Error Estimates with Sharp Constants for a Fading Memory Volterra Problem in Linear Solid Viscoelasticity

S. Shaw, M. K. Warby & J. R. Whiteman,
Brunel University, Uxbridge, England.

Abstract

The problem characterizing non-ageing linear isothermal quasistatic isotropic compressible solid viscoelasticity in the time interval $[0, T]$ is described. This is essentially a Volterra equation of the second kind arrived at by adding smooth fading memory to the elliptic linear elasticity equations. We analyze the errors resulting from: replacing the relaxation functions with practical approximations; semidiscrete finite element formulation; and, a fully discrete scheme derived by replacing the hereditary integral with the trapezoidal rule for numerical integration. The error estimates are sharp in the sense that if certain bounds on the data are independent of T then so also are the constants involved in them. This is a consequence of by-passing the usual Gronwall lemmas with comparison theorems that are more sensitive to the fading memory of the physical problem. Coupling these error bounds to weak stability estimates for a model problem we then investigate, by way of numerical experimentation, the possibility of providing a computable error bound for the time discretization error.

AMS(MOS) 1991 subject classifications: Primary 65M60, 73F15, 65M15. Secondary 45D05.

Keywords: Volterra equation, finite element method, fading memory, error estimate, comparison theorem, Gronwall, viscoelasticity.

1. Introduction. *A priori* stability and error estimates for time dependent problems posed in the time interval $[0, T]$ often include a constant which is of the order e^{CT}. Typically this is a consequence of applying Gronwall lemmas which do not always take proper account of any special information contained in the equations. In some cases it is known that the solution does not grow in this way. Such knowledge might, for example, derive from experience, exact solutions to model problems, or heuristic physical argument. In such cases it is arguable as to whether Gronwall type estimates actually contain any useful information. Of course they can be used to prove convergence on finite time intervals but this is only relevant as the time step approaches zero. This *never* happens in any practical computation. Rather, for long time numerical integration it is important to know that the error is not exponentially increasing. Then and only then, it seems, can any quantitative meaning be attached to the computed results.

For a specific problem in non-ageing quasistatic linear isothermal isotropic compressible solid viscoelasticity we will show that sharp data-stability estimates (in the sense of the constants) can be derived as well as correspondingly sharp error estimates for particular discrete schemes. Using the traditional "Gronwall approach" this problem has already been analyzed in [15]. However, the estimate given there is pessimistic for solid viscoelasticity problems because experience shows (and we will demonstrate below) that the error is far better controlled than is indicated by the derived error bound. Note that we are not addressing the convergence rate implied by this estimate, only the constant.

Before summarising the contents of this article it is necessary to introduce some notation. Define $\mathbb{R}_0 := [0, \infty)$. For $T \in \mathbb{R}_+ := \mathbb{R}_0 \setminus \{0\}$ we let $\mathcal{J} := [0, T]$ represent a time interval and, for $n = 1$, 2 or 3, we consider $\Omega \subset \mathbb{R}^n$ to be an open, connected and bounded domain with (for simplicity) polyhedral boundary $\partial\Omega$. As usual $W_p^m(\Omega)$ denotes the Sobolev space of functions whose derivatives of order up to and including m belong to $L_p(\Omega)$, $p \in [1, \infty]$, and we make the usual distinction: $H^m(\Omega) = W_2^m(\Omega)$. For the problem that we consider the following notation is convenient. If, for $i = 1, \ldots, n$, we have $u_i \in W_p^m(\Omega)$ then we write $u := (u_i)_{i=1}^n \in \mathcal{W}_p^m(\Omega) := W_p^m(\Omega) \times \cdots \times W_p^m(\Omega)$ (n times). This space is endowed with the product norm: $\|u\|^2_{\mathcal{W}_p^m(\Omega)} := \sum_i \|u_i\|^2_{W_p^m(\Omega)}$. We note that $\mathcal{L}_2(\Omega) = \mathcal{W}_2^0(\Omega)$ and again single out the Hilbert space: $\mathcal{H}^m(\Omega) = \mathcal{W}_2^m(\Omega)$. On occasion we will also use the seminorms $|\cdot|_{\mathcal{W}_p^m(\Omega)}$, these are defined by the obvious analogue of the definition given above for $\|\cdot\|_{\mathcal{W}_p^m(\Omega)}$. To allow for functions that are also time dependent we adopt the standard practice: if $\mathcal{B}$ is a normed linear space then we denote by $\mathcal{Q}(\mathcal{I}; \mathcal{B})$ the space of functions of class $\mathcal{Q}$, defined on $\mathcal{I}$, that take values in $\mathcal{B}$, if $\mathcal{B} = \mathbb{R}$ then we simplify this notation and just write $\mathcal{Q}(\mathcal{I})$. In the cases below we usually have $\mathcal{Q} = L_p$ or $\mathcal{Q} = C^m$ and then $\mathcal{Q}(\mathcal{I}; \mathcal{B})$ is normed by the function $\|\cdot\|_{\mathcal{Q}(\mathcal{I};\mathcal{B})} := \|\gamma(\cdot)\|_{\mathcal{Q}(\mathcal{I})}$ where $\gamma(\cdot) := \|\cdot\|_{\mathcal{B}}$. We also use the seminorm $|\cdot|_{\mathcal{Q}(\mathcal{I};\mathcal{B})}$ which is similarly defined except that now $\gamma(\cdot) := |\cdot|_{\mathcal{B}}$. Note that for appropriately smooth $u(x, t) = \big(u_i(x, t)\big)_{i=1}^n$, where $x = (x_i)_{i=1}^n \in \Omega$ and $t \in \mathcal{J}$, we write $u \in L_p\big(\mathcal{J}; \mathcal{W}_p^m(\Omega)\big)$ and not $u \in \big(L_p\big(\mathcal{J}; W_p^m(\Omega)\big)\big)^n$. This latter symbol implies that each component of u can be evaluated simultaneously at different values of t. Thus we think of u as the map $u : \mathcal{J} \mapsto \mathcal{W}_p^m(\Omega)$.

Our concern is with a particular instance of the generic *elliptic-Volterra* problem: find $u \in C(\mathcal{J}; \mathcal{V})$ such that

$$Au(t) = f(t) + \int_0^t B(t - s)u(s)\, ds, \quad t \in \mathcal{J}, \tag{1.1}$$

with Neumann and homogeneous Dirichlet boundary data specified on a time independent partition of $\partial\Omega$. The Dirichlet condition is implied by a suitable definition of a trial space $\mathcal{V} \subset \mathcal{H}^1(\Omega)$ and the Neumann data are adjoined in the loading term $f(t)$. Also: A and $B(t - s)$ are second-order differential operators with the former assumed self-adjoint and $\mathcal{V}$-elliptic, we therefore define an energy norm in a natural way as $\|\cdot\| := \sqrt{(A\,\cdot, \cdot)}$, where $(\cdot, \cdot)$ is the $\mathcal{L}_2(\Omega)$ inner product. All of our results are stated with respect to this norm. In fact, although time dependent, $B(t - s)$ is extremely close to A in nature. This will be evident in Section 2 where the physical origin of this problem is described more fully. This allows us to draw heavily on the *fading memory hypothesis* of viscoelasticity theory to justify our assumption on the continuity of $B(t-s)$. We assume: $(B(t-s)v, v) \leq \kappa_s(t-s)\|v\|^2$ for all $v \in \mathcal{V}$, where $\kappa(t)$ behaves in a qualitatively similar way to $1 - C(1 - e^{-\alpha t})$ for C, $\alpha \in \mathbb{R}_+$. Note that there is no generic constant on the right of this inequality, it is the fact that $B(t - s)$ is "close" to A in this way that renders the method of analysis described below so effective for this problem. In fact it is just this exploitation of the fading memory term (for viscoelastic solids) which sets these results apart from those already given in [15].

After elaborating on this brief introduction to the physical problem below in Section 2, we move on in Section 3 to consider in turn: the stability of the solution, Theorem 3.1; the effect of experimental or modelling errors in the viscoelastic functions, Theorem 3.2; and, in Theorem 3.3, derive an error estimate for a semidiscrete finite element approximation. The important feature of all these results is that the constants do not *a priori* contain exponentially large quantities and also, if the data can be suitably bounded independently of T, then the constants can be also. In this case all the estimates continue to hold uniformly as $T \to \infty$. These results are all built on a comparison theorem, first given by [4] and restated below in Lemma 3.1, which for the viscoelasticity problem yields far sharper results than the usual Gronwall lemma.

In Section 4 we form a fully discrete scheme by replacing the Volterra operator with the trapezoidal rule for numerical integration. Following the ideas in [5, Theorem 2.1], we then develop a comparison theorem for a discrete trapezoidal-Volterra equation (with fading memory) in Lemma 4.2, and use this to derive a sharp (again, in the sense of the constant) error estimate for the fully discrete scheme in Theorem 4.1. With such a realistic estimate to hand it is tempting to ask whether or not we can exploit it to provide a computable error bound, or even a time step selection criterion for the numerical algorithm. Some numerical experiments in this vein, and in the context of a model problem with realistic material data, are presented in Section 5. Finally, in Section 6 we make a few observations and conclusions about the results just discussed.

It seems proper to mention that the original stimulus to the authors for seeking estimates in this way was Theorem 3.7 in [13]. Also, it is interesting to note the recent work of [16]. They derive analogously sharp results for the parabolic counterpart of (1.1) which arises, for example, in the theory of hereditary heat conduction. The proof structure is however quite different from that employed here, but they also find the analysis more tractable in the case $B = A$.

2. Definition of the Problem. In this section we do not attempt to give a detailed description of the phenomenological theory of viscoelasticity but merely to outline the background and describe the mathematical model. For the relevant continuum mechanics concepts we refer to, for example, [9] or [14], and for background to viscoelasticity to, for example, [8] or [3], for a mathematical approach, and [6] for a physical description of the viscoelastic nature of polymeric materials. Also, the problem as described below is given with more detail in [15].

