AN ERROR BOUND VIA THE RITZ-VOLTERRA PROJECTION FOR A FULLY DISCRETE APPROXIMATION TO A HYPERBOLIC INTEGRODIFFERENTIAL EQUATION (LONG VERSION)*

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Abstract. A fully discrete scheme for approximating a second order hyperbolic Volterra integrodifferential equation of the second kind is proposed. The discretization is accomplished by applying the finite element method in the space variables, a finite difference replacement for the time derivative and the trapezoidal rule for the history integral. Using the Ritz–Volterra projection the error associated with the scheme is analyzed and an optimal error estimate derived. This bound depends crucially on the way the initial conditions are represented in the scheme in a similar way as for analogous discretizations of the wave equation.

Key words. Volterra equation, finite element method, error estimate, Ritz–Volterra projection.

AMS(MOS) subject classifications. 65M60, 65M15, 45K05

1. Introduction. Let Ω be an open bounded subset of $\mathbb{R}^n$ with polyhedral boundary $\partial\Omega$ and for some $T > 0$ define the time interval: $\mathcal{I} := (0, T] \subset \mathbb{R}$, with $\bar{\mathcal{I}} := [0, T]$. We consider the approximation of the classical problem of finding a solution $u \in \mathcal{C}^2(\mathcal{I} \times \Omega)$ to the second-order hyperbolic Volterra integrodifferential equation

$$u''(t) + Au(t) + \int_0^t B(t,s)u(s)\; ds = f(t) \text{ in } \Omega,\, t \in \mathcal{I}, \tag{1.1}$$

with prescribed smooth initial conditions $u(0)$ and $u'(0)$ and homogeneous Dirichlet boundary data. Our prime concern is with the temporal discretization of (1.1) and so throughout we refrain from explicitly indicating the space variables. Here f is assumed to be a smooth function, both spatially and temporally; A and $B(t,s)$ are linear, second-order partial differential operators with the former assumed positive-definite and self-adjoint. Also we assume that $B(t,s)$ is infinitely differentiable with respect to both t and s on the triangle $0 \le s \le t \le T$.

Our aim is to formulate a fully discrete analogue of (1.1) and to estimate the approximation error; to begin we define a weak formulation of (1.1) as follows: for $\mathcal{V} \subset \mathcal{H}_0^1(\Omega)$, find a twice differentiable map $u(t) : \mathcal{I} \mapsto \mathcal{V}$ such that, for $t \in \mathcal{I}$,

$$\big(u''(t), v\big) + a\big(u(t), v\big) + \int_0^t b\big(t,s;u(s),v\big)\; ds = \big(f(t), v\big) \qquad \forall v \in \mathcal{V}, \tag{1.2}$$

with initial data given by $u(0)$ and $u'(0)$. The symbol $(\cdot,\cdot)$ denotes the standard $\mathcal{L}_2(\Omega)$ inner product and $a(\cdot,\cdot)$ and $b(t,s;\cdot,\cdot)$ are continuous bilinear maps, each taking $\mathcal{V} \times \mathcal{V}$ into $\mathbb{R}$, associated, respectively, with A and $B(t,s)$. As a consequence of our assumptions $a(\cdot,\cdot)$ is symmetric and positive-definite and $b(t,s;\cdot,\cdot)$ has the same

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differentiability properties as $B(t,s)$. We shall denote these differentiations with subscripts so that, for example, $b_{tts}(t,s;\cdot,\cdot)$ corresponds to $\partial^3 b(t,s;\cdot,\cdot)/\partial t^2\partial s$. Also we assume that a unique $u \in \mathcal{C}^2(\mathcal{I};\mathcal{V})$ exists solving (1.2); for this we refer to (for example) [1] and [14] where the questions of existence, uniqueness and smoothness of solutions are addressed at length together with some indication of how (1.1) might arise in practical situations.

Although the analysis presented here is confined to "scalar" problems, this work has been motivated by the need to produce numerical models for the dynamic response of viscoelastic structures under time dependent loading. For example: in the case of a non-ageing linear viscoelastic material (1.2) is the prototype for the governing equations of motions of the structure; the scalar $u(t)$ is replaced by the displacement functions $u \in \mathcal{C}^2\big(\mathcal{I};\big(\mathcal{H}^1(\Omega)\big)^n\big)$ and the bilinear forms have the explicit representation

$$\begin{aligned} a\big(u(t),v\big) &:= \int_\Omega \lambda(0)\nabla\cdot u(t)\nabla\cdot v + \mu(0)\varepsilon_{ij}\big(u(t)\big)\varepsilon_{ij}(v)\; d\Omega, \\ b\big(t,s;u(t),v\big) &:= -\int_\Omega \lambda_s(t-s)\nabla\cdot u(s)\nabla\cdot v + \mu_s(t-s)\varepsilon_{ij}\big(u(s)\big)\varepsilon_{ij}(v)\; d\Omega, \end{aligned} \tag{1.3}$$

for v now belonging to an appropriate subspace of $\big(\mathcal{H}^1(\Omega)\big)^n$. The ε_{ij} denote first derivative quantities that arise from the infinitesimal strain tensor and so the Einstein summation convention is implied in (1.3). The linear form $f(v) := (f,v)$ represents the loading on the structure and the functions λ and μ are the viscoelastic analogues of the Lamé coefficients that appear in linear elasticity theory. Much more detail on the origin of (1.3) and formulations for quasistatic viscoelastic problems may be found in [4]. We have confined ourselves to the simpler problem (1.1), (1.2), primarily in the interest of clarity and ease of exposition although the numerical scheme presented here has been implemented in the viscoelastic context in [17].

Fully discrete and semidiscrete methods for the parabolic counterpart of (1.1)–(1.2) are dealt with at some length in the literature, see for example [6], where a semidiscrete estimate for the case of non-smooth initial data is given for a linear problem, and [7] where the history is considered non-linear and an estimate given for a fully discrete explicit time stepping scheme. Prior to these in [5] estimates are given both for an explicit (backward Euler) scheme and an implicit (Crank-Nicolson) scheme, each combined with a sparse quadrature rule. Also, in [8] and [9] we find an error analysis for a non-linear hereditary heat conduction problem. As far as the authors are aware however, relatively little has been presented in the way of fully discrete approximations to (1.2), with the exception of the analysis of a model problem by Douglas and Jones [10], upon which we loosely base our numerical scheme, and a Lax-Wendroff method, again for a model problem, given in [11]. Also, [18] gives a numerical scheme, with a quoted estimate, for a hyperbolic problem in which $B(t,s)$ is a first-order differential operator. Many examples of integrodifferential equations that arise in practical situations are also given in [18].

In Section 2 we present our fully implicit three level scheme for approximating (1.2), we use the standard finite difference replacement for $u''(t)$ and replace the time integral in the continuous problem by the trapezoidal rule for numerical integration, although there seems no reason why a sparse quadrature method such as that proposed in [5] could not be employed in order to reduce the storage and calculation requirements of the algorithm. Fortunately, however, in linear viscoelasticity the time dependent part of the history kernel is usually in the form of a Dirichlet series, $\sum_i C_i e^{\alpha_i(t-s)}$ (see [2] or [3]), in which case a simple recursion formula may be found which obviates the need for storing the entire solution history, again see [4] for more detail. Our scheme is only one of a number that could be chosen, we rely on averaging (1.2) at three consecutive time levels with the stencil $\left]\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4}\right[$, although more general schemes may be generated by using $\left]\Theta\quad(1-\Theta-\Phi)\quad\Phi\right[$, for the wave equation $\Theta=\Phi=1/2$ is a popular choice. Also in [15, Chap. 11] Linz demonstrates how the semidiscrete form of (1.2) may be rewritten as a system of standard Volterra equations (of the second kind) and therefore solved by standard methods.

Section 4 contains an error analysis of the scheme; this is accomplished with the aid of a relatively new analytical tool: the Ritz-Volterra projection of the solution u of (1.2) into the finite element space $\mathcal{V}^h$. This projection seems to have been first proposed by Cannon and Lin, eg [8], and Lin, eg [9], and is a generalization of the Ritz, or elliptic, projection, [12], to problems with memory. Recognizing this generality Lin, Thomée and Wahlbin in [13] studied the projection in greater generality and applied it to derive error estimates for semidiscrete schemes for (1.2) and its parabolic counterpart, among others. Accordingly, Section 3 prefixes Section 4 with an outline of some of the pertinent results presented in [8] and [13]. Our analysis yields an $O(h^r+k^2)$ type estimate, but the approximation of the initial conditions is seen to be of vital importance in attaining this predicted convergence rate; this we explore briefly in Section 5.

Throughout, $\|\cdot\|_r$ will denote the norm on $\mathcal{H}^r(\Omega)$ and $h<1$ the mesh size. As usual the symbol C will indicate the so-called generic positive constants, not necessarily the same at any given occurence and often depending upon unknown functions or quantities, when we wish to make this dependence explicit we shall use, for example, expressions of the form $C(t)$ or $C(u)$. Finally the fully discrete scheme employs a constant time step k which we assume to be bounded above, that is $k\le k_0$.

