathcal{J})$ *to* $(*)$ *given by the variation of constants formula:* $m = n + R * n$.

Comparison theorem. *For real* $p \in [1, \infty]$ *and for* $\phi \in C\big(\mathcal{J}; [0, \infty)\big)$ *let* m, $n \in L_p\big(\mathcal{J}; [0, \infty)\big)$ *satisfy the Volterra inequality* $m(t) \leqslant n(t) + (\phi * m)(t)$ a.e. *in* $\mathcal{J}$. *Then* $m(t) \leqslant M(t)$ a.e. *in* $\mathcal{J}$ *where* M *solves the comparison equation* $M = n + \phi * M$ *in* $\mathcal{J}$.

These results, with fading memory, prove **sharp** data-stability estimates for the dual quasistatic viscoelasticity problem.

Definition *We call a function* $\phi \in C\big(\mathcal{J};[0,\infty)\big)$ *a* function with fading memory *if*

$$\Phi(t) := \left(1 - \int_0^t \phi(s)\,ds\right)^{-1} \in (0,\infty) \quad \forall t \in \mathcal{J}.$$

The nondecreasing function Φ *is called the* long term effect *of* ϕ.

If $*$ denotes convolution, then we have the following stability estimates for $(\dagger)$ in this fading memory context:

Lemma. *If* $0 \leqslant m(t) \leqslant G(t) + (\phi * m)(t)$ *a.e. in* $\mathcal{J}$ *with:*

- $G \in L_\infty\big(\mathcal{J};[0,\infty)\big)$ *nondecreasing;*
- ϕ *a function with fading memory.*

Then $\|m\|_{L_\infty(0,t)} \leqslant G(t)\Phi(t)$ *for all* $t \in \mathcal{J}$.

Lemma. *For* m, $n \in L_1(\mathcal{J})$ *and* ϕ *a function with fading memory related by* $m = n + \phi * m$ *in* $\mathcal{J}$, *we have* $\|m\|_{L_1(0,t)} \leqslant \Phi(t)\|n\|_{L_1(0,t)}$ *in* $\mathcal{J}$.

In solid viscoelasticity: $\Phi(t) \leqslant 1/\varphi_0$ independently of $\mathcal{J}$.
Gronwall's inequality not required.

Theorem. *Assume that $B(t,s)$ is similar to A in the following two senses. For all v, $w \in \mathcal{H}$ and a.e. in $\{0 \leqslant s \leqslant t \in \mathcal{J}\}$:*

$$\|B(t,s)v\| \leqslant \phi(t-s)\|Av\|, \tag{a}$$

$$|B(t,s;w,v)| \leqslant \phi(t-s)\|w\|_{\mathcal{H}}\|v\|_{\mathcal{H}}, \tag{b}$$

where ϕ is a function with fading memory with long term effect Φ.

Then, for (MEV):

$$\left.\begin{array}{l} \|u\|_{L_p(0,t;L_2(\Omega))} \\ \|u\|_{L_p(0,t;\mathcal{H})} \\ \|u''\|_{L_p(0,t;L_2(\Omega))} \end{array}\right\} \sim \Phi(t) \left(\begin{array}{c} \|f\|_{L_p(0,t;L_2(\Omega))} \\ + \quad \sqrt{2}\|f_\Gamma\|_{L_p(0,t)} \end{array} \right).$$

Similar results hold for the dual backward problem:

$$\left.\begin{array}{l} \|\chi\|_{L_p(T-t,T;L_2(\Omega))} \\ \|\chi\|_{L_p(T-t,T;\mathcal{H})} \\ \|\chi''\|_{L_p(T-t,T;L_2(\Omega))} \end{array}\right\} \sim \Phi(t) \left(\begin{array}{c} \|g\|_{L_p(T-t,T;L_2(\Omega))} \\ + \quad \sqrt{2}\|g_\Gamma\|_{L_p(T-t,T)} \end{array} \right).$$

For all $t \in \mathcal{J}$ and for $p = 1$ or $p = \infty$ if f, g, f_Γ and g_Γ are regular enough.