For $t \in \mathcal{J}$ assume that body forces $\big(f_i(x,t)\big)_{i=1}^n$, for $x = (x_i)_{i=1}^n \in \Omega$, and surface tractions $\big(g_i(x,t)\big)_{i=1}^n$, for $x \in \Gamma_N \subset \partial\Omega$, are applied to a non-ageing solid linear viscoelastic isotropic compressible body $\mathcal{G}$, the interior of which occupies Ω and whose boundary coincides with $\partial\Omega$. Newtons law of motion, assuming a quasistatic displacement field, gives, for $i = 1, \ldots, n$, and $t \in \mathcal{J}$:

$$
\begin{gathered}
-\sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_j} = f_i, \quad x \in \Omega, \\
u_i = 0 \ \text{ for } \ x \in \Gamma_D \subseteq \partial\Omega \quad \text{ and } \quad \textstyle\sum_{j=1}^{n} \sigma_{ij}\hat{n}_j = g_i \ \text{ for } \ x \in \Gamma_N = \partial\Omega \setminus \Gamma_D.
\end{gathered}
\tag{2.1}
$$

Here $\hat{n} = (\hat{n}_i)_{i=1}^n$ is the unit outward normal to Γ_N and $\{\Gamma_N, \Gamma_D\}$ forms a time-independent partition of $\partial\Omega$ such that Γ_D has strictly positive measure. Also in (2.1), $(\sigma_{ij})_{i,j=1}^n$ denotes the symmetric stress tensor and $u = (u_i)_{i=1}^n$ are the displacements in the x_i directions. In the linear theory we assume that the deformation is "small" so that the symmetric strain tensor, $(\varepsilon_{ij})_{i,j=1}^n$, is given by

$$\varepsilon_{ij}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \text{for } 1 \le i, j \le n. \tag{2.2}$$

To close the problem we require a constitutive relationship between the stress and the strain, and hence the displacement. We assume that the body is quiescent for all $t < 0$ and take this relationship as (see e.g. [8, Chap. 1§2.4]),

$$\begin{aligned} \sigma_{ij}(x,t;u) &= \lambda(x,0)\Lambda_1(x)\nabla\cdot u(x,t)\delta_{ij} + \mu(x,0)\Lambda_2(x)\varepsilon_{ij}\big(u(x,t)\big) \\ &- \int_0^t \lambda_s(x,t-s)\Lambda_1(x)\nabla\cdot u(x,s)\delta_{ij} + \mu_s(x,t-s)\Lambda_2(x)\varepsilon_{ij}\big(u(x,s)\big)\,ds. \end{aligned} \tag{2.3}$$

In this expression: the Kronecker symbol has its usual meaning, $\delta_{ij} = 1$ for $i = j$ and 0 otherwise; repeated indices imply summation; $\nabla\cdot$ denotes the divergence operator and the subscript s denotes partial differentiation. The material functions Λ_1 and Λ_2 are the viscoelastic analogues to the Lamé functions of linear elasticity, see e.g. [14], that we shall assume belong to $L_\infty(\Omega;\mathbb{R}_+)$, and express the stiffness of the material in instantaneous bulk and shear deformation respectively. The time dependent functions $\lambda(x,t)$ and $\mu(x,t)$ are stress relaxation functions and, as a matter of terminology, we will say that a material for which $\lambda(x,t) = \mu(x,t)$ is *synchronous*. Such materials exhibit a constant Poisson's ratio, ν. A material which is not synchronous is called *asynchronous* and will have a time dependent Poisson ratio. Other stress relaxation functions also exist, e.g. $E(x,t)$ for longitudinal relaxation (compare Young's modulus of linear elasticity) and $K(x,t)$ for bulk relaxation. When discussing the properties of these functions it is convenient to introduce a *generic stress relaxation function*, denoted $\varphi(x,t)$, on the understanding that any assumptions that we make about φ will also apply to all the other relaxation functions. The assumptions that we shall require now follow.

Assumptions 2.1. *For the generic non-ageing stress relaxation function, and hence for every other relaxation function, we assume:*

(i) Normalization: $\varphi(x,0) = 1$, *æ* $x \in \Omega$.

(ii) Positive definiteness: $\exists C_\varphi > 0$ *such that* $\varphi(x,t) \ge C_\varphi$ $\forall t \in \mathcal{J}$ *and æ* $x \in \Omega$.

(iii) The fading memory hypothesis: $\varphi_t(x,t) < 0$ *and* $\varphi_{tt}(x,t) > 0$ $\forall t \in \mathcal{J}$ *and æ* $x \in \Omega$.

(iv) Regularity: $\varphi(x,t) \in C^3\big(\mathcal{J}; L_\infty(\Omega)\big)$.

We note that the normalization is always possible since the material stiffness can be hidden in the functions Λ_1 *and* Λ_2 *in (2.3). The physical basis of conditions (ii) and (iii) is covered, for example, in [8, Chap. 1§3]. Finally, the regularity assumption is prompted by the convergence requirement of the numerical scheme that we shall later describe.* ■

Note the implication contained in these assumptions that, for fixed x, $\varphi(x,t)$ is a monotonically decreasing function of t bounded below by a positive constant. Hence, if it is defined: $\lim_{t\to\infty}\varphi_t(x,t) = 0$. To "justify" the regularity assumption we note the following generalization of commonly chosen forms for the relaxation functions, see e.g. [8],

$$\varphi(t) = \varphi_0 + \sum_{i=1}^{N_D} \varphi_i^D e^{-\alpha_i^D t} + \sum_{i=1}^{N_P} \varphi_i^P (t + l_i^P)^{-\alpha_i^P}, \tag{2.4}$$

where the (possibly x-dependent) coefficients are chosen so as to satisfy Assumptions 2.1. Here the superscripts, D and P refer to "Dirichlet series" and "power law" respectively. Often in analytical studies the l_i^P are taken as zero; we do not allow this because it implies that the material has infinite stiffness at $t = 0$. Also we have that

$$\varphi_t(t) = -\sum_{i=1}^{N_D} \varphi_i^D \alpha_i^D e^{-\alpha_i^D t} - \sum_{i=1}^{N_P} \varphi_i^P \alpha_i^P (t + l_i^P)^{-(\alpha_i^P+1)}. \tag{2.5}$$

Remark 2.1. *Note that we do not allow the "stretched" type of relaxation function such as that described in [1]. Such functions are not smooth enough for the trapezoidal discretization introduced later in Section 4.* ■

As is well known for the linear elasticity problem (which would coincide with our problem if λ and μ were temporally constant) this classical formulation often fails to make sense because the displacements are not regular enough to permit the two space differentiations required. Thus we consider a weak formulation of the problem formally derived by firstly defining the trial space $\mathcal{V}$ as

$$\mathcal{V} := \Big\{ v \in \mathcal{H}^1(\Omega) : v = 0 \text{ on } \Gamma_D \Big\}, \tag{2.6}$$

and then taking the $\mathcal{L}_2(\Omega)$ inner product of (2.1) with a test function $v = (v_i)_{i=1}^n \in \mathcal{V}$. Now integrating by parts using Green's formula we obtain (omitting the x dependence):

$$\int_\Omega \sigma_{ij}(t;u)\varepsilon_{ij}(v)\, d\Omega = \int_\Omega f_i(t) v_i \, d\Omega + \int_{\Gamma_N} g_i(t) v_i \, d\Gamma =: L(t;v), \qquad \text{for } t \in \mathcal{J}, \tag{2.7}$$

where, as before, the repeated indices imply summation.

Using (2.3) to substitute for the stress in the leftmost term of this expression and interchanging the space and time integrations yields the weak problem: find $u \in C(\mathcal{J};\mathcal{V})$ such that

$$a(u(t), v) = L(t;v) + \int_0^t b(t-s; u(s), v)\, ds \qquad \forall v \in \mathcal{V},\ t \in \mathcal{J}, \tag{2.8}$$

where $\forall v, w \in \mathcal{V}$ and $t, t-s \in \mathcal{J}$:

$$a(w,v) := \int_\Omega \Lambda_1 \nabla\cdot w \nabla\cdot v + \Lambda_2 \varepsilon_{ij}(w)\varepsilon_{ij}(v)\, d\Omega, \tag{2.9}$$

$$b(t-s;w,v) := \int_\Omega \lambda_s(t-s)\Lambda_1 \nabla\cdot w \nabla\cdot v + \mu_s(t-s)\Lambda_2 \varepsilon_{ij}(w)\varepsilon_{ij}(v)\, d\Omega. \tag{2.10}$$

The derivation of these forms is essentially the same as for the linear elasticity problem and as such is standard, see e.g. [11]. We make the following assumptions concerning this formulation:

Assumptions 2.2.

(i) *The data are such that* $\exists! u \in C(\mathcal{J}; \mathcal{V})$ *solving (2.8); this requires that* $L(t; \cdot)$ *also exhibit certain regularity in addition to that assumed of the kernel functions in Assumption 2.1, see e.g. [13, Theorem 3.6].*

(ii) *The bilinear form* $a(\cdot, \cdot)$ *is continuous and also* $\mathcal{V}$*-coercive so that* $\exists c > 0$ *such that* $a(v, v) \geq c\|v\|^2_{\mathcal{H}^1(\Omega)}$ $\forall v \in \mathcal{V}$*. In this case* $a(\cdot, \cdot)$ *is a scalar product on* $\mathcal{V}$ *and induces the energy norm:* $\|\cdot\| := \sqrt{a(\cdot, \cdot)}$*. The coercivity is due to the non-trivial* Korn's inequality*. This holds providing that* $\mathcal{V}$ *does not contain any non-zero displacement fields composed solely of a linear combination of rigid body translations and/or rotations. For Korn's inequality we refer to [14, Chap. 6, Definition 3.1] and also to [7]. From a physical argument it is easy to see that a sufficient condition for these type of displacement fields to be excluded from the trial space, in our context, is the strictly positive measure of* Γ_D *insisted upon below equation (2.1). All of our results will be given in terms of the energy norm just defined.*

(iii) *For each* $t \in \mathcal{J}$*,* $L(t; \cdot)$ *is a continuous linear form on* $\mathcal{V}$*. That is, for the energy norm defined above there exists a positive constant,* C_L*, such that*

$$|L(t; v)| \leq C_L\|v\| \qquad \forall v \in \mathcal{V} \text{ and } \forall t \in \mathcal{J}.$$

(iv) *The bilinear form* $b(t - s; \cdot, \cdot)$ *is continuous on* $\mathcal{V}$ *and exhibits fading memory in the sense that there exists a function* κ *such that:*

$$|b(t - s; w, v)| \leq \kappa_s(t - s)\|w\|\,\|v\| \qquad \forall v, w \in \mathcal{V} \text{ and } \forall t, t - s \in \mathcal{J}.$$

And $\exists \ell \in \mathbb{R}_+$ *such that this function satisfies:*

(a) $\kappa(t) \in C^1[-\ell, \infty)$.

(b) *For every* $w \in C[-\ell, \infty)$ *we have* $\int_{-\ell}^{t} \kappa_{st}(t - s)w(s)\, ds \in C[-\ell, \infty)$.

(c) $\kappa(t) > 0$, $\kappa'(t) < 0$ *and* $\kappa''(t) > 0$ *in* $[-\ell, \infty)$.

(d) $0 < \kappa(-\ell) - \kappa(\infty) < 1$.

Of course, $\kappa_s(t-s) = -\kappa'(t-s)$ *and* $\kappa_{st}(t-s) = -\kappa''(t-s)$*. The* ℓ *is required in Section 4 for the fully discrete estimate. It seems to be necessary because the trapezoidal approximation to a definite integral of a positive convex function—compare (2.4)—is never an underestimate.* ■

3. Stability and a semidiscrete scheme. The key point when considering stability is that from the correspondence principle of viscoelasticity theory (see e.g. [8] or [3, Chap. 2§5]), applied to simple model problems, we can deduce that the solution to (2.8) is bounded by the solution to a linear elasticity problem. To show that this is always true we first need a comparison theorem.