2. Fully discrete implicit approximation. The spatial discretization is accomplished in the usual finite element way by replacing $\mathcal{V}$ with a finite dimensional subspace $\mathcal{V}^h\subset\mathcal{V}$ and then seeking $u(x,t)\in\mathcal{C}^2(\mathcal{I};\mathcal{V}^h)$ such that (1.2) holds for all $v\in\mathcal{V}^h$. Next we choose a constant time step $k\le k_0$ such that $T/k=N\ge 2$ is a positive integer and form the discrete time intervals:

$$\mathcal{I}_j^k := \left\{t_i = ik\in\bar{\mathcal{I}} : j\le i\le N-1\right\} \tag{2.1}$$

for $0\le j\le N-1$, and define $\bar{\mathcal{I}}_j^k := \mathcal{I}_j^k\cup\{T\}$. To deal with the history term in (1.2) we replace the integral with the trapezoidal rule approximation, where for $\varphi(t)\in\mathcal{C}^2(\mathcal{I})$ and $t_n\in\bar{\mathcal{I}}_0^k$ we have

$$\begin{aligned}\int_0^{t_n}\varphi(s)\,ds-\sum_{i=0}^{n}\varpi_{ni}\varphi(t_i)&=\delta_n(k;\varphi),\\ |\delta_n(k;\varphi)|\le\sum_{i=1}^{n}Ck^2\int_{t_{i-1}}^{t_i}&|\varphi''(s)|\,ds,\end{aligned} \tag{2.2}$$

and $\varpi_{n0}=\varpi_{nn}=k/2$, $\varpi_{ni}=k$ for $1\le i\le n-1$ and $\varpi_{00}:=0$. For a function w defined on $\bar{\mathcal{I}}_0^k$ we write $w_i\equiv w(t_i)$ and set up the following notation:

$$\left.\begin{aligned}\partial_t w_i &:= \frac{w_i-w_{i-1}}{k}\\ \bar{w}_i &:= \frac{1}{2}\left(w_i+w_{i-1}\right)\end{aligned}\right\}\qquad 1\le i\le N \tag{2.3}$$

$$
\left.\begin{aligned}
\bar{\partial}_t w_{i+1} &:= \frac{1}{2}\left(\partial_t w_{i+1} + \partial_t w_i\right) \\
\partial_t^2 w_{i+1} &:= \partial_t \partial_t w_{i+1} \\
&= \frac{w_{i+1} - 2w_i + w_{i-1}}{k^2} \\
\Delta w_i &:= \frac{1}{4}\left(w_{i+1} + 2w_i + w_{i-1}\right) \\
&= w_0 \text{ when } i = 0.
\end{aligned}\right\} \qquad 1 \le i \le N-1 \tag{2.4}
$$

We now define the fully discrete approximation to the solution $u(x,t)$ of (1.2) to be a sequence $\{U_i\}_{i=0}^N$, with U_0 and U_1 given, such that $\forall t_n \in \mathcal{I}_1^k$

$$
(\partial_t^2 U_{n+1}, v) + a(\Delta U_n, v) + \sum_{q=0}^{n} \omega_{nq} b(t_n, t_q; \Delta U_q, v) = (\Delta f_n, v) \qquad \forall v \in \mathcal{V}^h. \tag{2.5}
$$

The reason for the choice of differencing stencil, $\Delta \equiv \left]\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4}\right[$, is that for the wave equation, $(u'', v) + (\nabla u, \nabla v) = (f, v)$, discretized in this manner the time step k may be chosen independently of the space $\mathcal{V}^h$, i.e. the scheme is unconditionally stable. Also, since $k^{-2}(\cdot,\cdot) + a(\cdot,\cdot)$ is positive-definite we deduce that (2.5) possesses a unique solution for U_{n+1} provided that k is small enough.

3. Ritz–Volterra projection. In this section we outline those results presented in [8] and [13] which are pertinent to the analysis of the next section. The projection, or mapping $V \equiv V(t) : \mathcal{C}(\bar{\mathcal{I}}; \mathcal{V}) \to \mathcal{C}(\bar{\mathcal{I}}; \mathcal{V}^h)$, is defined through

$$
a\big(w(t) - u(t), v\big) + \int_0^t b\big(t, s; w(s) - u(s), v\big)\, ds = 0 \qquad \forall v \in \mathcal{V}^h \text{ and } \forall t \in \bar{\mathcal{I}}, \tag{3.1}
$$

where $w(t) := Vu(t)$. The existence and uniqueness of w follows from the fact that $a(\cdot,\cdot)$ is positive definite, see [13]. Also from that source we have the following

PROPOSITION 3.1. *Assume that the approximation space $\mathcal{V}^h$ is chosen so that for $r \ge 2$ $\|Ru\|_1 \le C\|u\|_1$ and $\|Ru - u\|_s \le Ch^{r-s}\|u\|_r$, $s = 0, 1$, where $R : \mathcal{V} \mapsto \mathcal{V}^h$ is the Ritz mapping, [12], defined by $a(u - Ru, v) = 0$ $\forall v \in \mathcal{V}^h$ and $h < 1$ is a small parameter, the so-called "mesh size". Then for $w(t)$ defined by (3.1) and for non-negative integers i, with D_t^i denoting i^{th} order time differentiation, there exist constants C_i such that for $t \in \bar{\mathcal{I}}$*

$$
\left\|D_t^i (w - u)(t)\right\|_0 + h\left\|D_t^i (w - u)(t)\right\|_1 \le C_i h^r |||u(t)|||_{i,r}
$$

where $|||\cdot|||_{i,r}$ is a special notation for

$$
|||u(t)|||_{i,r} := \sum_{j=0}^{i} \left(\left\|D_t^j u(t)\right\|_r + \int_0^t \left\|D_t^j u(s)\right\|_r ds \right).
$$

Clearly then: $u \in \mathcal{C}^i\big(\bar{\mathcal{I}}; \mathcal{H}^r(\Omega)\big) \Longrightarrow D_t^i w \in \mathcal{V}^h$ for $t \in \bar{\mathcal{I}}$ and $r \ge 2$.

Also we shall need to assume that

$$
\left\|D_t^i w\right\|_1 \le C|||u|||_{i,1}, \tag{3.2}
$$

for which we refer to Lemma 3.4 in [8].

4. Error analysis. We now exploit the Ritz–Volterra projection and derive an *a priori* error bound for the scheme outlined in Section 2. Firstly we define for $t_n \in \bar{\mathcal{I}}_0^k$, $\theta_n := U_n - w(t_n)$ and $\eta(t_n) := w(t_n) - u(t_n)$, then for suitable norm, $\|\cdot\|$, we have

$$\|U_n - u(t_n)\| \leq \|\theta_n\| + \|\eta(t_n)\| . \tag{4.1}$$

In view of the smoothness assumptions made in Section 1 we shall also assume that the continuity of the bilinear form, $b(t,s;\cdot,\cdot)$, also implies that, for a t independent function $v \in \mathcal{V}$ and non-negative integers i and j,

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial s^j} b(t,s;w(s),v)\right| \leq C\,\|v\|_1 \left(\sum_{\ell=0}^{j} \left\|w^{(\ell)}(s)\right\|_1\right) \qquad \forall t, t-s \in \mathcal{I}. \tag{4.2}$$

Also, in order to justify the ensuing analysis we shall require that f is such that the solution of (1.2), $u \in \mathcal{C}^4\big(\mathcal{I};\mathcal{V}\cap\mathcal{H}^r(\Omega)\big)$ for some $r \geq 2$. Then, rearranging (3.1) we arrive at a Volterra equation of the second kind for the unknown w with a forcing term dependent upon u, due to the regularity of u and our smoothness assumption on the history form we have that $w \in \mathcal{C}^4(\mathcal{I};\mathcal{V}^h)$, see for example Linz [15].

The results of [13], summarized in Section 3, already provide the necessary bounds for η and so our task is to bound θ. This is accomplished subsequently in Theorem 4.1. Initially we state four lemmas that will considerably shorten the proof of that theorem. Each of these lemmas illustrate that a substantial amount of cancellation occurs in the terms that arise when moving from (1.2) to (2.5). For completeness the proofs are given in detail, they are quite simple but rather long and routine. Also the statements of the lemmas are rather minimal since they will be given a context during the proof of Theorem 4.1. In the following $b(t,s;\cdot,\cdot)$ is as defined in Section 1 and the $O(k)$ quantities, ω_{nq}, represent the weights associated with the trapezoidal rule for numerical quadrature, also we employ the notation of (2.3) and (2.4). In what follows we shall make frequent use of the inequality

$$ab < a^2/\epsilon^2 + \epsilon^2 b^2 \qquad \forall a,b \in \mathbb{R} \text{ and } \epsilon > 0; \tag{4.3}$$

the discrete Gronwall lemma (quoted in, for example, [4]); and of the formulæ:

$$u = \int_a^x g(x,y)\,dy \implies \frac{du}{dx} = g(x,x) + \int_a^x \frac{\partial g}{\partial x}(x,y)\,dy \tag{4.4}$$

$$a^2 + b^2 \leq (a+b)^2 \tag{4.5}$$

$$a^2 + b^2 \leq \sum_i c_i^2 \implies a + b \leq \sqrt{2}\sum_i c_i \tag{4.6}$$

where a, b, c_1, c_2, c_3,... are non-negative real numbers (in the last two formulæ) and u and g are sufficiently regular.

LEMMA 4.1. *Let $\bar{\theta}_i \in \mathcal{V}^h$ for $1 \leq i \leq N$, then for the error in the trapezoidal approximation:*

$$\delta_i := \int_0^{t_i} b(t_i,s;w(s),\bar{\theta}_{i+1}-\bar{\theta}_i)\,ds - \sum_{j=0}^{i} \varpi_{ij} b(t_i,t_j;w(t_j),\bar{\theta}_{i+1}-\bar{\theta}_i),$$

there exist positive constants C such that for $m \leq N-1$,

$$\left|\sum_{i=1}^{m}\delta_i\right| \leq \frac{C}{\epsilon}\left\|\bar{\theta}_{m+1}\right\|_1^2 + Ck^4(1+\epsilon T + T^2)\int_0^T \sum_{\ell=0}^{2}\left\|w^{(\ell)}(s)\right\|_1^2 ds + Ck\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2$$

for all $\epsilon > 0$.