Partition $\mathcal{J} = [0, T]$ into N disjoint intervals $\mathcal{J}_i = (t_{i-1}, t_i)$, each of length $k_i := t_i - t_{i-1} > 0$, and such that

$$\mathcal{J} = \overline{\mathcal{J}_1 \cup \mathcal{J}_2 \cup \cdots \cup \mathcal{J}_N},$$

with $t_0 = 0$ and $t_N = T$. For each $\mathcal{J}_i$ partition $\Omega = \Omega^i$ into M^i intervals $\{\Omega^i_j := (x_{j-1}, x_j)\}_{j=1}^{M^i}$, each of length $h_{ij} := x_j - x_{j-1} > 0$, and such that

$$\overline{\Omega} = \overline{\Omega^i_1 \cup \Omega^i_2 \cup \cdots \cup \Omega^i_{M^i}}$$

$\forall i \in \mathbb{N}(1, N) := \{1, \ldots, N\}$, and $x_0 = 0$ and $x_{M^i} = 1$.

- $\mathbb{P}_r(I, K) = \Big\{ r^{\text{th}}$ degree polynomials on I, valued in $K \Big\}$.
- Semidiscrete (spatial) f.e. spaces $\mathcal{H}_i \subseteq \mathcal{H}$:

  $$\mathcal{H}_i := \{v \text{ continuous, p.w. linear on } \Omega^i\} \ \forall i \in \mathbb{N}(1, N).$$

- Space-time finite element spaces:

  $$V_r := \Big\{ v : v|_{\mathcal{J}_i} \in \mathbb{P}_r(\mathcal{J}_i; \mathcal{H}_i) \quad \forall i \in \mathbb{N}(1, N) \Big\},$$

- dG(0)cG(1) finite element approximation to (MEV):

  $$\text{find } U \in V_0 \text{ s.t.} \quad a(U, v) = L(v) \quad \forall v \in V_0.$$

- Galerkin orthogonality:

  $$a(u - U, v) = 0 \quad \forall v \in V_0.$$

To seperate space and time errors in the estimate we define

Definition. *For $w \in L_\infty(\mathcal{J};\mathcal{H})$ we define the* Volterra projection *of w in the following piecewise manner:*

$$(Y_i w(t), v) = -A(w(t), v) + v(1) f_\Gamma(t) + \int_0^t B(t, s; w(s), v)\, ds,$$

$\forall v \in \mathcal{H}_i$, a.e. in $\mathcal{J}_i$ and $\forall i \in \mathbb{N}(1, N)$.

$\widetilde{U}|_{\mathcal{J}_i} \in L_\infty(\mathcal{J}_i;\mathcal{H}_i)$ is a semidiscrete approximation to $u|_{\mathcal{J}_i}$ if

$$A(\widetilde{U}(t), v) = (f(t), v) + v(1) f_\Gamma(t) + \int_0^t B(t, s; \widetilde{U}(s), v)\, ds$$

for all $v \in \mathcal{H}_i$. Using this we obtain:

Proposition. *Let $\mathcal{P}_i : L_2(\Omega) \to \mathcal{H}_i$ denote the $L_2(\Omega)$ projection on to $\mathcal{H}_i$ for each $i \in \mathbb{N}(1, N)$. Then:*

- *$-Y_i \widetilde{U}(t) = \mathcal{P}_i f(t)$ a.e. in $\mathcal{J}_i$ for each i;*
- *$(Y_i(U - \widetilde{U}), v) = (Y_i U + \mathcal{P}_i f, v)$ $\forall v \in \mathcal{H}_i$, a.e. in $\mathcal{J}_i$.*

Thus we may study the "time error" $U - \widetilde{U}$ in isolation via the known quantities U and f.

- "Mass lumping" could be used here.