Lemma 3.1. *Suppose that for a constant* $G \in \mathbb{R}_0$ *and* $\psi(t, s)$ *nonnegative, bounded and continuous on the triangle* $\{0 \leq s \leq t \leq T\}$ *we have, for a bounded function* $y(t)$*, continuous almost everywhere on* $[0, T]$*, that*

$$y(t) \leq G + \int_0^t \psi(t, s)y(s)\, ds, \quad t \in \mathcal{J}.$$

Now, let $Y(t)$ be the unique bounded continuous solution to the Volterra equation:

$$Y(t) = G + \int_0^t \psi(t,s) Y(s)\, ds, \quad t \in \mathcal{J},$$

then $y(t) \le Y(t)$ for all $t \in \mathcal{J}$. ■

For continuous $y(t)$ this comparison theorem has been given by [4], that it continues to hold in the "almost everywhere" case is noted in [5, Lemma 1.3].

To utilize this lemma we need some quantitative information about the solution to a Volterra problem that, in some sense, characterizes the viscoelasticity problems that we want to consider. For this we use the following result.

Lemma 3.2. *For constants $G_1 \in \mathbb{R}_0$, $G_2 \in \mathbb{R}_+$, consider the Volterra problem of finding $u(t) \in C[0,T]$ such that*

$$u(t) = G_1 + G_2 \int_0^t \chi_s(t-s) u(s)\, ds, \qquad \text{for } 0 \le t \le T < \infty, \tag{A}$$

subject to the following conditions on the kernel:

(a) $\chi(t-s) \in C^1[0 \le s \le t < \infty)$.

(b) *For every $w \in C[0,\infty)$ we have $\int_0^t \chi_{st}(t-s) w(s)\, ds \in C[0,\infty)$.*

(c) $\chi(t-s) > 0$ *and* $\chi_s(t-s) > 0$ *in* $\{0 \le s \le t < \infty\}$.

(d) $0 < \chi(0) - \chi(\infty) < 1/G_2$.

Then there exists a unique $u \in C^1[0,T]$, for every choice of $T \in \mathbb{R}_+$, such that

$$G_1 \le u(t) \le \frac{G_1}{1 - G_2\big(\chi(0) - \chi(\infty)\big)} \qquad \forall t \in [0,T]. \tag{B}$$

Furthermore u is monotonic so that equality on the left is attained only at $t = 0$ and equality on the right only in the limit $t = T \to \infty$.

Proof. The existence and uniqueness of $u \in C[0,T]$ follows from the theory of Volterra equations, see e.g. [13, Theorem 3.1]. Formally differentiating (A) we obtain

$$u'(t) = G_2 \chi_s(0) u(t) + G_2 \int_0^t \chi_{st}(t-s) u(s)\, ds$$

and, from (b), it follows that $u' \in C[0,T]$ also. Now, integrating this expression by parts gives:

$$u'(t) = G_2 G_1 \chi_s(t) + G_2 \int_0^t \chi_s(t-s) u'(s)\, ds,$$

since $u(0) = G_1$ and $\chi_{st}(t-s) = -\chi_{ss}(t-s)$. It is clear that if $G_1 = 0$ then $u(t) = 0$ is the unique continuous solution to (A) and in this case (B) holds trivially. From now on we assume $G_1 > 0$ then, using (c), it follows that $u'(t) > 0$ and so $u(t)$ is an increasing function. This, with $u(0) = G_1$, proves the left side of (B). Also, from (c) we find that $\chi(\infty) \le \chi(t)$ for $t \ge 0$ and so integrating (A) by parts gives:

$$\begin{aligned} G_1 &= u(t)\big(1 - G_2\chi(0)\big) + G_2 G_1 \chi(t) + G_2 \int_0^t \chi(t-s) u'(s)\, ds \\ &\ge u(t)\big(1 - G_2\chi(0)\big) + G_2 G_1 \chi(\infty) + G_2 \chi(\infty)(u(t) - G_1) \\ &= u(t)\Big(1 - G_2\big(\chi(0) - \chi(\infty)\big)\Big) > 0 \end{aligned}$$

by (d). This gives the right hand side of (B) and completes the proof. ■

As an example, if we take in equation (A) of Lemma 3.2 the kernel $\chi(t) = A_1 + \epsilon e^{-\alpha t}$, with A_1 and α positive constants and ϵ such that: $0 < \epsilon < 1/G_2$, then it is easy to verify that

$$u(t) = \frac{G_1}{1 - G_2\epsilon}\Big(1 - G_2\epsilon\, e^{-\alpha(1-G_2\epsilon)t}\Big), \tag{3.1}$$

and we can, of course, always take $\epsilon = (NG_2)^{-1}$, for arbitrary $N > 1$. It is easy to see that $u(t)$ behaves in exactly the way described by the lemma.

To illustrate the utility of these results we first show that the weak solution of the viscoelasticity problem, described earlier in Section 2, can be bounded without recourse to the standard Gronwall lemma. These bounds are independent of T if the bounds on the data are also.

Theorem 3.1. *Let the requirements and assumptions of Section 2 hold, then for smooth enough loading there exist stability constants, C_s, C_s' and C_s'', such that for $t \in \mathcal{J}$:*

$$\begin{aligned}
\|u(t)\| &\le C_s := C_L/C_\kappa \quad \textit{(see Assumption 2.2iii)}, && (A)\\
\|u'(t)\| &\le C_s' := C_L'/C_\kappa = \big(c' + C_s\kappa_s(0)\big)/C_\kappa, && (B)\\
\|u''(t)\| &\le C_s'' := C_L''/C_\kappa = \big(c'' + C_sC_{\lambda\mu} + C_s'\kappa_s(0)\big)/C_\kappa. && (C)
\end{aligned}$$

Where: $C_\kappa := 1 - \kappa(0) + \kappa(\infty)$; c' and c'' bound the first and second derivative (resp.) of the loading, $L(t;\cdot)$; and, $C_{\lambda\mu}$ depends upon the second derivatives of λ and μ. Therefore, if the data are smooth enough and can be bounded independently of T—in the way required in Assumptions 2.2—these stability constants are also independent of T.

Proof. For arbitrary $t \in \mathcal{J}$ pick $v = u(t) \in \mathcal{V}$ in (2.8) and use Assumptions 2.2. Excluding the trivial case $\|u(t)\| = 0$ we get

$$\|u(t)\| \le C_L + \int_0^t \kappa_s(t-s)\|u(s)\|\,ds.$$

This Volterra inequality conforms to the requirements of Lemma 3.1 and so we infer that $\|u(t)\| \le Y(t)$, where $Y(t)$ solves

$$Y(t) = C_L + \int_0^t \kappa_s(t-s)Y(s)\,ds.$$

In addition, due to Assumption 2.2iv, this Volterra equation conforms to the requirements of Lemma 3.2 and therefore:

$$\|u(t)\| \le Y(t) \le \frac{C_L}{C_\kappa},$$

which proves (A). Now, differentiate (2.8) to get

$$a(u'(t), v) = L'(t; v) + b(0; u(t), v) + \int_0^t b_t(t-s; u(s), v)\,ds,$$

and make the replacement: $b_t(t-s;\cdot,\cdot) = -b_s(t-s;\cdot,\cdot)$. Noting (2.10) we integrate by parts in the final term to obtain:

$$a(u'(t), v) = L'(t; v) + b(t; u(0), v) + \int_0^t b(t-s; u'(s), v)\, ds. \tag{i}$$

Picking $v = u'(t)$ and proceeding as above this gives

$$\|u'(t)\| \leq C_L' + \int_0^t \kappa_s(t-s)\|u'(s)\|\, ds$$

where $C_L' := c' + C_s\kappa_s(0)$. Exactly the same comparison argument as used above then yields: $\|u(t)\| \leq C_L'/C_\kappa$. This proves (B). To establish (C) we simply repeat these steps. Differentiating (i) we obtain:

$$a(u''(t), v) = L''(t; v) + b_t(t; u(0), v) + b(0; u'(t), v) + \int_0^t b_t(t-s; u'(s), v)\, ds.$$

Shifting the t-derivative under the integral to an s-derivative, as before, and partially integrating now gives

$$a(u''(t), v) = L''(t; v) + b_t(t; u(0), v) + b(t; u'(0), v) + \int_0^t b(t-s; u''(s), v)\, ds.$$

Now picking $v = u''(t)$ we obtain:

$$\|u''(t)\| \leq C_L'' + \int_0^t \kappa_s(t-s)\|u''(s)\|\, ds,$$

where $C_L'' := c'' + C_s C_{\lambda\mu} + C_s'\kappa_s(0)$. One more application of the comparison argument then yields (C) and completes the proof. ■

Remark 3.1. *Theorem 3.1 does not apply to a viscoelastic fluid, see e.g [8], because in this case $\kappa(\infty) = 0$ and, due to the normalization, $\kappa(0) = 1$. In general a result like Theorem 3.1 cannot hold for a fluid and this is why we have restricted our attention to viscoelastic solids.* ■

Theorem 3.1 gives a basic stability result for (2.8) and holds if the data are known exactly. This is usually not the case. In particular the stress relaxation functions λ and μ would normally have to determined by curve fitting experimental data and this would obviously entail some error. Clearly it is important to know how these data errors are reflected in the solution to the "fitted" problem. We will assume that the stiffnesses Λ_1 and Λ_2 are known precisely so that the primary source of uncertainty is in the loading, $L(t;\cdot)$, and the stress relaxation functions λ and μ. To be precise, let the fitted relaxation functions be denoted by $\widetilde{\lambda}$ and $\widetilde{\mu}$ and define the data errors:

$$\epsilon^\lambda(t) := \widetilde{\lambda}(t) - \lambda(t) \qquad \text{and} \qquad \epsilon^\mu(t) := \widetilde{\mu}(t) - \mu(t) \qquad \text{for } t \in \mathcal{J}. \tag{3.2}$$

Also, suppose there is an error in the applied loading,

$$\epsilon^L(t; v) := \widetilde{L}(t; v) - L(t; v) \qquad \text{for } t \in \mathcal{J} \text{ and } v \in \mathcal{V}, \tag{3.3}$$

which is continuous on $\mathcal{V}$. That is, there is a constant, c_L, such that:

$$|\epsilon^L(t;v)| \leq c_L\|v\| \qquad \forall v \in \mathcal{V} \text{ and } t \in \mathcal{J}. \tag{3.4}$$

Now, instead of (2.8) we consider the perturbed problem: find $\widetilde{u} \in C(\mathcal{J};\mathcal{V})$ such that

$$a(\widetilde{u}(t), v) = \widetilde{L}(t;v) + \int_0^t \widetilde{b}(t-s;\widetilde{u}(s), v)\, ds, \qquad \forall v \in \mathcal{V},\ t \in \mathcal{J}. \tag{3.5}$$

This time, instead of (2.10), we have, for $w, v \in \mathcal{V}$:

$$\widetilde{b}(t-s;w,v) := b(t-s;w,v) + b_\epsilon(t-s,w,v) \tag{3.6}$$

and

$$b_\epsilon(t-s,w,v) := \int_\Omega \epsilon_s^\lambda(t-s)\Lambda_1\nabla\cdot w\nabla\cdot v + \epsilon_s^\mu(t-s)\Lambda_2\varepsilon_{ij}(w)\,\varepsilon_{ij}(v)\, d\Omega. \tag{3.7}$$

Note that because we assume certain knowledge of the instantaneous stiffnesses, Λ_1 and Λ_2, the bilinear form $a(\cdot,\cdot)$, and hence the energy norm, $\|\cdot\|$, remain unchanged. We also expect Assumptions 2.1 and 2.2 to hold in the context of (3.5) for, if they did not, we would not admit the fitted functions $\widetilde{\lambda}$ and $\widetilde{\mu}$ as suitable relaxation functions. We therefore have: $|\widetilde{L}(t;v)| \leq (C_L + c_L)\|v\|\ \forall v \in \mathcal{V}$ and $t \in \mathcal{J}$; and, $|\widetilde{b}(t-s;w,v)| \leq \widetilde{\kappa}_s(t-s)\|w\|\,\|v\|\ \forall w, v \in \mathcal{V}$ and $t, t-s \in \mathcal{J}$. Here $\widetilde{\kappa}$ also satisfies the conditions on κ given in Assumptions 2.2iv. Then, from Theorem 3.1 we have immediately that

$$\|\widetilde{u}(t)\| \leq \frac{C_L + c_L}{1 - \widetilde{\kappa}(0) + \widetilde{\kappa}(\infty)}. \tag{3.8}$$

Now we are able to state a theorem on the distance between u and $\widetilde{u}$.