Proof. Consider the integral first, let

$$\begin{aligned}
I &:= \sum_{i=1}^{m} \int_0^{t_i} b\big(t_i, s; w(s), \bar\theta_{i+1}\big)\ ds - \sum_{i=1}^{m} \int_0^{t_i} b\big(t_i, s; w(s), \bar\theta_{i}\big)\ ds \\
&= \int_0^{t_m} b\big(t_m, s; w(s), \bar\theta_{m+1}\big)\ ds - \int_0^{t_1} b\big(t_1, s; w(s), \bar\theta_{1}\big)\ ds \\
&\quad + \sum_{i=1}^{m-1} \left(\int_0^{t_i} b\big(t_i, s; w(s), \bar\theta_{i+1}\big)\ ds - \int_0^{t_{i+1}} b\big(t_{i+1}, s; w(s), \bar\theta_{i+1}\big)\ ds \right) \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= \int_0^{t_m} b\big(t_m, s; w(s), \bar\theta_{m+1}\big)\ ds \\
I_2 &:= -\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} b\big(t_{i+1}, s; w(s), \bar\theta_{i+1}\big)\ ds \\
I_3 &:= \sum_{i=1}^{m-1} \int_0^{t_i} \int_{t_{i+1}}^{t_i} b_t\big(t, s; w(s), \bar\theta_{i+1}\big)\ dtds.
\end{aligned}$$

Turning now to the summation term in δ_i we define

$$B\big(t_i, t_j; w_j, \cdot\big) := b\big(t_i, t_j; w_j, \cdot\big) + b\big(t_i, t_{j-1}; w_{j-1}, \cdot\big) \tag{i}$$

and write the trapezoidal rule summation in an equivalent manner to obtain

$$\begin{aligned}
S &:= \sum_{i=1}^{m} \sum_{j=1}^{i} \frac{k}{2} B\big(t_i, t_j; w_j, \bar\theta_{i+1} - \bar\theta_i\big) \\
\Longrightarrow \frac{2S}{k} &= \sum_{i=1}^{m} \sum_{j=1}^{i} B\big(t_i, t_j; w_j, \bar\theta_{i+1}\big) - \sum_{i=1}^{m} \sum_{j=1}^{i} B\big(t_i, t_j; w_j, \bar\theta_{i}\big) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{i} B\big(t_i, t_j; w_j, \bar\theta_{i+1}\big) - \sum_{i=0}^{m-1} \sum_{j=1}^{i+1} B\big(t_{i+1}, t_j; w_j, \bar\theta_{i+1}\big) \\
&= \sum_{i=1}^{m} B\big(t_m, t_i; w_i, \bar\theta_{m+1}\big) - B\big(t_1, t_1; w_1, \bar\theta_1\big) - \sum_{i=1}^{m-1} B\big(t_{i+1}, t_{i+1}; w_{i+1}, \bar\theta_{i+1}\big) \\
&\quad + \sum_{i=1}^{m-1} \sum_{j=1}^{i} \big(B\big(t_i, t_j; w_j, \bar\theta_{i+1}\big) - B\big(t_{i+1}, t_j; w_j, \bar\theta_{i+1}\big)\big) \\
\Longrightarrow S &= S_1 + S_2 + S_3,
\end{aligned}$$

where

$$\begin{aligned}
S_1 &:= \sum_{j=1}^{m} \frac{k}{2} \Big(b\big(t_m, t_j; w_j, \bar\theta_{m+1}\big) + b\big(t_m, t_{j-1}; w_{j-1}, \bar\theta_{m+1}\big) \Big) \\
S_2 &:= -\sum_{j=0}^{m-1} \frac{k}{2} \Big(b\big(t_{j+1}, t_{j+1}; w_{j+1}, \bar\theta_{j+1}\big) + b\big(t_{j+1}, t_j; w_j, \bar\theta_{j+1}\big) \Big) \\
S_3 &:= \sum_{i=1}^{m-1} \sum_{j=1}^{i} \frac{k}{2} \left(\int_{t_{i+1}}^{t_i} b_t\big(t, t_j; w_j, \bar\theta_{i+1}\big) + b_t\big(t, t_{j-1}; w_{j-1}, \bar\theta_{i+1}\big)\ dt \right).
\end{aligned}$$

Recall that for the trapezoidal rule the error is of the form, for $\tau = Mk \in \mathcal{I}_0^k$,

$$\left|\int_0^\tau u(s)\ ds - \sum_{i=0}^{M} \varpi_{Mi} u(t_i)\right| \leq Ck^2 \int_0^\tau |u''(s)|\ ds,$$

and with this in mind we now pair off the terms of S and I: firstly,

$$\begin{aligned}
|I_1 - S_1| = &\ \Bigg| \int_0^{t_m} b\big(t_m, s; w(s), \bar\theta_{m+1}\big)\ ds \\
&- \sum_{j=1}^{m} \frac{k}{2}\Big(b\big(t_m, t_j; w_j, \bar\theta_{m+1}\big) + b\big(t_m, t_{j-1}; w_{j-1}, \bar\theta_{m+1}\big)\Big) \Bigg| \\
&\leq Ck^2 \int_0^{t_m} \left|\frac{\partial^2}{\partial s^2} b\big(t_m, s; w(s), \bar\theta_{m+1}\big)\right|\ ds \\
&\leq Ck^2 \int_0^{T} \left\|\bar\theta_{m+1}\right\|_1 \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1\ ds \\
&\leq \frac{C}{\epsilon}\left\|\bar\theta_{m+1}\right\|_1^2 + \epsilon C T k^4 \int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2\ ds \quad \forall \epsilon > 0.
\end{aligned}$$

Also,

$$\begin{aligned}
|I_2 - S_2| = &\ \Bigg| \sum_{i=0}^{m-1} \bigg(\int_{t_i}^{t_{i+1}} b\big(t_{i+1}, s; w(s), \bar\theta_{i+1}\big)\ ds \\
&- \frac{k}{2}\Big(b\big(t_{i+1}, t_{i+1}; w_{i+1}, \bar\theta_{i+1}\big) + b\big(t_{i+1}, t_i; w_i, \bar\theta_{i+1}\big)\Big)\bigg) \Bigg| \\
&\leq \sum_{i=0}^{m-1} Ck^2 \int_{t_i}^{t_{i+1}} \left|\frac{\partial^2}{\partial s^2} b\big(t_{i+1}, s; w(s), \bar\theta_{i+1}\big)\right|\ ds \\
&\leq \sum_{i=0}^{m-1} Ck^2 \int_{t_i}^{t_{i+1}} C(s)\left\|\bar\theta_{i+1}\right\|_1 \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1\ ds \\
&\leq \sum_{i=0}^{m-1} Ck\left\|\bar\theta_{i+1}\right\|_1^2 + \sum_{i=0}^{m-1} Ck^4 \int_{t_i}^{t_{i+1}} \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2\ ds \\
&\leq Ck \sum_{i=0}^{m-1} \left\|\bar\theta_{i+1}\right\|_1^2 + Ck^4 \int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2\ ds,
\end{aligned}$$

and,

$$\begin{aligned}
|I_3 - S_3| = &\ \Bigg| \sum_{i=1}^{m-1} \int_0^{t_i} \int_{t_{i+1}}^{t_i} b_t\big(t, s; w(s), \bar\theta_{i+1}\big)\ dtds \\
&- \sum_{i=1}^{m-1} \sum_{j=1}^{i} \frac{k}{2}\left(\int_{t_{i+1}}^{t_i} b_t\big(t, t_j; w_j, \bar\theta_{i+1}\big) + b_t\big(t, t_{j-1}; w_{j-1}, \bar\theta_{i+1}\big)\ dt\right) \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg| \sum_{i=1}^{m-1} \sum_{j=1}^{i} \left(\int_{t_{j-1}}^{t_j} \int_{t_{i+1}}^{t_i} b_t\big(t,s;w(s),\bar\theta_{i+1}\big)\,dtds \right. \\
&\quad \left. -\frac{k}{2} \left(\int_{t_{i+1}}^{t_i} b_t\big(t,t_j;w_j,\bar\theta_{i+1}\big) + b_t\big(t,t_{j-1};w_{j-1},\bar\theta_{i+1}\big)\,dt \right) \right) \Bigg| \\
&\le \sum_{i=1}^{m-1} \sum_{j=1}^{i} Ck^2 \int_{t_{j-1}}^{t_j} \left| \frac{\partial^2}{\partial s^2} \int_{t_{i+1}}^{t_i} b_t\big(t,s;w(s),\bar\theta_{i+1}\big)\,dt \right| \,ds \\
&\le \sum_{i=1}^{m-1} \sum_{j=1}^{i} Ck^2 \int_{t_{j-1}}^{t_j} \left| \int_{t_{i+1}}^{t_i} C(t,s) \left\|\bar\theta_{i+1}\right\|_1 \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1 \,dt \right| \,ds \\
&\le \sum_{i=1}^{m-1} Ck^3 \left\|\bar\theta_{i+1}\right\| \int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1 \,ds \\
&\le \sum_{i=1}^{m-1} Ck \left\|\bar\theta_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} Ck^5 \left(\int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1 \,ds \right)^2 \\
&\le Ck \sum_{i=1}^{m-1} \left\|\bar\theta_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} CTk^5 \int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2 \,ds \\
&\le Ck \sum_{i=1}^{m-1} \left\|\bar\theta_{i+1}\right\|_1^2 + CT^2k^4 \int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2 \,ds.
\end{aligned}
$$

Finally we combine these three results to give

$$
\begin{aligned}
&|I_1 - S_1| + |I_2 - S_2| + |I_3 - S_3| \\
&\qquad \le \frac{C}{\epsilon} \left\|\bar\theta_{m+1}\right\|_1^2 + Ck^4(1+\epsilon T + T^2) \int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2 \,ds + Ck \sum_{i=0}^{m-1} \left\|\bar\theta_{i+1}\right\|_1^2,
\end{aligned}
$$

and the lemma is proven. ∎

LEMMA 4.2. *Let $\bar\theta_i \in \mathcal{V}^h$ for $1 \le i \le N$ and define*

$$
J(t_i;\,\cdot\,) \equiv J_i(\,\cdot\,) := \int_0^{t_i} b(t_i,s;w(s),\,\cdot\,)\,ds,
$$

then for $m \le N-1$ there exist positive constants C such that

$$
\begin{aligned}
&\sum_{i=1}^{m} \left| J_i(\bar\theta_{i+1} - \bar\theta_i) - \Delta J_i(\bar\theta_{i+1} - \bar\theta_i) \right| \\
&\qquad \le Ck^4 \left(\int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(s)\right\|_1^2 \,ds + T^2 \int_0^T \left\|w(s)\right\|_1^2 \,ds \right) + Ck \sum_{i=0}^{m} \left\|\bar\theta_{i+1}\right\|_1^2.
\end{aligned}
$$

We emphasize here that the notation is such that Δ is not *applied to $\bar\theta_{i+1} - \bar\theta_i$.*

Proof. Note first that for $\phi(t) \in \mathcal{C}^2(\mathcal{I})$ we have, from Taylor's series, that

$$
\begin{aligned}
4\big(\Delta\phi(t_i) - \phi(t_i)\big) &= \big(\phi(t_{i-1}) - \phi(t_i)\big) + \big(\phi(t_{i+1}) - \phi(t_i)\big) \\
&= \int_{t_i}^{t_{i+1}} (t_{i+1} - t)\phi''(t)\,dt + \int_{t_i}^{t_{i-1}} (t_{i-1} - t)\phi''(t)\,dt.
\end{aligned}
$$