Lemma. *For each* $i \in \mathbb{N}(1, N)$ *let* $P_i : C(\overline{\Omega}) \to \mathcal{H}_i$ *denote the spatial-nodal* $\mathcal{H}_i$*-interpolant map. Define*

- θ *such that* $\theta|_{\mathcal{J}_i} = \theta_i := \chi - P_i\chi \in L_1(\mathcal{J}_i; \mathcal{H})$,
- ρ *such that* $\rho|_{\mathcal{J}_i} = \rho_i := P_i\chi - LP_i\chi \in L_1(\mathcal{J}_i; \mathcal{H}_i)$,

where L *is the* L_2 *projection (in time) on to* V_0*. Then for* $e := u - U$*:*

$$L^*(e) = \int_0^T \Big(f - AU + \int_0^t B(t,s)U(s)\,ds, \theta\Big)\,dt + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} (\mathcal{P}_i f + Y_i U, \rho_i)\,dt,$$

where $AU(t)$, $B(t,s)U(s)$ *must be interpreted as smooth functions in a piecewise manner on each space mesh.*

Proof. Take $v = e \in L_\infty(\mathcal{J}; \mathcal{H})$ in the dual problem. Duality and Galerkin orthogonality imply:

$$\begin{aligned} L^*(e) &= a^*(\chi, e) = a(e, \chi) = a(e, \chi - \pi\chi) \\ &= L(\chi - \pi\chi) - a(U, \chi - \pi\chi) \qquad \forall \pi\chi \in V_0. \end{aligned}$$

Choose $\pi\chi|_{\mathcal{J}_i} = LP_i\chi$ for all $i \in \mathbb{N}(1, N)$ then

$$L^*(e) = \Big[L(\theta) - a(U, \theta)\Big] + \Big[L(\rho) - a(U, \rho)\Big] = \mathsf{I} + \mathsf{II}.$$

For I integrate by parts over each spatial element, for II use the Volterra projection.

Using some interpolate stability and error estimates we have:

Theorem. *Impose the mesh condition $\mathcal{H}_{i-1} \subseteq \mathcal{H}_i$ for all $i \in \mathbb{N}(2, N)$, then for all $m \in \mathbb{N}(1, N)$ the Galerkin solution satisfies the a posteriori error bound:*

$$\|u - U\|_{L_\infty(0,t_m;L_2(\Omega))} \leqslant \max_{1 \leqslant i \leqslant m} \left\{ \mathcal{C}_h(t_i)\mathcal{E}_i^h + \mathcal{C}_k(t_i, \hat{h}_i)\mathcal{E}_i^k \right\},$$

where,

$$\begin{aligned} &(a) \quad \mathcal{E}_i^h := \|h_i^2(f - AU + \Lambda U)\|_{L_\infty(\mathcal{J}_i;L_2(\Omega))}, \\ &(b) \quad \mathcal{E}_i^k := \|\mathcal{P}_i f + Y_i U\|_{L_\infty(\mathcal{J}_i;L_2(\Omega))}, \end{aligned}$$

and $f - AU + \Lambda U$ is interpreted piecewise on each space mesh. In addition:

$$\left.\begin{array}{c} \mathcal{C}_h(t_m) \\ \mathcal{C}_k(t_m, \hat{h}_m) \end{array}\right\} \sim \Phi(t),$$

$$\hat{h}_m := \max\left\{h_{ij} : j \in \mathbb{N}(1, M^i) \text{ and } i \in \mathbb{N}(1, m)\right\},$$

- The time dependence of the constants is known from the fading memory estimates.
- The difficulty in higher space dimensions lies in finding the elliptic regularity constant.
- For energy estimates we need the Poincare (Friedrich or Korn) constant.

Let M^i and h_i be the number of elements and the meshwidth function at the time slab $\mathcal{J}_i$. Equidistribution of the error leads to a new mesh width for element j given by:

$$h_{ij}^{\mathrm{new}} = \sqrt{\frac{\mathsf{TOL}_\Omega}{\mathcal{C}_h(t_i)\sqrt{M^i_{\mathrm{old}}\int_{\Omega^i_j}\|\mathcal{R}_h\|^2_{L_\infty(\mathcal{J}_i;L_2)}\,dx}}},$$

where TOL_Ω is a user-specified tolerance and, interpreted as a bounded discontinuous function: $\mathcal{R}_h := f - AU + \Lambda U$.