Theorem 3.2. *In the context of the foregoing discussion the energy distance between the exact solution, u, of (2.8) and the solution, $\widetilde{u}$, of the fitted problem (3.5) satisfies:*

$$\|u(t) - \widetilde{u}(t)\| \leq C_1 + C_2\left\| \int_0^\tau \max\left\{\|\epsilon_s^\lambda(\tau-s)\|_{L_\infty(\Omega)}, \|\epsilon_s^\mu(\tau-s)\|_{L_\infty(\Omega)}\right\} ds\right\|_{L_\infty(0,t)}$$

$\forall t \in \mathcal{J}$. Also, $C_1 \sim c_L$ and C_2 are constants that depend only upon the loading and data errors: $\kappa(0)$, $\widetilde{\kappa}(0)$, $\kappa(\infty)$, $\widetilde{\kappa}(\infty)$, C_L and c_L.

Proof. Subtract (2.8) from (3.5), using the fact that the energy form is identical in each, to give for an arbitrary $t \in \mathcal{J}$,

$$a(\widetilde{u}(t) - u(t), v) = \widetilde{L}(t;v) - L(t;v) + \int_0^t \widetilde{b}(t-s;\widetilde{u}(s), v) - b(t-s;u(s),v)\, ds \qquad \forall v \in \mathcal{V},$$

or, simplifying this using (3.3) and (3.6),

$$a(\widetilde{u}(t) - u(t), v) = \epsilon^L(t;v) + \int_0^t b_\epsilon(t-s;\widetilde{u}(s),v)\, ds + \int_0^t b(t-s;\widetilde{u}(s) - u(s), v)\, ds.$$

Assuming that $\widetilde{u}(t) - u(t)$ is non-trivial (otherwise no proof is necessary) we can take $v = \widetilde{u}(t) - u(t) \in \mathcal{V}$ to give

$$\|\widetilde{u}(t) - u(t)\| \leq c_L + \int_0^t \widehat{\epsilon}(t-s)\|\widetilde{u}(s)\|\,ds + \int_0^t \kappa_s(t-s)\|\widetilde{u}(s) - u(s)\|\,ds,$$

where

$$\widehat{\epsilon}(t-s) := \max\Big\{\|\epsilon_s^\lambda(t-s)\|_{L_\infty(\Omega)}, \|\epsilon_s^\mu(t-s)\|_{L_\infty(\Omega)}\Big\}.$$

Using the bound on $\widetilde{u}$ given by (3.8) we get

$$c_L + \int_0^t \widehat{\epsilon}(t-s)\|\widetilde{u}(s)\|\,ds \leq c_L + \frac{C_L + c_L}{1 - \widetilde{\kappa}(0) + \widetilde{\kappa}(\infty)} \left\| \int_0^\tau \widehat{\epsilon}(\tau - s)\,ds \right\|_{L_\infty(0,t)} = \Psi \qquad \text{(say)}.$$

From Lemma 3.1 we know that $\|\widetilde{u}(t) - u(t)\| \leq Y(t)$ where $Y(\tau)$ is the unique continuous solution to

$$Y(\tau) = \Psi + \int_0^\tau \kappa_s(\tau - s)Y(s)\,ds, \qquad \text{for } \tau \in [0,t],$$

and from Lemma 3.2: $Y(\tau) \leq \Psi/\big(1 - \kappa(0) + \kappa(\infty)\big)$. Consequently:

$$\|\widetilde{u}(t) - u(t)\| \leq \frac{c_L}{1 - \kappa(0) + \kappa(\infty)} + \frac{(C_L + c_L)\big\| \int_0^\tau \widehat{\epsilon}(\tau - s)\,ds \big\|_{L_\infty(0,t)}}{\big(1 - \kappa(0) + \kappa(\infty)\big)\big(1 - \widetilde{\kappa}(0) + \widetilde{\kappa}(\infty)\big)}.$$

The original choice of t was arbitrary and so this proves the theorem. ∎

Theorem 3.2 demonstrates that if ϵ_s^λ and ϵ_s^μ are bounded then the accumulated error due to curve fitting is also bounded, in the worst case, by linear growth. But more realistically, due to the monotonicity implication of Assumption 2.1: $\lambda(t)$, $\mu(t) \to$ constant as $t \to \infty$ and so we would usually have $\epsilon_s^\lambda(t)$, $\epsilon_s^\mu(t) \to 0$. Therefore the integral in the bound may converge to a constant independent of t.

We form a semidiscrete finite element approximation in the usual way: replacing $\mathcal{V}$ with an appropriate finite dimensional subspace $\mathcal{V}^h \subset \mathcal{V}$ we seek $u^h \in C(\mathcal{J}; \mathcal{V}^h)$ such that

$$a(u^h(t), v) = L(t; v) + \int_0^t b(t-s; u^h(s), v)\,ds \qquad \forall v \in \mathcal{V}^h,\ t \in \mathcal{J}, \tag{3.9}$$

for the details of this discretization we refer to [15]. This can also be written as a system of Volterra equations:

$$AU(t) = F(t) + \int_0^t B(t-s)U(s)\,ds, \quad t \in \mathcal{J}, \tag{3.10}$$

where: A and B are symmetric matrices generated by the bilinear forms, with A positive definite due to Assumption 2.2ii; F is the load vector; and, $U(t)$ is a vector of time dependent coordinates (with respect to the basis of $\mathcal{V}^h$) of the semidiscrete solution $u^h(t)$. In the case of a homogeneous synchronous material (where λ and μ

are x-independent and identical) this system is actually just a collection of uncoupled Volterra equations each of the form:

$$w(t) = r(t) + \int_0^t \varphi_s(t-s) w(s)\, ds. \tag{3.11}$$

Where: $w(t)$ is an element of $U(t)$; $r(t)$ is the corresponding element of $A^{-1}F(t)$; and, $\varphi(t-s)$ is the generic stress relaxation function referred to in Section 2.

It may be possible to obtain an analytic solution to these problems in which case no further approximation is necessary. An error estimate is then merely a distance between the weak and semidiscrete solutions. The energy distance is estimated in the next theorem.

Theorem 3.3. *Let Assumptions 2.2 hold along with the additional regularity:* $u \in C\big(\mathcal{J}; \mathcal{V} \cap \mathcal{H}^r(\Omega)\big)$ *for some* $r \geq 2$. *Assume also that the finite element space is chosen so that for this* r: $\|Ru - u\| \leq C_R h^{r-1} |u|_{\mathcal{H}^r(\Omega)}$, *where* $R : \mathcal{V} \mapsto \mathcal{V}^h$ *is the Ritz map, [17], defined by* $a(u - Ru, v) = 0\ \forall v \in \mathcal{V}^h$ *and the mesh size,* h, *is a small parameter. Then there exists a constant,* C, *such that*

$$\|u(t) - u^h(t)\| \leq C h^{r-1} |u(t)|_{L_\infty(0,t;\mathcal{H}^r(\Omega))} \qquad \forall t \in \mathcal{J},$$

where $C = C_R / \big(1 - \kappa(0) + \kappa(\infty)\big)$, *and is therefore independent of* T *if the necessary bounds on the data are.*

Proof. Subtract the weak problem (2.8) from the semidiscrete formulation (3.9) to give

$$a(u(t) - u^h(t), v) = \int_0^t b(t-s; u(s) - u^h(s), v)\, ds \qquad \forall v \in \mathcal{V}^h,\ t \in \mathcal{J},$$

and set: $\eta(t) := u(t) - Ru(t)$ and $\zeta(t) := Ru(t) - u^h(t)$ for $t \in \mathcal{J}$, where $Ru(t) \in \mathcal{V}^h$, for each given $t \in \mathcal{J}$, is the Ritz projection of u on $\mathcal{V}^h$. In this case $\|\eta(t)\| \leq C_R h^{r-1} |u(t)|_{\mathcal{H}^r(\Omega)}$ for $r \geq 2$. Therefore

$$a(\zeta(t), v) = \int_0^t b(t-s; \eta(s), v)\, ds + \int_0^t b(t-s; \zeta(s), v)\, ds \qquad \forall v \in \mathcal{V}^h,\ t \in \mathcal{J},$$

since, by definition, $a(\eta(t), v) = 0\ \forall v \in \mathcal{V}^h$. Picking $v = \zeta(t) \in \mathcal{V}^h$ we find that

$$\|\zeta(t)\|^2 \leq \int_0^t \kappa_s(t-s) \|\eta(s)\|\, ds\ \|\zeta(t)\| + \int_0^t \kappa_s(t-s) \|\zeta(s)\|\, ds\ \|\zeta(t)\|,$$

and excluding the case $\|\zeta(t)\| = 0$ (in which case the theorem is trivially proven at this t since $C > C_R$ by Assumption 2.2iv.d) we obtain

$$\begin{aligned} \|\zeta(t)\| &\leq \int_0^t \kappa_s(t-s) \|\eta(s)\|\, ds + \int_0^t \kappa_s(t-s) \|\zeta(s)\|\, ds \\ &\leq \big(\kappa(0) - \kappa(t)\big) \|\eta(s)\|_{L_\infty(0,t;\mathcal{V})} + \int_0^t \kappa_s(t-s) \|\zeta(s)\|\, ds. \end{aligned}$$

From Assumption 2.2iv.c we have $0 < \kappa(0) - \kappa(t) < \kappa(0) - \kappa(\infty)$ and so from Lemma 3.1 we see that $\|\zeta(t)\| \leq Y(t)$ where $Y(t)$ is the continuous solution of

$$Y(t) = \Psi + \int_0^t \kappa_s(t-s) Y(s)\, ds,$$

where $\Psi := \big(\kappa(0) - \kappa(\infty)\big)\|\eta\|_{L_\infty(0,t;\mathcal{V})}$. Applying Lemma 3.2 to this gives

$$Y(t) \leq \frac{\Psi}{1 - \big(\kappa(0) - \kappa(\infty)\big)}$$

and so, since $\|u(t) - u^h(t)\| \leq \|\eta(t)\| + \|\zeta(t)\|$, we have

$$\begin{aligned} \|u(t) - u^h(t)\| &\leq C_R h^{r-1} |u(t)|_{\mathcal{H}^r(\Omega)} + \frac{\big(\kappa(0) - \kappa(\infty)\big) C_R h^{r-1} |u(t)|_{L_\infty(0,t;\mathcal{H}^r(\Omega))}}{1 - \big(\kappa(0) - \kappa(\infty)\big)} \\ &= \frac{C_R h^{r-1} |u(t)|_{L_\infty(0,t;\mathcal{H}^r(\Omega))}}{1 - \big(\kappa(0) - \kappa(\infty)\big)} \end{aligned}$$

which was to be proven. ■

Remark 3.2. *This result is also a special case of an elliptic projection with fading memory which has been introduced by [2], and studied further in [12], as an aid in analyzing Volterra problems.* ■

4. A fully discrete scheme. In simple cases it may be possible to find an analytic solution to the semidiscrete problem (3.9) or (3.10) and then there will be no time discretization error. The convolution form of the history integral seems to make the application of the Laplace transform an attractive proposition, but this is tempered by the fact that the subsequent inversion would have to be performed numerically in all but the simplest cases. This is not always a straightforward task and would necessarily introduce a "time" discretization error. Also, [13, Chaps. 1 & 6] notes that for certain types of kernel the Volterra equation (3.10) may be re-cast as an ODE problem. This does not seem to offer any great advantage. In fact quite the opposite is true: it is noted in [10] that this approach to the viscoelasticity problem can result in a great deal more computational work than is necessary.