Using (4.4) we have also

$$
\begin{aligned}
\frac{d}{dt}J(t;\,\cdot\,) &= b(t,t;w(t),\,\cdot\,) + \int_0^t b_t(t,s;w(s),\,\cdot\,)\,ds \\
\frac{d^2}{dt^2}J(t;\,\cdot\,) &= \frac{d}{dt}b(t,t;w(t),\,\cdot\,) + b_t(t,t;w(t),\,\cdot\,) + \int_0^t b_{tt}(t,s;w(s),\,\cdot\,)\,ds,
\end{aligned}
$$

and hence

$$
\begin{aligned}
4\big(\Delta J(t_i;\,\cdot\,) - J(t_i;\,\cdot\,)\big) &= \int_{t_i}^{t_{i+1}} (t_{i+1}-t)\alpha(t;\,\cdot\,)\,dt + \int_{t_i}^{t_{i-1}} (t_{i-1}-t)\alpha(t;\,\cdot\,)\,dt \\
&\quad + \int_{t_i}^{t_{i+1}} (t_{i+1}-t)\beta(t;\,\cdot\,)\,dt + \int_{t_i}^{t_{i-1}} (t_{i-1}-t)\beta(t;\,\cdot\,)\,dt \\
&\quad + \int_{t_i}^{t_{i+1}} (t_{i+1}-t) \int_0^t \gamma(t,s;\,\cdot\,)\,dsdt \\
&\quad + \int_{t_i}^{t_{i-1}} (t_{i-1}-t) \int_0^t \gamma(t,s;\,\cdot\,)\,dsdt
\end{aligned}
$$

where we have introduced the expedient notation:

$$
\begin{aligned}
\alpha(t;\,\cdot\,) &= \frac{d}{dt}b(t,t;w(t),\,\cdot\,) \\
\beta(t;\,\cdot\,) &= b_t(t,t;w(t),\,\cdot\,) \\
\gamma(t,s;\,\cdot\,) &= b_{tt}(t,s;w(s),\,\cdot\,).
\end{aligned}
$$

Consider the terms $\alpha(t;\bar\theta_{i+1}-\bar\theta_i)$, $\beta(t;\bar\theta_{i+1}-\bar\theta_i)$ and $\gamma(t,s;\bar\theta_{i+1}-\bar\theta_i)$ (each of which are clearly linear in their last arguments) separately and sum over $i=1,\dots,m\le N-1$. For α:

$$
\begin{aligned}
&\sum_{i=1}^m \left(\int_{t_i}^{t_{i+1}} (t_{i+1}-t)\alpha(t;\bar\theta_{i+1}-\bar\theta_i)\,dt + \int_{t_i}^{t_{i-1}} (t_{i-1}-t)\alpha(t;\bar\theta_{i+1}-\bar\theta_i)\,dt \right) \\
&\quad = \sum_{i=1}^m \left(\int_{t_i}^{t_{i+1}} (t_{i+1}-t)\alpha(t;\bar\theta_{i+1})\,dt - \int_{t_i}^{t_{i+1}} (t_{i+1}-t)\alpha(t;\bar\theta_i)\,dt \right. \\
&\qquad\qquad \left. + \int_{t_i}^{t_{i-1}} (t_{i-1}-t)\alpha(t;\bar\theta_{i+1})\,dt - \int_{t_i}^{t_{i-1}} (t_{i-1}-t)\alpha(t;\bar\theta_i)\,dt \right) \\
&\quad = \sum_{i=1}^m \left(\int_{t_i}^{t_{i+1}} (t_{i+1}-t)\alpha(t;\bar\theta_{i+1})\,dt + \int_{t_i}^{t_{i-1}} (t_{i-1}-t)\alpha(t;\bar\theta_{i+1})\,dt \right) \\
&\qquad - \sum_{i=0}^{m-1} \left(\int_{t_{i+1}}^{t_{i+2}} (t_{i+2}-t)\alpha(t;\bar\theta_{i+1})\,dt + \int_{t_{i+1}}^{t_i} (t_i-t)\alpha(t;\bar\theta_{i+1})\,dt \right) \\
&\quad = \sum_{i=1}^m \left(\int_{t_i}^{t_{i+1}} (t_{i+1}-t)\alpha(t;\bar\theta_{i+1})\,dt - \int_{t_{i-1}}^{t_i} (t_{i-1}-t)\alpha(t;\bar\theta_{i+1})\,dt \right) \\
&\qquad - \sum_{i=0}^{m-1} \left(\int_{t_{i+1}}^{t_{i+2}} (t_{i+2}-t)\alpha(t;\bar\theta_{i+1})\,dt - \int_{t_i}^{t_{i+1}} (t_i-t)\alpha(t;\bar\theta_{i+1})\,dt \right)
\end{aligned}
$$

$$= \sum_{i=1}^{m} \frac{k^2}{2}\Big(\alpha(t_i^{i+1};\bar\theta_{i+1}) - \alpha(t_{i-1}^{i};\bar\theta_{i+1})\Big) - \sum_{i=0}^{m-1} \frac{k^2}{2}\Big(\alpha(t_{i+1}^{i+2};\bar\theta_{i+1}) - \alpha(t_i^{i+1};\bar\theta_{i+1})\Big)$$
$$= \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} \frac{d}{dt}\alpha(t;\bar\theta_{i+1})\,dt - \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \frac{d}{dt}\alpha(t;\bar\theta_{i+1})\,dt$$
$$= \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} \frac{d^2}{dt^2}b(t,t;w(t),\bar\theta_{i+1})\,dt - \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \frac{d^2}{dt^2}b(t,t;w(t),\bar\theta_{i+1})\,dt.$$

Where $t_i^{i+1} \in (t_i, t_{i+1})$ indicates the application of the mean value theorem. Clearly an identical argument may be applied to the term in β and so we have

$$\sum_{i=1}^{m}\left(\int_{t_i}^{t_{i+1}}(t_{i+1}-t)\beta(t;\bar\theta_{i+1}-\bar\theta_i)\,dt + \int_{t_i}^{t_{i-1}}(t_{i-1}-t)\beta(t;\bar\theta_{i+1}-\bar\theta_i)\,dt\right)$$
$$= \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} \frac{d}{dt}b_t(t,t;w(t),\bar\theta_{i+1})\,dt - \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \frac{d}{dt}b_t(t,t;w(t),\bar\theta_{i+1})\,dt.$$

And for the term in γ we set

$$\psi(t;\,\cdot\,) := \int_0^t \gamma(t,s;\,\cdot\,)\,ds,$$

then, again by reasoning as for the α term,

$$\sum_{i=1}^{m}\left(\int_{t_i}^{t_{i+1}}(t_{i+1}-t)\psi(t;\bar\theta_{i+1}-\bar\theta_i)\,dt + \int_{t_i}^{t_{i-1}}(t_{i-1}-t)\psi(t;\bar\theta_{i+1}-\bar\theta_i)\,dt\right)$$
$$= \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} \frac{d}{dt}\psi(t;\bar\theta_{i+1})\,dt - \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \frac{d}{dt}\psi(t;\bar\theta_{i+1})\,dt$$
$$= \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} \frac{d}{dt}\int_0^t b_{tt}(t,s;w(s),\bar\theta_{i+1})\,dsdt$$
$$- \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \frac{d}{dt}\int_0^t b_{tt}(t,s;w(s),\bar\theta_{i+1})\,dsdt$$
$$= \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} b_{tt}(t,t;w(t),\bar\theta_{i+1})\,dt$$
$$+ \frac{k^2}{2}\sum_{i=1}^{m}\int_{t_{i-1}^{i}}^{t_i^{i+1}} \int_0^t b_{ttt}(t,s;w(s),\bar\theta_{i+1})\,dsdt$$
$$- \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} b_{tt}(t,t;w(t),\bar\theta_{i+1})\,dt$$
$$- \frac{k^2}{2}\sum_{i=0}^{m-1}\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \int_0^t b_{ttt}(t,s;w(s),\bar\theta_{i+1})\,dsdt.$$

And so, collecting these three results together gives

$$\Big|4\big(\Delta J(t_i;\bar\theta_{i+1}-\bar\theta_i) - J(t_i;\bar\theta_{i+1}-\bar\theta_i)\big)\Big|$$

$$
\begin{aligned}
&= \Bigg| \frac{k^2}{2} \sum_{i=1}^{m} \int_{t_{i-1}^{i}}^{t_i^{i+1}} \bigg(\frac{d^2}{dt^2} b(t,t;w(t),\bar\theta_{i+1}) + \frac{d}{dt} b_t(t,t;w(t),\bar\theta_{i+1}) \\
&\qquad\qquad + b_{tt}(t,t;w(t),\bar\theta_{i+1}) + \int_0^t b_{ttt}(t,s;w(s),\bar\theta_{i+1})\,ds \bigg)\,dt \\
&\quad - \frac{k^2}{2} \sum_{i=0}^{m-1} \int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \bigg(\frac{d^2}{dt^2} b(t,t;w(t),\bar\theta_{i+1}) + \frac{d}{dt} b_t(t,t;w(t),\bar\theta_{i+1}) \\
&\qquad\qquad + b_{tt}(t,t;w(t),\bar\theta_{i+1}) + \int_0^t b_{ttt}(t,s;w(s),\bar\theta_{i+1})\,ds \bigg)\,dt \Bigg| \\
&\le Ck^2 \sum_{i=1}^{m} \int_{t_{i-1}^{i}}^{t_i^{i+1}} \bigg(C(t) \left\|\bar\theta_{i+1}\right\|_1 \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1 \\
&\qquad\qquad + \left\|\bar\theta_{i+1}\right\|_1 \int_0^t C(t,s) \left\|w(s)\right\|_1\,ds \bigg)\,dt \\
&\quad + Ck^2 \sum_{i=0}^{m-1} \int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \bigg(C(t) \left\|\bar\theta_{i+1}\right\|_1 \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1 \\
&\qquad\qquad + \left\|\bar\theta_{i+1}\right\|_1 \int_0^t C(t,s) \left\|w(s)\right\|_1\,ds \bigg)\,dt \\
&\le Ck \sum_{i=1}^{m} \left\|\bar\theta_{i+1}\right\|_1^2 + Ck \sum_{i=0}^{m-1} \left\|\bar\theta_{i+1}\right\|_1^2 \\
&\quad + Ck^3 \sum_{i=1}^{m} \left(\int_{t_{i-1}^{i}}^{t_i^{i+1}} C(t) \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1 + \int_0^t C(t,s) \left\|w(s)\right\|_1\,ds\,dt \right)^2 \\
&\quad + Ck^3 \sum_{i=0}^{m-1} \left(\int_{t_i^{i+1}}^{t_{i+1}^{i+2}} C(t) \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1 + \int_0^t C(t,s) \left\|w(s)\right\|_1\,ds\,dt \right)^2 \\
&\le Ck^4 \sum_{i=1}^{m} \int_{t_{i-1}^{i}}^{t_i^{i+1}} \left(C(t) \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1^2 + C(t)t \int_0^t \left\|w(s)\right\|_1^2\,ds \right)\,dt \\
&\quad + Ck^4 \sum_{i=0}^{m-1} \int_{t_i^{i+1}}^{t_{i+1}^{i+2}} \left(C(t) \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1^2 + C(t)t \int_0^t \left\|w(s)\right\|_1^2\,ds \right)\,dt \\
&\quad + Ck \sum_{i=0}^{m} \left\|\bar\theta_{i+1}\right\|_1^2 \\
&\le Ck \sum_{i=0}^{m} \left\|\bar\theta_{i+1}\right\|_1^2 + Ck^4 \left(\int_0^T \sum_{\ell=0}^{2} \left\|w^{(\ell)}(t)\right\|_1^2\,dt + T^2 \int_0^T \left\|w(s)\right\|_1^2\,ds \right).
\end{aligned}
$$