Test results when solving a problem for which the time discretization is exact. The error is over estimated by a factor of around 13

TOL_Ω	$\widehat{M}$	$\|e\|_{L^h_\infty(\mathcal{J};L^h_2(\Omega))}$	$\max_i\{C_h(t_i)\mathcal{E}^h_i\}$	$\max_i\{C_k(t_i)\mathcal{E}^k_i\}$
2.500(−1)	11	5.728(−3)	7.374(−2)	2.005(−13)
6.250(−2)	20	1.540(−3)	2.069(−2)	3.436(−13)
1.563(−2)	41	3.233(−4)	4.336(−3)	1.355(−12)
3.906(−3)	78	9.807(−5)	1.251(−3)	3.219(−11)
9.766(−4)	158	3.003(−5)	2.911(−4)	1.823(−10)
2.441(−4)	312	1.674(−5)	7.548(−5)	3.717(−10)
6.104(−5)	629	1.393(−6)	1.854(−5)	1.955(−9)

The error estimate does not provide an obvious means by which a new time step can be selected. We assume the bound is proportional to the current time step k_i and allow a growth in the user-specified tolerance $\mathsf{TOL}_{\mathcal{J}}$ governed by the stability factor. The new time step is thus chosen as:

$$k_i^{\text{new}} = \gamma k_i \frac{\mathsf{tol}_{\mathcal{J}}}{\mathcal{C}_k(t_i)\mathcal{E}_i^k} \quad \text{where} \quad \mathsf{tol}_{\mathcal{J}} := \mathsf{TOL}_{\mathcal{J}}\Phi(t)/\Phi(\infty).$$

Here $\gamma \in (0, 1]$ is a safety factor.

Test results when solving a problem for which the space discretization is exact. The error is over estimated by a factor of around 40 for $\gamma = 0.9$.

$\mathrm{TOL}_{\mathcal{J}}$	$\|e\|_{L^h_\infty(\mathcal{J};L^h_2(\Omega))}$	$\max_i\{C_k(t_i)\mathcal{E}_i^k\}$	time (hours)
$4.000(-1)$	$9.987(-3)$	$3.558(-1)$	0.012
$2.000(-1)$	$4.724(-3)$	$1.772(-1)$	0.052
$1.000(-1)$	$2.236(-3)$	$8.842(-2)$	0.212
$5.000(-2)$	$1.090(-3)$	$4.417(-2)$	0.86
$2.500(-2)$	$5.405(-4)$	$2.207(-2)$	3.22
$1.250(-2)$	$2.669(-4)$	$1.103(-2)$	12.85

- The timings indicate how important the N^2 problem is.

FUTURE WORK

- Obtain *a posteriori* estimates for **computed** solution by estimating quadrature errors.
- Consider, and account for, **mass lumping** in $\mathcal{E}_i^k$.
- Use the $L_2(\Omega)$ projection instead of the nodal interpolant in the error representation formula, this will:
  - remove the mesh constraint $\mathcal{H}_{i-1} \subseteq \mathcal{H}_i$;
  - remove dependence of $\mathcal{C}_k$ on $\hat{h}_m$;
  - introduce derivative "jumps" at element interfaces.
- Derive energy-norm estimates:
  - appropriate to higher space-dimensional problems;
  - *a priori* estimates on the Korn's constants are available for some geometries, see Horgan, SIAM Rev, 37.
- Provide *a priori* error analysis. This may reveal extra requirements for convergence, e.g. $\nabla h \leqslant$ constant.
- Derive and implement dG(1)cG(1).
- Can Sloan & Thomée's sparse quadrature techniques be applied? Numerical tests suggest **yes**.