We take a much more direct route to a fully discrete approximation of (2.8) by replacing the history form in (3.9) with a numerical quadrature rule. In principle many choices are possible, for example: the rectangle rule, which is only of first order accuracy; the midpoint rule, which is inconvenient since it does not sample the integrand at the discrete time levels; and, the Simpson rules which, although of potentially higher accuracy, are also inconvenient since not only do they require a starting value, they also require an alternating solution algorithm according to whether the time level is even or odd, see e.g. [13, Chap. 7]. In the light of these observations we choose the composite trapezoidal rule which is potentially second-order accurate and is simple to implement. In this case, providing the maximum time step is bounded below a certain value, we can again derive an error estimate with sharp constant for the numerical algorithm.

To form the fully discrete scheme we firstly partition $\mathcal{J}$ into

$$\mathcal{J}^k := \big\{t_i \in \mathcal{J} : 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_N = T\big\} \tag{4.1}$$

for a positive integer N, and define the time steps $k_i := t_i - t_{i-1}$ for $i = 1, \dots, N$. Since we require these time steps to be bounded we set $\widehat{k} = \max_{1\le i\le N} k_i$. The history integral in (3.9) is then replaced with the composite trapezoidal rule for numerical quadrature wherein: for $w \in C^2[0,T]$ and $t_i \in \mathcal{J}^k$ we write

$$\int_0^{t_i} w(s)\,ds - \sum_{j=1}^{i} \frac{k_j}{2}\Big(w(t_j) + w(t_{j-1})\Big) = \delta(t_i), \tag{4.2}$$

where the error estimate is given by

$$|\delta(t_i)| \le \frac{1}{8}\sum_{j=1}^{i} k_j^2 \int_{t_{j-1}}^{t_j} |w''(s)|\,ds, \tag{4.3}$$

and the approximation is formed by setting δ to zero and replacing $w(t_j)$ with $w_j \approx w(t_j)$, for $j = 0, \dots, i$, in (4.2).

The fully discrete problem is now defined to be: find, for each $t_i \in \mathcal{J}^k$, a $u_i^{hk} \in \mathcal{V}^h$ such that

$$a(u_i^{hk}, v) = L(t_i; v) + \sum_{j=0}^{i} \varpi_{ij} b(t_i - t_j; u_j^{hk}, v) \qquad \forall v \in \mathcal{V}^h. \tag{4.4}$$

That is, u_i^{hk} is the fully discrete approximation to $u(t_i)$, for $i = 0, \dots, N$. The ϖ_{ij} are the weights associated with the trapezoidal rule and are given by: $\varpi_{00} = 0$, and for $i > 0$: $\varpi_{i0} = k_1/2$, $\varpi_{ii} = k_i/2$ and, for $0 < j < i$, $\varpi_{ij} = (k_{j+1} + k_j)/2$.

In the computational scheme we seek, at each time level t_i, an array of coordinates (taken with respect to the basis of $\mathcal{V}^h$) U_i such that, for $i > 0$:

$$\begin{aligned}\Big(A - \frac{k_i}{2}B(0)\Big)U_i &= F(t_i) + \frac{k_i}{2}B(t_i - t_{i-1})U_{i-1}\\ &\quad + \sum_{j=1}^{i-1}\frac{k_j}{2}\Big(B(t_i - t_j)U_j + B(t_i - t_{j-1})U_{j-1}\Big)\end{aligned} \tag{4.5}$$

which, for k_i small enough, possesses a unique solution since A is positive definite. The actual restriction on the maximum time step is dealt with in Lemma 4.3 and given in Theorem 4.1 below, and in this case the constant in the error bound depends upon T only in so much as Assumptions 2.2 (iii) and (iv) do. However, before we can state this theorem we need some preliminary results.

The first is a discrete Gronwall lemma appropriate to the trapezoidal discretization of a second kind Volterra equation with variable time step.

Lemma 4.1. *Suppose that the sequence of nonnegative real numbers $\{y_i\}_{i=0}^N$ satisfies the trapezoidal-Volterra inequality:*

$$y_i \leq g(t_i) + \sum_{j=1}^{i} \frac{k_j}{2}\Big(\psi(t_i,t_j)y_j + \psi(t_i,t_{j-1})y_{j-1}\Big), \tag{A}$$

where $g(t) \in C[0,T]$ and $\psi(t,s) \in C[0 \leq s \leq t \leq T]$ with sup *norms $\widetilde{G}$ and $\widetilde{M}$ respectively. Then if $\max_{1\leq i\leq N} k_i \leq \widehat{k} < 2/\max_{t\in[0,T]}\psi(t,t)$ these iterates are bounded. Specifically:*

$$y_0 \leq \widetilde{G} \quad \text{and} \quad y_i \leq \frac{\widetilde{G}\exp\left(\frac{\widetilde{M}t_{i-1/2}}{1-\widehat{k}\psi(t_i,t_i)/2}\right)}{1-\widehat{k}\psi(t_i,t_i)/2} \quad \text{for } i>0,$$

where $t_{i-1/2} := (t_i + t_{i-1})/2$.

Proof. Define $\beta := 1 - \widehat{k}\psi(t_i,t_i)/2 \in (0,1)$. Then, for an arbitrary $i \in \{1,\ldots,N\}$, a rearrangement of (A) gives:

$$y_i \leq G + \frac{1}{2}Mk_i y_{i-1} + M\sum_{j=1}^{i-1}\frac{k_j}{2}(y_j + y_{j-1}), \tag{i}$$

where $G = \widetilde{G}/\beta$, $M = \widetilde{M}/\beta$ and we insist that $\sum_{j=m_1}^{m_2} \equiv 0$ for $m_2 < m_1$. From this it is clear that

$$y_0 \leq \widetilde{G} \leq G \quad \text{and} \quad y_1 \leq G + \frac{1}{2}Mk_1y_0 \leq G\exp(Mt_{1/2}).$$

The first part of this establishes the result for $i = 0$ and we now proceed by an inductive argument. Postulating that for $i = 1,\ldots,I < N$

$$y_i \leq G\exp(Mt_{i-1/2}), \tag{ii}$$

which has just been shown for $I = 1$, and using this in (i) we find:

$$\begin{aligned} y_{i+1} &\leq G + \frac{1}{2}Mk_{i+1}y_i + M\sum_{j=1}^{i}\frac{k_j}{2}(y_j + y_{j-1}) \\ &\leq G\left(1 + \frac{1}{2}Mk_{i+1}e^{Mt_{i-1/2}} + M\sum_{j=1}^{i}\frac{k_j}{2}\left(e^{Mt_{j-1/2}} + e^{Mt_{j-3/2}}\right)\right). \end{aligned}$$

This sum may be bounded by a definite integral:

$$\frac{k_j}{2}\left(e^{Mt_{j-1/2}} + e^{Mt_{j-3/2}}\right) \leq \int_{t_{j-1/2}}^{t_j} e^{M\xi}\,d\xi + \int_{t_{j-3/2}}^{t_{j-3/2}+\frac{1}{2}k_j} e^{M\xi}\,d\xi \leq \int_{t_{j-1}}^{t_j} e^{M\xi}\,d\xi$$

since $e^{-\frac{1}{2}Mk_{j-1}} \leq 1$. Using this in the estimate for y_{i+1} we can now see that

$$y_{i+1} \leq G\left(1 + \frac{1}{2}Mk_{i+1}e^{Mt_i} + M\sum_{j=1}^{i}\int_{t_{j-1}}^{t_j} e^{M\xi}\,d\xi\right) \leq G\exp\Big(Mt_{(i+1)-1/2}\Big).$$

Since the base clause (ii) is true for $I = 1$ this establishes (ii) for all $i = 1,\ldots,N$. The lemma follows by substituting back for M and G in terms of $\widetilde{M}$, $\widetilde{G}$ and β. ∎

The second preliminary result that we require is an analogue of [5, Theorem 2.1]. We cannot apply that result here because it holds only for constant time steps, however, we base the following proof on theirs. It is here that we require the ℓ in Assumptions 2.2.

Lemma 4.2. *Let $g(t)$ be a nonnegative function defined on $\mathcal{J}^k$ with $\max_{t\in\mathcal{J}^k} g(t) = G$, and, for some $\ell \in \mathbb{R}_+$, let $\psi(t)$ be a continuous, positive-valued and monotone decreasing function on $[-\ell, T]$. Also, let $\{y_i\}_{i=0}^N$ be a sequence of bounded nonnegative real numbers satisfying the trapezoidal-Volterra inequality:*

$$\begin{aligned} &y_0 \le g(0), \\ &y_i \le g(t_i) + \sum_{j=1}^{i} \frac{k_j}{2}\Big(\psi(t_i - t_j)y_j + \psi(t_i - t_{j-1})y_{j-1}\Big), \qquad 1 \le i \le N. \end{aligned} \tag{A}$$

Suppose now that $Y(t) \in C[0,T]$ is the unique continuous solution to the Volterra equation:

$$Y(t) = \frac{G}{1 - \widehat{k}\psi(0)/2} + \frac{1}{1 - \widehat{k}\psi(0)/2}\int_0^t \psi(t - \widehat{k} - s)Y(s)\,ds, \qquad t \in \mathcal{J}, \tag{B}$$

then, if the time steps satisfy: $\max_{1\le i\le N} k_i = \widehat{k} < \min\big\{\ell, 2/\psi(0)\big\}$ we have $y_i \le Y(t_i)$ for $i = 0, \dots, N$.