This completes the proof. ∎

LEMMA 4.3. *Let $\bar\theta_i \in \mathcal{V}^h$ for $1 \le i \le N$ and note that by definition $\Delta w(0) \equiv w(0)$, then there exist positive constants C such that for $m \le N-1$*

$$
\left| \sum_{i=1}^{m} \sum_{j=0}^{i} \varpi_{ij} b(t_i,t_j;\Delta w(t_j) - w(t_j), k\bar\partial_t \theta_{i+1}) \right|
$$

$$\leq \frac{C}{\epsilon}\left\|\bar{\theta}_{m+1}\right\|_1^2 + Ck^4(1+\epsilon T+T^2)\int_0^T \|w''(s)\|_1^2\ ds + Ck\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2$$

for all $\epsilon > 0$.

Proof. Set $\chi_j := \Delta w(t_j) - w(t_j)$, we know that $\chi_0 = 0$ and for $w \in \mathcal{C}^2\big(\mathcal{I};\mathcal{H}^1(\Omega)\big)$ we also know that

$$\|\chi_j\|_1^2 \leq \frac{k^3}{24}\int_{t_{j-1}}^{t_{j+1}} \|w''(s)\|_1^2\ ds.$$

Utilizing the notation introduced in the proof of Lemma 4.1 $\big($definition (i)$\big)$ we may re-write the trapezoidal rule and obtain, since $k\bar{\partial}_t\theta_{i+1} = \bar{\theta}_{i+1} - \bar{\theta}_i$,

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{i}\frac{k}{2}B\big(t_i,t_j;\chi_j,\bar{\theta}_{i+1}-\bar{\theta}_i\big)\\
&\quad= \sum_{i=1}^{m}\sum_{j=1}^{i}\frac{k}{2}B\big(t_i,t_j;\chi_j,\bar{\theta}_{i+1}\big) - \sum_{i=1}^{m}\sum_{j=1}^{i}\frac{k}{2}B\big(t_i,t_j;\chi_j,\bar{\theta}_i\big)\\
&\quad= \sum_{i=1}^{m}\sum_{j=1}^{i}\frac{k}{2}B\big(t_i,t_j;\chi_j,\bar{\theta}_{i+1}\big) - \sum_{i=0}^{m-1}\sum_{j=1}^{i+1}\frac{k}{2}B\big(t_{i+1},t_j;\chi_j,\bar{\theta}_{i+1}\big)\\
&\quad= \sum_{j=1}^{m}\frac{k}{2}B\big(t_m,t_j;\chi_j,\bar{\theta}_{m+1}\big) - \frac{k}{2}B\big(t_1,t_1;\chi_1,\bar{\theta}_1\big)\\
&\qquad+ \sum_{i=1}^{m-1}\left(\sum_{j=1}^{i}\frac{k}{2}B\big(t_i,t_j;\chi_j,\bar{\theta}_{i+1}\big) - \sum_{j=1}^{i+1}\frac{k}{2}B\big(t_{i+1},t_j;\chi_j,\bar{\theta}_{i+1}\big)\right)\\
&\quad= \sum_{j=1}^{m}\frac{k}{2}B\big(t_m,t_j;\chi_j,\bar{\theta}_{m+1}\big) - \sum_{j=0}^{m-1}\frac{k}{2}B\big(t_{j+1},t_{j+1};\chi_{j+1},\bar{\theta}_{j+1}\big)\\
&\qquad+ \sum_{i=1}^{m-1}\sum_{j=1}^{i}\frac{k}{2}\Big(B\big(t_i,t_j;\chi_j,\bar{\theta}_{i+1}\big) - B\big(t_{i+1},t_j;\chi_j,\bar{\theta}_{i+1}\big)\Big)\\
&\quad= \sum_{j=1}^{m}\frac{k}{2}B\big(t_m,t_j;\chi_j,\bar{\theta}_{m+1}\big) - \sum_{j=0}^{m-1}\frac{k}{2}B\big(t_{j+1},t_{j+1};\chi_{j+1},\bar{\theta}_{j+1}\big)\\
&\qquad+ \sum_{i=1}^{m-1}\sum_{j=1}^{i}\frac{k}{2}\int_{t_{i+1}}^{t_i} B_t\big(t,t_j;\chi_j,\bar{\theta}_{i+1}\big)\ dt. \qquad \text{(i)}
\end{aligned}$$

Now, estimating this expression term by term we see that for the first term in (i)

$$\begin{aligned}
\left|\sum_{j=1}^{m}\frac{k}{2}B\big(t_m,t_j;\chi_j,\bar{\theta}_{m+1}\big)\right| &= \left|\sum_{j=1}^{m}\frac{k}{2}\left(b\big(t_m,t_j;\chi_j,\bar{\theta}_{m+1}\big) + b\big(t_m,t_{j-1};\chi_{j-1},\bar{\theta}_{m+1}\big)\right)\right|\\
&\leq \sum_{j=1}^{m}Ck\|\chi_j\|_1\left\|\bar{\theta}_{m+1}\right\|_1 + \sum_{j=1}^{m}Ck\|\chi_{j-1}\|_1\left\|\bar{\theta}_{m+1}\right\|_1\\
&\leq \sum_{j=1}^{m}Ck\|\chi_j\|_1\left\|\bar{\theta}_{m+1}\right\|_1 \qquad \text{(since } \chi_0 = 0)
\end{aligned}$$

$$\leq \frac{C}{\epsilon}\left\|\bar{\theta}_{m+1}\right\|_1^2 + \epsilon C k^2\Big(\sum_{j=1}^{m}\|\chi_j\|_1\Big)^2 \quad \forall \epsilon > 0.$$
$$\leq \frac{C}{\epsilon}\left\|\bar{\theta}_{m+1}\right\|_1^2 + \epsilon C T k^4 \int_0^T \|w''(s)\|_1^2 \; ds \quad \forall \epsilon > 0.$$

And for the second term,

$$\left|\sum_{j=0}^{m-1}\frac{k}{2}\left(b\big(t_{j+1},t_{j+1};\chi_{j+1},\bar{\theta}_{j+1}\big) + b\big(t_{j+1},t_j;\chi_j,\bar{\theta}_{j+1}\big)\right)\right|$$
$$\leq \sum_{j=0}^{m-1} Ck\|\chi_{j+1}\|_1\left\|\bar{\theta}_{j+1}\right\|_1 + \sum_{j=1}^{m-1} Ck\|\chi_j\|_1\left\|\bar{\theta}_{j+1}\right\|_1 \qquad \text{(since } \chi_0 = 0\text{)}$$
$$\leq 2\sum_{j=0}^{m-1} Ck\left\|\bar{\theta}_{j+1}\right\|_1^2 + 2\sum_{j=1}^{m} Ck\|\chi_j\|_1^2$$
$$\leq \sum_{j=0}^{m-1} Ck\left\|\bar{\theta}_{j+1}\right\|_1^2 + Ck^4\int_0^T \|w''(s)\|_1^2 \; ds.$$

And finally, for the double sum in (i),

$$\left|\sum_{i=1}^{m-1}\sum_{j=1}^{i}\frac{k}{2}\int_{t_{i+1}}^{t_i} B_t\big(t,t_j;\chi_j,\bar{\theta}_{i+1}\big)\; dt\right|$$
$$\leq \left|\sum_{i=1}^{m-1}\sum_{j=1}^{i}\frac{k}{2}\int_{t_{i+1}}^{t_i} b_t\big(t,t_j;\chi_j,\bar{\theta}_{i+1}\big) + b_t\big(t,t_{j-1};\chi_{j-1},\bar{\theta}_{i+1}\big)\; dt\right|$$
$$\leq \left|\sum_{i=1}^{m-1}\sum_{j=1}^{i}\frac{k}{2}\int_{t_{i+1}}^{t_i} C(t)\left\|\bar{\theta}_{i+1}\right\|_1\|\chi_j\|_1 + C(t)\left\|\bar{\theta}_{i+1}\right\|_1\|\chi_{j-1}\|_1\; dt\right|$$
$$= \sum_{i=1}^{m-1} Ck\left\|\bar{\theta}_{i+1}\right\|_1\sum_{j=1}^{i}\int_{t_i}^{t_{i+1}}\|\chi_j\|_1 + \|\chi_{j-1}\|_1\; dt$$
$$\leq \sum_{i=1}^{m-1} Ck\left\|\bar{\theta}_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} Ck\Big(\sum_{j=1}^{i}\int_{t_i}^{t_{i+1}}\|\chi_j\|_1\; dt\Big)^2 \quad \text{(since } \chi_0 = 0\text{)}$$
$$\leq \sum_{i=1}^{m-1} Ck\left\|\bar{\theta}_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} Cik\sum_{j=1}^{i}\Big(\int_{t_i}^{t_{i+1}}\|\chi_j\|_1\; dt\Big)^2$$
$$\leq \sum_{i=1}^{m-1} Ck\left\|\bar{\theta}_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} Cik^2\sum_{j=1}^{i}\int_{t_i}^{t_{i+1}}\|\chi_j\|_1^2\; dt$$
$$\leq \sum_{i=1}^{m-1} Ck\left\|\bar{\theta}_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} Ckt_i\int_{t_i}^{t_{i+1}}\sum_{j=1}^{i}\frac{k^3}{24}\int_{t_{j-1}}^{t_{j+1}}\|w''(\xi)\|_1^2\; d\xi dt$$
$$\leq \sum_{i=1}^{m-1} Ck\left\|\bar{\theta}_{i+1}\right\|_1^2 + \sum_{i=1}^{m-1} Ck^5T\int_0^T\|w''(\xi)\|_1^2\; d\xi$$

$$\leq Ck\sum_{j=1}^{m-1}\left\|\bar{\theta}_{j+1}\right\|_1^2+CT^2k^4\int_0^T\left\|w''(s)\right\|_1^2\ ds,$$

assembling these results proves the lemma. ∎

LEMMA 4.4. *$\Delta\theta_0\equiv\theta_0$, $\bar{\theta}_i\in\mathcal{V}^h$ for $1\leq i\leq N$, $\forall\epsilon>0$ and $\forall m\leq N-1$*

$$\left|\frac{k}{2}\sum_{i=1}^{m}\sum_{j=1}^{i}\big(b(t_i,t_j;\Delta\theta_j,k\bar{\partial}_t\theta_{i+1})+b(t_i,t_{j-1};\Delta\theta_{j-1},k\bar{\partial}_t\theta_{i+1})\big)\right|$$
$$\leq CT\left\|\theta_0\right\|_1^2+C(k+T/\epsilon)\left\|\bar{\theta}_{m+1}\right\|_1^2+C(1+T+\epsilon)k\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2.$$