Proof. Since the magnitude of each time step is bounded so that $1 - k_i\psi(0)/2 \in (0,1)$ for $i = 1, \dots, N$ we can verify, by Lemma 4.1, that the iterates are indeed bounded. Therefore we can define a step function, $y(t)$, as

$$y(t) := \begin{cases} y_{i-1}, & \text{for } t \in [t_{i-1}, t_{i-1/2}) \text{ and } i = 1, \dots, N; \\ y_i, & \text{for } t \in [t_{i-1/2}, t_i) \text{ and } i = 1, \dots, N; \\ y_N, & \text{for } t = t_N = T, \end{cases}$$

where $t_{i-1/2} := (t_i + t_{i-1})/2$ for $i = 1, \dots, N$.

Now, for $1 \le j \le i$ and $i = 1, \dots, N$ we can exploit the monotonic nature of ψ to see that

$$\frac{k_j}{2}\psi(t_i - t_{j-1})y_{j-1} \le \int_{t_{j-1}}^{t_{j-1/2}} \psi(t - \frac{1}{2}\widehat{k} - s)y(s)\,ds \quad \forall t \in [t_{j-1/2}, \min\{t_{i+1/2}, T\}],$$

and

$$\frac{k_j}{2}\psi(t_i - t_j)y_j \le \int_{t_{j-1/2}}^{t_j} \psi(t - \widehat{k} - s)y(s)\,ds \quad \forall t \in [t_j, \min\{t_{i+1/2}, T\}],$$

with the convenient definition: $t_{N+\frac{1}{2}} = T + \widehat{k}/2$. Setting $\beta := 1 - \widehat{k}\psi(0)/2$ and using these results in (A) we find, for $i = 1, \dots, N-1$ and $t \in [t_{i-1/2}, t_{i+1/2})$, or, $i = N$ and $t \in [t_{N-1/2}, T]$, that

$$\begin{aligned} \beta y(t) = \beta y_i &\le g(t_i) + \frac{1}{2}k_i\psi(t_i - t_{i-1})y_{i-1} + \sum_{j=1}^{i-1}\frac{k_j}{2}\Big(\psi(t_i - t_j)y_j + \psi(t_i - t_{j-1})y_{j-1}\Big) \\ &\le G + \int_0^{t_{i-1/2}} \psi(t - \widehat{k} - s)y(s)\,ds \\ &\le G + \int_0^t \psi(t - \widehat{k} - s)y(s)\,ds. \end{aligned}$$

The last step is permitted since $t \geq t_{i-1/2}$ and the integrand is nonnegative. In a similar fashion: for $i = 0$ and $t \in [0, t_{1/2})$ we get

$$y(t) = y_0 \leq G \leq \frac{G}{\beta} \leq \frac{G}{\beta} + \frac{1}{\beta} \int_0^t \psi(t - \widehat{k} - s) y(s)\, ds.$$

So, for $t \in \mathcal{J}$ we have the Volterra inequality:

$$y(t) \leq \frac{G}{\beta} + \frac{1}{\beta} \int_0^t \psi(t - \widehat{k} - s) y(s)\, ds, \qquad t \in \mathcal{J}.$$

From Lemma 3.1 it follows that $y_i = y(t_i) \leq Y(t_i)$ where $Y(t)$ solves (B). Finally we note that $Y(t)$ is actually a continuous function even though $y(t)$ is not—this follows from the theory of Volterra equations. ■

The last preparatory result in this section deals with the fact that we can always choose a non-trivial time step. The context of this result is Theorem 4.1 which follows immediately afterward.

Lemma 4.3. *Let Assumptions 2.2iv.a—d hold; then it is always possible to choose a non-trivial maximum time step $\widehat{k}$ such that, for positive z:*

$$0 < \kappa(-z) - \kappa(\infty) < 1 - \frac{1}{2} z \kappa_s(0) \quad \forall z \leq \widehat{k} < \min\{\ell, 2/\kappa_s(0)\}. \qquad (A)$$

For example: setting

$$\overline{z} := \frac{\kappa(0) - \kappa(-\ell)}{\kappa'(-\ell) + \frac{1}{2}\kappa'(0)} > 0,$$

we can choose $\widehat{k} = \epsilon \min\{\ell, 2/\kappa_s(0), \overline{z}\}$ for any $\epsilon \in (0, 1)$.

Proof. Firstly: $\ell > 0$ and $\kappa_s(0) = -\kappa'(0) > 0$ from the assumptions, so choosing a positive $\widehat{k} < \min\{\ell, 2/\kappa_s(0)\}$ is easy. From Assumption 2.2iv.d it follows that the left hand inequality in (A) is satisfied for any positive $\widehat{k} \leq \ell$. Now, for $z \in [0, \ell]$ we get from 2.2iv.f that

$$\kappa(-z) + \big(\kappa(-\ell) - \kappa(-z)\big) - \kappa(\infty) < 1,$$

which implies

$$\kappa(-z) - \kappa(\infty) + \frac{1}{2} z \kappa_s(0) < 1 \qquad \text{(i)}$$

$$\text{if} \quad \frac{1}{2} z \kappa_s(0) \leq \kappa(-\ell) - \kappa(-z). \qquad \text{(ii)}$$

Setting $F(z) := \kappa(-z) + \frac{1}{2} z \kappa_s(0) - \kappa(-\ell)$ for $z \in [0, \ell]$ we see from 2.2iv.c that $F(0) < 0$ and $F(\ell) > 0$. Also, from 2.2iv.a, $F \in C^1[0, \ell]$ and from from 2.2iv.c it follows that

$$F'(z) = -\kappa'(-z) + \frac{1}{2} \kappa_s(0) > 0 \qquad \forall z \in [0, \ell].$$

It is now clear that $\exists! z^* \in (0, \ell)$ such that $F(z^*) = 0$, and also that any $z \in [0, z^*]$ satisfies (ii) and hence (i) and also (A). Therefore we may choose $\widehat{k} < \min\{\ell, 2/\kappa_s(0), z^*\}$. If this z^* is difficult to find we can easily obtain a useful approximation as $\overline{z} < z^*$ from the following argument. For $0 \le z_1 < z_2 \le \ell$ note, from 2.2iv.c, that $\kappa'(-z_1) > \kappa'(-z_2)$. This implies that $F'(z_1) - F'(z_2) < 0$ and hence $\max F'(z) = F'(\ell)$ for $z \in [0, \ell]$. A simple graphical argument now shows that a suitable $\overline{z}$ may be defined through: $-F(0)/\overline{z} = F'(\ell)$, or

$$\overline{z} := \frac{-F(0)}{F'(\ell)} = \frac{\kappa(0) - \kappa(-\ell)}{\kappa'(-\ell) + \frac{1}{2}\kappa'(0)}.$$

We now choose $\widehat{k} < \min\{\ell, 2/\kappa_s(0), \overline{z}\}$ as indicated in the lemma. This completes the proof. ■

With these preliminaries established we can now give the error estimate for the fully discrete scheme.

Theorem 4.1. *Let Assumptions 2.2 hold along with the additional regularity:* $u \in C^2\big(\mathcal{J}; \mathcal{V} \cap \mathcal{H}^r(\Omega)\big)$ *for some* $r \ge 2$. *Assume also that the finite element space is chosen so that for this* r*:* $\|Ru - u\| \le C_R h^{r-1}|u|_{\mathcal{H}^r(\Omega)}$, *where* $R : \mathcal{V} \mapsto \mathcal{V}^h$ *is the Ritz map, [17], defined by* $a(u - Ru, v) = 0\ \forall v \in \mathcal{V}^h$ *and the mesh size,* h, *is a small parameter. Additionally, let the (positive) time steps be such that* $\max_{1\le i\le N} k_i \le \widehat{k}$ *where*

$$0 < \kappa(-k) - \kappa(\infty) < 1 - \frac{1}{2}k\kappa_s(0) \quad \forall k \le \widehat{k} < \min\{\ell, 2/\kappa_s(0)\}.$$

Then there exist data and solution dependent constants, C_1 *and* C_2, *such that, for* $0 \le i \le N$*:*

$$\|u(t_i) - u_i^{hk}\| \le C_1(t_i; u)h^{r-1} + \max_{1\le j\le i} \sum_{q=1}^{j} C_2(t_j, t_q; u)k_q^2.$$

The form of the C_i *are exhibited in the proof: they do not a-priori contain exponentially large quantities and* $C_2 = O(k_q)$.

Proof. For an arbitrary $i \in \{0, \ldots, N\}$ we define the Ritz projection, $Ru(t_i)$, in the usual way:

$$a\big((u - Ru)(t_i), v\big) = 0 \qquad \forall v \in \mathcal{V}^h,\ t_i \in \mathcal{J}^k,$$

and then set, for $t_i \in \mathcal{J}^k$, $\eta(t_i) := u(t_i) - Ru(t_i)$ and $\zeta_i := Ru(t_i) - u_i^{hk}$. This gives $u(t_i) - u_i^{hk} = \eta(t_i) + \zeta_i$ and, by assumption, $\|\eta(t)\| \le C_R h^{r-1}|u(t)|_{\mathcal{H}^r(\Omega)}$.

Subtracting the fully discrete scheme, (4.4), from the weak formulation, (2.8), gives, with this $t_i \in \mathcal{J}^k$,

$$a(\zeta_i, v) = \int_0^{t_i} b(t_i - s; u(s), v)\, ds - \sum_{j=0}^{i} \varpi_{ij} b(t_i - t_j; u(t_j), v)$$
$$+ \sum_{j=0}^{i} \varpi_{ij} b(t_i - t_j; u(t_j) - u_j^{hk}, v) \qquad \forall v \in \mathcal{V}^h.$$

Analogously to (4.3) we define the quadrature error as

$$\delta(t_i; v) := \int_0^{t_i} b(t_i - s; u(s), v)\, ds - \sum_{j=0}^{i} \varpi_{ij} b(t_i - t_j; u(t_j), v),$$

and assume for the moment that

$$|\delta(t_i;v)| \le Q(t_i)\|v\| \le \widehat{Q}(t_i)\|v\| \quad \forall v \in \mathcal{V}^h \text{ where } \widehat{Q}(t_i) = \max_{1\le j\le i} Q(t_j). \tag{i}$$

The error equation then becomes:

$$a(\zeta_i, v) = \delta(t_i;v) + \sum_{j=0}^{i} \varpi_{ij} b(t_i - t_j; \eta(t_j), v) + \sum_{j=0}^{i} \varpi_{ij} b(t_i - t_j; \zeta_j, v), \qquad \forall v \in \mathcal{V}^h,$$

and taking $v = \zeta_i \in \mathcal{V}^h$ (excluding the trivial case $\zeta_i = 0$) we can write this as

$$\|\zeta_i\| \le Q(t_i) + \sum_{j=0}^{i} \varpi_{ij} \kappa_s(t_i - t_j)\|\eta(t_j)\| + \sum_{j=0}^{i} \varpi_{ij} \kappa_s(t_i - t_j)\|\zeta_j\|,$$

after using Assumption 2.2iv. To deal with the η term note that

$$\sum_{j=0}^{i} \varpi_{ij} \kappa_s(t_i - t_j) = \sum_{j=1}^{i} \frac{1}{2} k_j \Big(\kappa_s(t_i - t_j) + \kappa_s(t_i - t_{j-1})\Big),$$

and, for $0 < j \le i$ and $t_{j-1/2} := (t_j + t_{j-1})/2$,

$$\begin{aligned} \frac{1}{2} k_j \Big(\kappa_s(t_i - t_j) + \kappa_s(t_i - t_{j-1})\Big) &\le \int_{t_{j-1}}^{t_{j-1/2}} \kappa_s(t_i - s)\, ds + \int_{t_{j-1/2}}^{t_j} \kappa_s(t_i - k_j/2 - s)\, ds \\ &\le \int_{t_{j-1}}^{t_j} \kappa_s(t_i - \widehat{k}/2 - s)\, ds, \end{aligned}$$

due to Assumption 2.2iv.c. From this it follows that

$$\begin{aligned} \sum_{j=0}^{i} \varpi_{ij} \kappa_s(t_i - t_j)\|\eta(t_j)\| &\le C_R h^{r-1} |u|_{L_\infty(0,t_i;\mathcal{H}^r(\Omega))} \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \kappa_s(t_i - \widehat{k}/2 - s)\, ds, \\ &= C_R h^{r-1} |u|_{L_\infty(0,t_i;\mathcal{H}^r(\Omega))} \big(\kappa(-\widehat{k}/2) - \kappa(t_i - \widehat{k}/2)\big). \end{aligned}$$

Returning now to the estimate for ζ_i we can use this to write:

$$\|\zeta_i\| \le G + \sum_{j=0}^{i} \varpi_{ij} \kappa_s(t_i - t_j)\|\zeta_j\|,$$

with G defined by

$$G := \widehat{Q}(t_i) + C_R h^{r-1} \big(\kappa(-\widehat{k}/2) - \kappa(\infty)\big) |u|_{L_\infty(0,t_i;\mathcal{H}^r(\Omega))},$$

which follows since $\kappa(-\widehat{k}/2) - \kappa(t_i - \widehat{k}/2) \le \kappa(-\widehat{k}/2) - \kappa(\infty)$ from Assumption 2.2iv.c. Now we set $y_i := \|\zeta_i\|$ and $\psi(t_i, t_j) := \kappa_s(t_i - t_j)$ and use the bound on $\widehat{k}$ to see from Lemma 4.1 that the iterates, $\|\zeta_i\|$, are bounded, therefore by Lemma 4.2 we find that $\|\zeta_i\| \le Y(t_i)$, where $Y(t)$ is the unique continuous solution to

$$Y(t) = \frac{G}{\beta} + \frac{1}{\beta} \int_0^t \kappa_s(t - s - \widehat{k}) Y(s)\, ds, \qquad t \in \mathcal{J},$$

where $\beta := 1-\widehat{k}\kappa_s(0)/2$. To bound Y we invoke Lemma 3.2 with $G_1 = G/\beta$, $G_2 = 1/\beta$ and $\chi(t-s) = \kappa(t-s-\widehat{k})$ to find that

$$Y(t) \leq \frac{G/\beta}{1-\big(\kappa(-\widehat{k})-\kappa(\infty)\big)/\beta} = \frac{G}{\big(1-\frac{1}{2}\widehat{k}\kappa_s(0)\big)-\big(\kappa(-\widehat{k})-\kappa(\infty)\big)}, \tag{ii}$$

which is well defined due to our requirement on the time steps.

Turning now to the quadrature error we get, from (4.3), that

$$\big|\delta(t_i;v)\big| \leq \frac{1}{8}\sum_{j=1}^{i} k_j^2 \int_{t_{j-1}}^{t_j} \left|\frac{\partial^2}{\partial s^2} b(t_i-s;u(s),v)\right| ds.$$

After isolating and squaring the integrand, using: the definition of the bilinear form given in (2.10), the Cauchy-Schwartz inequality and, finally, taking the square root (Using the fact that for nonnegative real numbers a, b and c: $a^2 \leq b^2+c^2 \;\Rightarrow\; a \leq b+c$.) we obtain the bound

$$\left|\frac{\partial^2}{\partial s^2} b(t_i-s;u(s),v)\right| \leq \Upsilon(t_i,s;u)\|v\|,$$

where

$$\Upsilon(t_i,s;u) := \left\|\Lambda_1^{\frac{1}{2}}\Big(\lambda_s(t_i-s)\nabla\cdot u(s)\Big)_{ss}\right\|_{L_2(\Omega)} + \sum_{mn}\left\|\Lambda_2^{\frac{1}{2}}\Big(\mu_s(t_i-s)\;\epsilon_{mn}\big(u(s)\big)\Big)_{ss}\right\|_{L_2(\Omega)}.$$

Therefore, in (i), we can set

$$\widehat{Q}(t_i) := \frac{1}{8}\max_{1\leq j\leq i}\sum_{q=1}^{j} k_q^2 \int_{t_{q-1}}^{t_q} \Upsilon(t_j,s;u)\,ds.$$

Combining these results with (ii), and using the facts: $\|u(t_i)-u_i^{hk}\| \leq \|\eta(t_i)\| + \|\zeta_i\|$ and $\|\zeta_i\| \leq Y(t)$, we find, after some rearrangement, that for the terms in h^{r-1}:

$$C_R|u(t_i)|_{\mathcal{H}^r(\Omega)} + \frac{C_R|u|_{L_\infty(0,t_i;\mathcal{H}^r(\Omega))}\big(\kappa(-\widehat{k}/2)-\kappa(\infty)\big)}{1-\frac{1}{2}\widehat{k}\kappa_s(0)-\kappa(-\widehat{k})+\kappa(\infty)} \leq C_1(t_i;u),$$

where

$$C_1(t_i;u) := \frac{C_R|u|_{L_\infty(0,t_i;\mathcal{H}^r(\Omega))}\Big(1-\frac{1}{2}\widehat{k}\kappa_s(0)\Big)}{1-\frac{1}{2}\widehat{k}\kappa_s(0)-\kappa(-\widehat{k})+\kappa(\infty)},$$

which follows since $\kappa(-\widehat{k}/2)-\kappa(-\widehat{k}) < 0$ from Assumption 2.2iv.c. Also, for the k_q terms:

$$\frac{\widehat{Q}(t_i)}{1-\frac{1}{2}\widehat{k}\kappa_s(0)-\kappa(-\widehat{k})+\kappa(\infty)} = \max_{1\leq j\leq i}\sum_{q=1}^{j} C_2(t_j,t_q;u)k_q^2,$$

where

$$C_2(t_j,t_q;u) := \frac{\frac{1}{8}\int_{t_{q-1}}^{t_q}\Upsilon(t_j,s;u)\,ds}{1-\frac{1}{2}\widehat{k}\kappa_s(0)-\kappa(-\widehat{k})+\kappa(\infty)}.$$

Assembling these results then demonstrates that

$$\|u(t_i)-u_i^{hk}\| \leq C_1(t_i;u)h^{r-1} + \max_{1\leq j\leq i}\sum_{q=1}^{j} C_2(t_j,t_q;u)k_q^2,$$

as required. ∎

In the next section we apply this same methodology to a model problem to investigate the potential of these theoretical error bounds for providing numerical feedback during the time stepping.

5. Exploiting the estimates. We have termed the estimates derived in the preceeding sections "sharp" because they do not *a priori* contain the exponentially large quantities which would result from applying the usual Gronwall lemma. It is pertinent therefore to enquire whether we could use the estimate to provide a computable error bound which would indicate how to choose the current time step so that $\|u(t_i) - u_i^{hk}\| \leq TOL + O(h^{r-1})$, where TOL is some user-defined tolerance level. It seems that, in general, this is not possible unless we allow an, at best, linear growth with t in the computed bound. However, there does seem some potential for providing a computable bound which would at least give some indication of how "good" the numerical solution is.

In this section we will illustrate these observations by adopting a model scalar Volterra problem to capture the essence of (2.8): find $u \in C^2(\mathcal{J})$ such that

$$u(t) = g(t) + \int_0^t \lambda_s(t-s)u(s)\,ds \quad \text{for } t \in \mathcal{J} := [0,T],\ T \in \mathbb{R}_+. \tag{5.1}$$

The assumptions of Section 2 are assumed to apply here also, in particular: λ is a relaxation function, as defined in Assumption 2.1, and $\exists\kappa(t)$ such that $0 < \lambda_s(t-s) \leq \kappa_s(t-s)$ in the same way as in Assumption 2.2. We also note that for a synchronous homogeneous material (3.11) shows that (5.1) is the semidiscrete approximation to a given nodal displacement, and so this simpler problem is directly relevant to the general one. Replacing the integral in (5.1) with the trapezoidal rule and $u(t_i)$ with u_i^k for $t_i \in \mathcal{J}^k$ (recall (4.1)), we then carry out an error analysis in exactly the same way as in Theorem 4.1. This yields the bound:

$$|u(t_i) - u_i^k| \leq E_u(t_i) := C_E \max_{1\leq j\leq i} \sum_{q=1}^{j} k_q^2 \int_{t_{q-1}}^{t_q} \left|\big(\lambda_s(t_j - s)u(s)\big)_{ss}\right| ds, \tag{5.2}$$

where

$$C_E := \frac{8^{-1}}{1 - \frac{1}{2}\widehat{k}\kappa_s(0) - \kappa(-\widehat{k}) + \kappa(\infty)} \quad \text{and} \quad \widehat{k} < \min\{\ell, 2/\kappa_s(0), \overline{z}\}, \tag{5.3}$$

with the data-dependent $\overline{z}$ given by Lemma 4.2.

The reason why we cannot choose k_i, at time t_i, such that $|u(t_i) - u_i^k| \leq TOL$ is now clear. This error bound contains history and so, even if we had some means of calculating the integrals on the right of (5.2) to an arbitrary degree of accuracy, the indicated "max" in (5.2) may occur for some $j < i$. Therefore k_i may not be involved in $E_u(t_i)$ and so there would be no rule for choosing k_i such that $E_u(t_i) \leq TOL$. A way around this might be to turn the clock back to some $j^* < j$, discard the solutions $\{u_i^k\}_{i\geq j^*}$, and then recompute through t_{j^*} with smaller time steps. Such a scheme is potentially rather expensive and it is not clear whether or not the algorithm would ever halt (i.e. $t_i = T$) without having to choose exceedingly small time steps. Another strategy would be to remove the history from the integrals in (5.2) by expanding the derivative, using the triangle inequality and taking the suprema of the λ_s, λ_{ss} and λ_{sss}

out of the sum. The price of such a move could be considerable: even for a solution $u =$constant$\neq 0$ the bound would grow monotonically (while error probably does not—from (5.2)) and so the k_i would be forced progressively smaller for no good reason. We conclude that there cannot be any reliable and economic adaptivity criterion derived from (5.2) which would provide a rule for choosing k_i such that $|u(t_i) - u_i^k| \leq TOL$.