Proof. Firstly we introduce the shorthand $b_{i\,j}(\cdot,\cdot):=b(t_i,t_j;\cdot,\cdot)$, then, for $j\geq 1$,

$$\begin{aligned}
b_{i\,j}(\Delta\theta_j,k\bar{\partial}_t\theta_{i+1}) &= \frac{1}{2}\frac{1}{4}b_{i\,j}\big(\theta_{j-1}+2\theta_j+\theta_{j+1},(\theta_{i+1}-\theta_i)+(\theta_i-\theta_{i-1})\big)\\
&= \frac{1}{8}b_{i\,j}\big((\theta_j+\theta_{j-1})+(\theta_{j+1}+\theta_j),(\theta_{i+1}+\theta_i)-(\theta_i+\theta_{i-1})\big)\\
&= \frac{1}{2}b_{i\,j}(\bar{\theta}_{j+1}+\bar{\theta}_j,\bar{\theta}_{i+1}-\bar{\theta}_i)\\
&= \frac{1}{2}\Big(b_{i\,j}(\bar{\theta}_j,\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_j,\bar{\theta}_i)+b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_i)\Big),
\end{aligned}$$

and also the abbreviations

$$\begin{aligned}
S_1 &:= \frac{k}{2}\sum_{j=1}^{i}b(t_i,t_j;\Delta\theta_j,k\bar{\partial}_t\theta_{i+1})\\
&= \frac{k}{4}\sum_{j=1}^{i}\left(b_{i\,j}(\bar{\theta}_j,\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_j,\bar{\theta}_i)+b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_i)\right),
\end{aligned}$$

and, since $\Delta\theta_0=\theta_0$ by (2.4),

$$\begin{aligned}
S_2 &:= \frac{k}{2}b_{i\,0}(\theta_0,\bar{\theta}_{i+1}-\bar{\theta}_i)\ \overset{i\geq 2}{+}\ \frac{k}{2}\sum_{j=1}^{i-1}b_{i\,j}(\Delta\theta_j,k\bar{\partial}_t\theta_{i+1})\\
&= \frac{k}{2}b_{i\,0}(\theta_0,\bar{\theta}_{i+1}-\bar{\theta}_i)\ \overset{i\geq 2}{+}\ \frac{k}{4}\sum_{j=1}^{i-1}\Big(b_{i\,j}(\bar{\theta}_j,\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_j,\bar{\theta}_i)\\
&\qquad\qquad +b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_i)\Big)\\
&= \frac{k}{2}b_{i\,0}(\theta_0,\bar{\theta}_{i+1}-\bar{\theta}_i)\ \overset{i\geq 2}{+}\ \frac{k}{4}\sum_{j=1}^{i-1}\Big(b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_{i+1})-b_{i\,j}(\bar{\theta}_{j+1},\bar{\theta}_i)\Big)\\
&\qquad \overset{i\geq 2}{+}\ \frac{k}{4}\sum_{j=0}^{i-2}\Big(b_{i\,j+1}(\bar{\theta}_{j+1},\bar{\theta}_{i+1})-b_{i\,j+1}(\bar{\theta}_{j+1},\bar{\theta}_i)\Big),
\end{aligned}$$

where $\overset{i\geq j}{+}$ means that the addition is only to occur if the condition (i.e. $i \geq j$) is satisfied. Now, the inner sum in the lemma is given by

$$\begin{aligned}
S_1 + S_2 = {} & \frac{k}{2} b_{i\,0}(\theta_0, \bar\theta_{i+1} - \bar\theta_i) \\
& + \frac{k}{4} \sum_{j=1}^{i} \big(b_{i\,j}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j}(\bar\theta_{j+1}, \bar\theta_i) \big) \\
& + \frac{k}{4} \sum_{j=0}^{i-1} \big(b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_i) \big) \\
& \overset{i\geq 2}{+} \frac{k}{4} \sum_{j=1}^{i-1} \big(b_{i\,j}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j}(\bar\theta_{j+1}, \bar\theta_i) \big) \\
& \overset{i\geq 2}{+} \frac{k}{4} \sum_{j=0}^{i-2} \big(b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_i) \big) \\
= {} & \frac{k}{2} b_{i\,0}(\theta_0, \bar\theta_{i+1} - \bar\theta_i) + \frac{k}{4} \Big(b_{i\,i}(\bar\theta_{i+1}, \bar\theta_{i+1}) - b_{i\,i}(\bar\theta_{i+1}, \bar\theta_i) \Big) \\
& + \frac{k}{4} \Big(b_{i\,1}(\bar\theta_1, \bar\theta_{i+1}) - b_{i\,1}(\bar\theta_1, \bar\theta_i) \Big) \\
& \overset{i\geq 2}{+} \frac{k}{4} \sum_{j=1}^{i-1} \Big(b_{i\,j}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j}(\bar\theta_{j+1}, \bar\theta_i) \\
& \qquad\qquad + b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_i) \Big) \\
& \overset{i\geq 2}{+} \frac{k}{4} \sum_{j=1}^{i-1} \big(b_{i\,j}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j}(\bar\theta_{j+1}, \bar\theta_i) \big) \\
& \overset{i\geq 2}{+} \frac{k}{4} \sum_{j=0}^{i-2} \big(b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_i) \big) \\
= {} & \frac{k}{2} b_{i\,0}(\theta_0, \bar\theta_{i+1} - \bar\theta_i) + \frac{k}{4} \big(b_{i\,i}(\bar\theta_{i+1}, \bar\theta_{i+1}) - b_{i\,i}(\bar\theta_{i+1}, \bar\theta_i) \big) && \text{(i)} \\
& + \frac{k}{4} \big(b_{i\,1}(\bar\theta_1, \bar\theta_{i+1}) - b_{i\,1}(\bar\theta_1, \bar\theta_i) \big) && \text{(ii)} \\
& \overset{i\geq 2}{+} \frac{k}{4} \sum_{j=1}^{i-1} \Big(b_{i\,j}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j}(\bar\theta_{j+1}, \bar\theta_i) \\
& \qquad\qquad + b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_i) \Big) && \text{(iii)} \\
& \overset{i\geq 2}{+} \frac{k}{4} \big(b_{i\,i-1}(\bar\theta_i, \bar\theta_{i+1}) - b_{i\,i-1}(\bar\theta_i, \bar\theta_i) + b_{i\,1}(\bar\theta_1, \bar\theta_{i+1}) - b_{i\,1}(\bar\theta_1, \bar\theta_i) \big) && \text{(iv)} \\
& \overset{i\geq 3}{+} \frac{k}{4} \sum_{j=1}^{i-2} \Big(b_{i\,j}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j}(\bar\theta_{j+1}, \bar\theta_i) \\
& \qquad\qquad + b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_{i+1}) - b_{i\,j+1}(\bar\theta_{j+1}, \bar\theta_i) \Big). && \text{(v)}
\end{aligned}$$

To progress we must examine $\sum_{i=1}^{m}(S_1 + S_2)$. Taking this last expression line-by-line we have for (iii), because of the conditional addition $\sum_{i=1}^{m} \equiv \sum_{i=2}^{m}$, that