So, even if we can compute the definite integrals of $\left|\big(\lambda_s(t_j - s)u(s)\big)_{ss}\right|$ to any desired degree of accuracy in (5.2) we still seem unable to exercise economical control over the error. The best we can hope for, it would seem, is that (5.2) could be used to yield a computable error estimate that would at least indicate the integrity of the numerical solution. A fundamental problem still remains: we cannot, of course, compute the integrals in (5.2)—but here there are at least two possibilities. We could make finite difference and quadrature approximations to the integrals and then replace $u(t_q)$ with the numerical solution u_q^k, or, we could expand the derivative and remove the $u(t)$ terms from the integrand by using the weak stability estimates analogous to those given in Theorem 3.1. In this case these are:

$$|u(t)| \leq C_s := \|g\|_{L_\infty(0,t)}/C_\kappa, \tag{5.4}$$

$$|u'(t)| \leq C_s' := \big(\|g'\|_{L_\infty(0,t)} + C_s\lambda_s(0)\big)/C_\kappa, \tag{5.5}$$

$$|u''(t)| \leq C_s'' := \big(\|g''\|_{L_\infty(0,t)} + C_s\|\lambda_{ss}\|_{L_\infty(0,t)} + C_s'\lambda_s(0)\big)/C_\kappa. \tag{5.6}$$

In this way we still integrate derivatives of the relaxation function λ, but can reasonably expect the term to remain bounded—at least for relaxation functions of the form given below in (5.9).

We still do not have a rule for choosing the current time step k_i. If k_i is not involved in (5.2) then it does not seem to matter how it is chosen, on the other hand, if it is involved then it appears reasonable to select k_i so that the newest contribution to the error is controlled, that is:

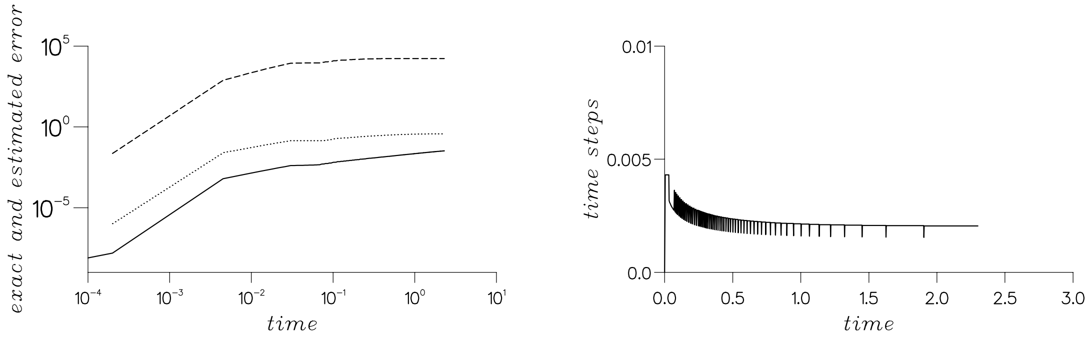
$$k_i^2 \int_{t_{i-1}}^{t_i} \left|\big(\lambda_s(t_i - s)u(s)\big)_{ss}\right| ds \leq TOL. \tag{5.7}$$

Employing a finite difference approximation to the integrand, a backward rectangle rule approximation to the integral (i.e. sampling at the endpoint t_{i-1}), and then replacing $u(t_{i-2})$, $u(t_{i-1})$ and $u(t_i)$ with u_{i-2}^k, u_{i-1}^k and u_i^k, respectively, we are then led to choose k_i iteratively so that:

$$k_i^3 \leq \frac{1}{2}TOL\left(\left|\frac{w(t_i,t_i) - w(t_i,t_{i-1})}{k_i(k_i + k_{i-1})} - \frac{w(t_i,t_{i-1}) - w(t_i,t_{i-2})}{k_{i-1}(k_i + k_{i-1})}\right| + \epsilon\right)^{-1}, \tag{5.8}$$

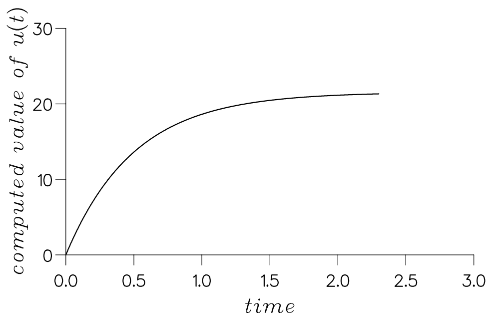
where: $w(t_i,t_j) := \lambda_s(t_i - t_j)u_j^k$ and $0 < \epsilon \ll 1$ is a defensive programming parameter that ensures the quotient is always defined.

In Figures 5.1 and 5.2 we give the results of some numerical experiments based upon using (5.8) as the time step selection criterion and each of the methods outlined above for computing the error estimate. The finite-difference-and-quadrature method for approximating the integral in (5.2) proceeds as follows: the rectangle rule for numerical integration is used over the intervals $(0,t_1)$ and (t_{i-1},t_i) and the trapezoidal rule over (t_1,t_{i-1}). Then the integrand, $\big(\lambda_s(t_i - s)u(s)\big)_{ss}$, is replaced with a finite difference quotient (as in (5.8)) of the numerical solution $\lambda_s(t_i - t_q)u_q^k$. Finally the



(a) (b)

Parameters: $a_1 = -a_2 = 21.5$,
$\omega = 0, \quad \xi = \pi/2$,
$\beta = -2$



(c)

Figure 5.1. Results of the test problem (5.1) with (5.9) and exact solution given by (5.10). The constants in (5.10) are shown in the figure.

sum is calculated at each time step and the maximum retained, as required in (5.2). For computing the weak stability constants in (5.4—6) we employ a finite difference replacement to approximate the derivatives of g (this is for ease of programming), the derivatives of λ are, however, programmed exactly.

For the numerical experiments we use the relaxation function

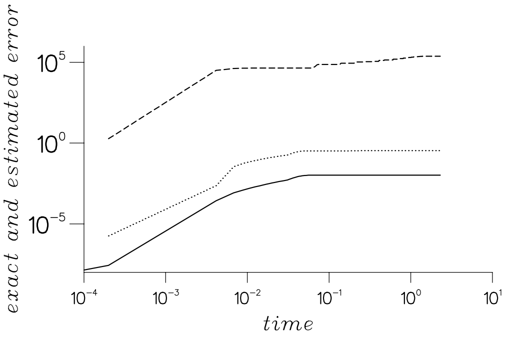
$$\lambda(t) = 0.18343 + 0.430766e^{-1.593t} + 0.385804e^{-53.821t}, \tag{5.9}$$

which is based upon real creep data for a nylon 66 material, see [15]. The units of time here are “years”. In this case we can set $\kappa(t) = \lambda(t)$ and find the values: $\ell = 7 \times 10^{-3}$, $\widehat{k} \approx 4.307 \times 10^{-3}$, $C_E \approx 3.718$, sufficient to satisfy the requisite assumptions. Also, we set $TOL = 0.01$ and compute to the final time of $T = 2.3$ years, and since (5.8) cannot be used for k_1 we choose a “small” initial time step as $k_{\text{init}} = k_1 = 0.0001$. Using λ as above the loading term is then set so that the exact solution to (5.1) is

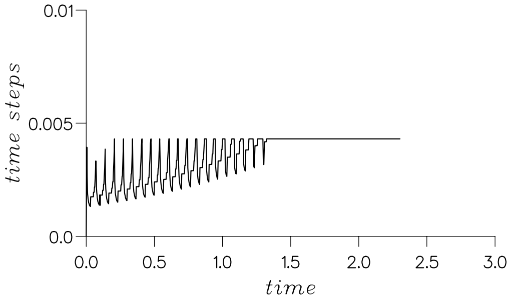
$$u(t) = (a_1 + a_2 e^{\beta t}) \sin(\omega t + \xi), \tag{5.10}$$

where a_1, a_2, β, ω and ξ are constants with the particular values indicated in the figures.

Obtained from (5.10), the solid line in diagram (a) of the figures shows the exact value of $\max_{0 \le j \le i} |u(t_j) - u_j^k|$, while the dotted and dashed lines show respectively the approximation to $E_u(t_i)$ obtained from the finite-difference-and-quadrature and weak stabilty schemes. Note that the axes are log-log and that the weak stability results are very poor—grossly over estimating the actual error. Diagrams (b) and (c) show,

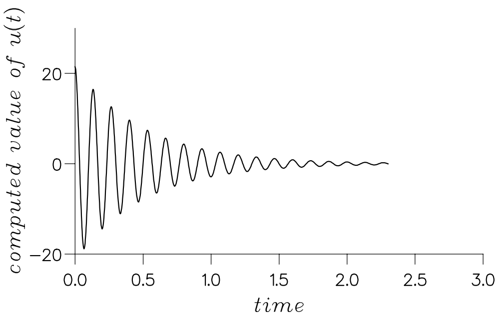


(a)



(b)

Parameters: $a_1 = 0, \quad a_2 = 21.5,$
$\omega = 15\pi, \quad \xi = \pi/2,$
$\beta = -2$



(c)

Figure 5.2. Results of the test problem (5.1) with (5.9) and exact solution given by (5.10). The constants in (5.10) are shown in the figure.

respectively, the variation of the time step size and the numerically computed solution. It is clear in Figure 5.2 that the time steps are forced smaller when computing through the local extrema of $u(t)$ although (5.8) does not seem to have worked so well for the case in Figure 5.1.

6. Closure. We believe that this is the first time the approach adopted above has been applied to the viscoelasticity problem. It is clear that the *a priori* bounds derived in the preceeding study are far sharper, and therefore more realistic, that those obtainable by the standard "Gronwall" approach. Before closing there are a few observations worth making.

The presence of the unknown u (with its time derivatives), could be removed completely from the time discretization error part of the bound given in Theorem 4.1 (i.e. the term C_2) by invoking the weak stability estimates given in Theorem 3.1. This is unlikely to be computationally useful however since the numerical results in the previous section show that such a bound can give a very poor indication of how the actual error behaves.

Examining the proof of Theorem 4.1 we see that the "time error", $\|\zeta_i\|$, can be bounded by $Y(t)$—the solution to a relatively simple Volterra equation. Rather than use the *a priori* information of Lemma 3.2 it may be more useful, from a computational standpoint, to actually solve for $Y(t)$ either exactly, if possible, or using a reliable numerical method. In this respect the results of the previous section for the simple problem (5.1) are directly relevant. Also, such an approach should remove the necessity of taking the "max" over the quadrature error terms since Lemmas 3.1 and 4.2 continue to hold when G is time dependent.

The logarithmic scale used in part (a) of Figures 5.1 and 5.2 is somewhat misleading. We should emphasize that the computed error bound based on the weak stability constants, (5.4—6), is hopelessly poor. The error is overestimated by a factor of around 10^{10}! On the other hand the bound produced by the finite-difference-and-quadrature method is, in comparison, much closer to the real error. However, at the moment we cannot justify using the discrete solution to approximate $u''(t)$. The numerical tests go some way to providing assurance that this can be an acceptable practice but it is far from clear whether or not this is always true.

Finally, the simple property: $f(t+k) = e^k f(t)$ enjoyed by the exponential function $f(t) := e^t$, means that an efficient history storage scheme for (4.5) can be devised when the relaxation function is of the form (5.9). It would be interesting to determine whether or not a result such as Lemma 4.2 can hold for sparse quadrature schemes of the type discussed in [16]. If so then it would become practical to use more general relaxation functions and still have a degree of confidence in the numerical results.

7. References.

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