$$\begin{aligned}
&\left|\frac{k}{4}\sum_{i=2}^{m}\sum_{j=1}^{i-1}\big(b_{i\,j}(\bar\theta_{j+1},\bar\theta_{i+1})+b_{i\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})-b_{i\,j}(\bar\theta_{j+1},\bar\theta_{i})-b_{i\,j+1}(\bar\theta_{j+1},\bar\theta_{i})\big)\right|\\
&\quad=\left|\frac{k}{4}\sum_{i=2}^{m}\sum_{j=1}^{i-1}\big(b_{i\,j}(\bar\theta_{j+1},\bar\theta_{i+1})+b_{i\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})\big)\right.\\
&\qquad\left.-\frac{k}{4}\sum_{i=1}^{m-1}\sum_{j=1}^{i}\big(b_{i+1\,j}(\bar\theta_{j+1},\bar\theta_{i+1})+b_{i+1\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})\big)\right|\\
&\quad=\left|\frac{k}{4}\sum_{j=1}^{m-1}\big(b_{m\,j}(\bar\theta_{j+1},\bar\theta_{m+1})+b_{m\,j+1}(\bar\theta_{j+1},\bar\theta_{m+1})\big)-\frac{k}{4}\big(b_{2\,1}(\bar\theta_{2},\bar\theta_{2})+b_{2\,2}(\bar\theta_{2},\bar\theta_{2})\big)\right.\\
&\qquad+\frac{k}{4}\sum_{i=2}^{m-1}\sum_{j=1}^{i-1}\Big(b_{i\,j}(\bar\theta_{j+1},\bar\theta_{i+1})+b_{i\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})\\
&\qquad\qquad-b_{i+1\,j}(\bar\theta_{j+1},\bar\theta_{i+1})-b_{i+1\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})\Big)\\
&\qquad\left.-\frac{k}{4}\sum_{i=2}^{m-1}\big(b_{i+1\,i}(\bar\theta_{i+1},\bar\theta_{i+1})+b_{i+1\,i+1}(\bar\theta_{i+1},\bar\theta_{i+1})\big)\right|\\
&\quad=\left|\frac{k}{4}\sum_{j=1}^{m-1}\big(b_{m\,j}(\bar\theta_{j+1},\bar\theta_{m+1})+b_{m\,j+1}(\bar\theta_{j+1},\bar\theta_{m+1})\big)\right.\\
&\qquad-\frac{k}{4}\sum_{i=1}^{m-1}\big(b_{i+1\,i}(\bar\theta_{i+1},\bar\theta_{i+1})+b_{i+1\,i+1}(\bar\theta_{i+1},\bar\theta_{i+1})\big)\\
&\qquad-\frac{k}{4}\sum_{i=2}^{m-1}\sum_{j=1}^{i-1}\Big(b_{i+1\,j}(\bar\theta_{j+1},\bar\theta_{i+1})-b_{i\,j}(\bar\theta_{j+1},\bar\theta_{i+1})\\
&\qquad\qquad\left.+b_{i+1\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})-b_{i\,j+1}(\bar\theta_{j+1},\bar\theta_{i+1})\Big)\right|\\
&\quad=\left|\frac{k}{4}\sum_{j=1}^{m-1}\big(b_{m\,j}(\bar\theta_{j+1},\bar\theta_{m+1})+b_{m\,j+1}(\bar\theta_{j+1},\bar\theta_{m+1})\big)\right.\\
&\qquad-\frac{k}{4}\sum_{i=1}^{m-1}\big(b_{i+1\,i}(\bar\theta_{i+1},\bar\theta_{i+1})+b_{i+1\,i+1}(\bar\theta_{i+1},\bar\theta_{i+1})\big)\\
&\qquad\left.-\frac{k}{4}\sum_{i=2}^{m-1}\sum_{j=1}^{i-1}\Big(\int_{t_i}^{t_{i+1}}b_t(t,t_j;\bar\theta_{j+1},\bar\theta_{i+1})\,dt+\int_{t_i}^{t_{i+1}}b_t(t,t_{j+1};\bar\theta_{j+1},\bar\theta_{i+1})\,dt\Big)\right|\\
&\quad\le Ck\sum_{i=1}^{m-1}\left\|\bar\theta_{i+1}\right\|_1\left\|\bar\theta_{m+1}\right\|_1+Ck\sum_{i=1}^{m-1}\left\|\bar\theta_{i+1}\right\|_1^2+Ck^2\sum_{i=2}^{m-1}\sum_{j=1}^{i-1}\left\|\bar\theta_{j+1}\right\|_1\left\|\bar\theta_{i+1}\right\|_1\\
&\quad\le Ck(1+\epsilon)\sum_{i=1}^{m-1}\left\|\bar\theta_{i+1}\right\|_1^2+Ck/\epsilon\sum_{i=1}^{m-1}\left\|\bar\theta_{m+1}\right\|_1^2+CTk\sum_{i=2}^{m-1}\left\|\bar\theta_{i+1}\right\|_1^2
\end{aligned}$$

$$\leq C(1+T+\epsilon)k\sum_{i=1}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2+\frac{CT}{\epsilon}\left\|\bar{\theta}_{m+1}\right\|_1^2 \qquad \forall \epsilon>0.$$

Clearly this estimate also holds for (v), because, apart from the summation limits, (iii) and (v) are identical. It remains to consider (i), (ii) and (iv) which we do in one piece:

$$\begin{aligned}
&\Bigg|\sum_{i=1}^{m}\Big(\frac{k}{2}b_{1\,0}(\theta_0,\bar{\theta}_{i+1}-\bar{\theta}_i)+\frac{k}{4}\left(b_{i\,i}(\bar{\theta}_{i+1},\bar{\theta}_{i+1})-b_{i\,i}(\bar{\theta}_{i+1},\bar{\theta}_i)\right)\\
&\qquad\qquad +\frac{k}{4}\left(b_{i\,1}(\bar{\theta}_1,\bar{\theta}_{i+1})-b_{i\,1}(\bar{\theta}_1,\bar{\theta}_i)\right)\Big)\\
&\overset{i\geq 2}{+}\sum_{i=2}^{m}\frac{k}{4}\Big(b_{i\,i-1}(\bar{\theta}_i,\bar{\theta}_{i+1})-b_{i\,i-1}(\bar{\theta}_i,\bar{\theta}_i)+b_{i\,1}(\bar{\theta}_1,\bar{\theta}_{i+1})-b_{i\,1}(\bar{\theta}_1,\bar{\theta}_i)\Big)\Bigg|\\
&\leq Ck\sum_{i=1}^{m}\Big(\left\|\theta_0\right\|_1\left\|\bar{\theta}_{i+1}\right\|_1+\left\|\theta_0\right\|_1\left\|\bar{\theta}_i\right\|_1+\left\|\bar{\theta}_{i+1}\right\|_1^2\\
&\qquad\qquad +\left\|\bar{\theta}_{i+1}\right\|_1\left\|\bar{\theta}_i\right\|_1+\left\|\bar{\theta}_1\right\|_1\left\|\bar{\theta}_{i+1}\right\|_1+\left\|\bar{\theta}_1\right\|_1\left\|\bar{\theta}_i\right\|_1\Big)\\
&\qquad\qquad +Ck\sum_{i=2}^{m}\Big(\left\|\bar{\theta}_i\right\|_1\left\|\bar{\theta}_{i+1}\right\|_1+\left\|\bar{\theta}_i\right\|_1^2+\left\|\bar{\theta}_1\right\|_1\left\|\bar{\theta}_{i+1}\right\|_1+\left\|\bar{\theta}_1\right\|_1\left\|\bar{\theta}_i\right\|_1\Big)\\
&\leq Ck\sum_{i=1}^{m}\Big(\left\|\theta_0\right\|_1^2+\left\|\bar{\theta}_1\right\|_1^2+\left\|\bar{\theta}_i\right\|_1^2+\left\|\bar{\theta}_{i+1}\right\|_1^2\Big)\\
&\leq CT\left\|\theta_0\right\|_1^2+Ck\sum_{i=0}^{m}\left\|\bar{\theta}_{i+1}\right\|_1^2.
\end{aligned}$$

Assembling these results together we have:

$$\begin{aligned}
\left|\sum_{i=1}^{m}(S_1+S_2)\right| &\equiv \left|\frac{k}{2}\sum_{i=1}^{m}\sum_{j=1}^{i}\Big(b(t_i,t_j;\Delta\theta_j,k\bar{\partial}_t\theta_{i+1})+b(t_i,t_{j-1};\Delta\theta_{j-1},k\bar{\partial}_t\theta_{i+1})\Big)\right|\\
&\leq\ CT\left\|\theta_0\right\|_1^2+C(k+T/\epsilon)\left\|\bar{\theta}_{m+1}\right\|_1^2+C(1+T+\epsilon)k\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2 \quad \forall\epsilon>0,
\end{aligned}$$

which was to be proved. ∎

Equipped with these preliminaries we can now state:

Theorem 4.1. *Let Lemmas 4.1,2,3 & 4 hold and suppose* $u\in\mathcal{C}^4\big(\mathcal{I};\mathcal{H}_0^1(\Omega)\cap\mathcal{H}^r(\Omega)\big)$ *for some* $r\geq 2$. *Then there is a* $k_0>0$ *such that for* $k\in(0,k_0]$

$$\max_{1\leq i\leq N-1}\left\{\left\|\partial_t\theta_{i+1}\right\|_0+\left\|\frac{\theta_{i+1}+\theta_i}{2}\right\|_1\right\}\leq C\left(\left\|\partial_t\theta_1\right\|_0+\left\|\theta_0\right\|_1+\left\|\theta_1\right\|_1+h^r+k^2\right),$$

where C *is a positive constant depending upon* u *but not on* h *and* k.

Proof. Average (1.2) at the time t_i, $i\geq 1$, using the stencil $\Delta=\,]\frac{1}{4}\ \frac{1}{2}\ \frac{1}{4}[$ and subtract the discrete scheme (2.5) from the result, this gives

$$\begin{aligned}
&(\Delta u''(t_i)-\partial_t^2U_{i+1},v)+a(\Delta u(t_i)-\Delta U_i,v)\\
&\qquad +\Delta\int_0^{t_i}b(t_i,s;u(s),v)\,ds-\sum_{j=0}^{i}\varpi_{ij}b(t_i,t_j;\Delta U_j,v)=0\quad\forall v\in\mathcal{V}^h. \qquad \text{(i)}
\end{aligned}$$

After some manipulation we find that

$$(\Delta u''(t_i) - \partial_t^2 U_{i+1}, v) = (\Lambda(t_i), v) - (\partial_t^2 \theta_{i+1}, v) - (\partial_t^2 \eta(t_{i+1}), v)$$

where

$$\Lambda(t_i) := \big(\Delta u''(t_i) - u''(t_i)\big) + \big(u''(t_i) - \partial_t^2 u(t_{i+1})\big).$$

And

$$\begin{aligned}\Delta \int_0^{t_i} b(t_i,s; u(s), v)\, ds &- \sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; \Delta U_j, v) \\ &= Q_i(v) - \sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; \Delta\theta_j, v) - \Delta \int_0^{t_i} b(t_i, s; \eta(s), v)\, ds \qquad \text{(ii)}\end{aligned}$$

where

$$\begin{aligned}Q_i(v) := &\left(\int_0^{t_i} b(t_i, s; w(s), v)\, ds - \sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; w(t_j), v)\right) \\ &- \left(\int_0^{t_i} b(t_i, s; w(s), v)\, ds - \Delta \int_0^{t_i} b(t_i, s; w(s), v)\, ds\right) \qquad \text{(iii)} \\ &+ \sum_{j=1}^{i} \varpi_{ij} b(t_i, t_j; w(t_j) - \Delta w(t_j), v)\end{aligned}$$

(since $w(0) = \Delta w(0)$). We now write (i) as

$$\begin{aligned}(\partial_t^2 \theta_{i+1}, v) &+ (\partial_t^2 \eta(t_{i+1}), v) - (\Lambda(t_i), v) + a(\Delta\theta_i, v) + a(\Delta\eta(t_i), v) \\ &- Q_i(v) + \sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; \Delta\theta_j, v) + \Delta \int_0^{t_i} b(t_i, s; \eta(s), v)\, ds = 0 \quad \forall v \in \mathcal{V}^h.\end{aligned}$$

But, by definition $a(\Delta\eta(t_i), v) + \Delta \int_0^{t_i} b(t_i, s; \eta(s), v)\, ds = 0$ and so this simplifies to

$$\begin{aligned}(\partial_t^2 \theta_{i+1}, v) + a(\Delta\theta_i, v) = Q_i(v) &- (\partial_t^2 \eta(t_{i+1}), v) + (\Lambda(t_i), v) \\ &- \sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; \Delta\theta_j, v). \qquad \text{(iv)}\end{aligned}$$

We now pause for a moment to consider the simpler consistency errors that appear on the right hand side of (iv). Firstly

$$\begin{aligned}\|\Lambda(t_i)\|_0^2 &= \big\| \big(\Delta u''(t_i) - u''(t_i)\big) + \big(u''(t_i) - \partial_t^2 u(t_{i+1})\big) \big\|_0^2 \\ &\le 2\, \|\Delta u''(t_i) - u''(t_i)\|_0^2 + 2\, \|u''(t_i) - \partial_t^2 u(t_{i+1})\|_0^2 \\ &\le Ck^3 \int_{t_{i-1}}^{t_{i+1}} \|u''''(s)\|_0^2\, ds, \qquad \text{(v)}\end{aligned}$$

from Taylor's series. Also for $\eta(t)$ we see

$$\begin{aligned}
\|\partial_t^2\eta(t_{i+1})\|_0^2 &= \left\|\eta''(t_i) + \left(\partial_t^2\eta(t_{i+1}) - \eta''(t_i)\right)\right\|_0^2 \\
&\leq 2\left\|\eta''(t_i)\right\|_0^2 + 2\left\|\partial_t^2\eta(t_{i+1}) - \eta''(t_i)\right\|_0^2 \\
&\leq Ch^{2r}|||u|||_{2,r}^2(t_i) + Ck^3\int_{t_{i-1}}^{t_{i+1}} \|\eta''''(s)\|_0^2\,ds \\
&\leq Ch^{2r}|||u|||_{2,r}^2(t_i) + Ch^{2r}k^3\int_{t_{i-1}}^{t_{i+1}} |||u|||_{4,r}^2(s)\,ds, \qquad \text{(vi)}
\end{aligned}$$

using the Ritz-Volterra projection (Proposition 3.1). Returning to (iv) we choose $v = k\bar{\partial}_t\theta_{i+1} = 1/2\big((\theta_{i+1}+\theta_i) - (\theta_i + \theta_{i-1})\big) = \bar{\theta}_{i+1} - \bar{\theta}_i \in \mathcal{V}^h$, and observe that

$$\begin{aligned}
(\partial_t^2\theta_{i+1}, k\bar{\partial}_t\theta_{i+1}) &= \frac{k}{2}\partial_t \left\|\partial_t\theta_{i+1}\right\|_0^2 \\
a(\Delta\theta_i, k\bar{\partial}_t\theta_{i+1}) &= \frac{k}{2}\partial_t \left\|\bar{\theta}_{i+1}\right\|_A^2
\end{aligned}$$

where the energy norm is defined in the usual way: $\|\cdot\|_A := \sqrt{a(\cdot,\cdot)}$. Then we have

$$\begin{aligned}
\frac{k}{2}\partial_t\left(\left\|\partial_t\theta_{i+1}\right\|_0^2 + \left\|\bar{\theta}_{i+1}\right\|_A^2\right) &= (\Lambda(t_i) - \partial_t^2\eta(t_{i+1}), k\bar{\partial}_t\theta_{i+1}) + Q_i(k\bar{\partial}_t\theta_{i+1}) \\
&\quad - \sum_{j=0}^{i}\varpi_{ij}b(t_i,t_j;\Delta\theta_j, k\bar{\partial}_t\theta_{i+1}).
\end{aligned}$$

Summing over $i = 1,\ldots,m \leq N-1$ and noting that, generically, $k\sum_{i=1}^m \partial_t\chi_{i+1} = \chi_{m+1} - \chi_1$, we obtain

$$\begin{aligned}
\frac{1}{2}\left\|\partial_t\theta_{m+1}\right\|_0^2 + \frac{1}{2}\left\|\bar{\theta}_{m+1}\right\|_A^2 &= \frac{1}{2}\left\|\partial_t\theta_1\right\|_0^2 + \frac{1}{2}\left\|\bar{\theta}_1\right\|_A^2 + \sum_{i=1}^m Q_i(k\bar{\partial}_t\theta_{i+1}) \\
&\quad + \sum_{i=1}^m(\Lambda(t_i) - \partial_t^2\eta(t_{i+1}), k\bar{\partial}_t\theta_{i+1}) \\
&\quad - \sum_{i=1}^m\sum_{j=0}^{i}\varpi_{ij}b(t_i,t_j;\Delta\theta_j, k\bar{\partial}_t\theta_{i+1}).
\end{aligned}$$

Now, to estimate the fourth term on the right-hand side we use our previous results from the consistency errors,

$$\begin{aligned}
&\left|\sum_{i=1}^m(\Lambda(t_i) - \partial_t^2\eta(t_{i+1}), k\bar{\partial}_t\theta_{i+1})\right| \\
&\qquad \leq \sum_{i=1}^m k\left(\|\Lambda(t_i)\|_0 + \|\partial_t^2\eta(t_{i+1})\|_0\right)\left(\|\partial_t\theta_{i+1}\|_0 + \|\partial_t\theta_i\|_0\right) \\
&\qquad \leq \sum_{i=1}^m \frac{k}{2}\Big(\left(\|\partial_t\theta_{i+1}\|_0 + \|\partial_t\theta_i\|_0\right)^2 + \left(\|\Lambda(t_i)\|_0 + \|\partial_t^2\eta(t_{i+1})\|_0\right)^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{m} k\left(\|\partial_t\theta_{i+1}\|_0^2 + \|\partial_t\theta_i\|_0^2\right) + \sum_{i=1}^{m} k\left(\|\Lambda(t_i)\|_0^2 + \|\partial_t^2\eta(t_{i+1})\|_0^2\right) \\
&\le \sum_{i=0}^{m} 2k\|\partial_t\theta_{i+1}\|_0^2 + \sum_{i=1}^{m} k\left(Ck^3\int_{t_{i-1}}^{t_{i+1}} \|u''''(s)\|_0^2\,ds\right) \\
&\qquad + \sum_{i=1}^{m} k\left(Ch^{2r}|\!|\!|u|\!|\!|_{2,r}^2(t_i) + Ch^{2r}k^3\int_{t_{i-1}}^{t_{i+1}} |\!|\!|u|\!|\!|_{4,r}^2(s)\,ds\right) \\
&\le Ck\sum_{i=0}^{m} \|\partial_t\theta_{i+1}\|_0^2 + Ck^4\int_0^T \|u''''(s)\|_0^2\,ds \\
&\qquad + CTh^{2r}\sup_{t\in\mathcal{I}} |\!|\!|u|\!|\!|_{2,r}^2(t) + Ch^{2r}k^4\int_0^T |\!|\!|u|\!|\!|_{4,r}^2(s)\,ds,
\end{aligned}
$$

(i.e. we have used (v) and (vi)). Using this now gives:-

$$
\begin{aligned}
\frac{1}{2}\|\partial_t\theta_{m+1}\|_0^2 + \frac{1}{2}\left\|\bar{\theta}_{m+1}\right\|_A^2 \le{}& \frac{1}{2}\|\partial_t\theta_1\|_0^2 + \frac{1}{2}\left\|\bar{\theta}_1\right\|_A^2 + Ck\sum_{i=0}^{m}\|\partial_t\theta_{i+1}\|_0^2 \\
&+ C(u)(h^{2r} + k^4 + h^{2r}k^4) \\
&+ \left|\sum_{i=1}^{m} Q_i(k\bar{\partial}_t\theta_{i+1})\right| \\
&+ \left|\sum_{i=1}^{m}\sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; \Delta\theta_j, k\bar{\partial}_t\theta_{i+1})\right|. \qquad \text{(vii)}
\end{aligned}
$$

Using now the definition of $Q_i(\cdot)$ from (iii) and Lemmas 4.1, 4.2 & 4.3 we then obtain, for all $\epsilon > 0$:

$$
\begin{aligned}
\left|\sum_{i=1}^{m} Q_i(k\bar{\partial}_t\theta_{i+1})\right| &\le C(k + 1/\epsilon)\left\|\bar{\theta}_{m+1}\right\|_1^2 + Ck^4\int_0^T\sum_{\ell=0}^{2}\left\|w^{(\ell)}(s)\right\|_1^2 ds + Ck\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2 \\
&\le C(k + 1/\epsilon)\left\|\bar{\theta}_{m+1}\right\|_1^2 + C(u)k^4 + Ck\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2.
\end{aligned}
$$

Here the constant indicated by $C(u)$ is formed by using the assumptions in (3.2) to bound derivatives of w independently of the space $\mathcal{V}^h$. Also, from Lemma 4.4 we have

$$
\left|\sum_{i=1}^{m}\sum_{j=0}^{i} \varpi_{ij} b(t_i, t_j; \Delta\theta_j, k\bar{\partial}_t\theta_{i+1})\right| \le CT\|\theta_0\|_1^2 + C(k + T/\epsilon)\left\|\bar{\theta}_{m+1}\right\|_1^2 + Ck\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2.
$$

Our assumptions on $a(\cdot,\cdot)$ give positive constants c and C such that $c\|\cdot\|_1^2 \le a(\cdot,\cdot) \le C\|\cdot\|_1^2$, so collecting these last few results together gives (vii) as:

$$
\begin{aligned}
\frac{1}{2}\|\partial_t\theta_{m+1}\|_0^2 + \frac{c}{2}\left\|\bar{\theta}_{m+1}\right\|_1^2 \le{}& \frac{1}{2}\|\partial_t\theta_1\|_0^2 + \frac{C}{2}\left\|\bar{\theta}_1\right\|_1^2 + C\|\theta_0\|_1^2 \\
&+ C\left(k + \frac{1}{\epsilon}\right)\left\|\bar{\theta}_{m+1}\right\|_1^2 + C(h^{2r} + k^4) \\
&+ Ck\sum_{i=0}^{m}\|\partial_t\theta_{i+1}\|_0^2 + Ck\sum_{i=0}^{m-1}\left\|\bar{\theta}_{i+1}\right\|_1^2, \quad \forall\epsilon > 0.
\end{aligned}
$$

Finally then it remains to choose k_0 small enough and ϵ large enough so that the above inequality may be meaningfully re-expressed as

$$\|\partial_t\theta_{m+1}\|_0^2 + \left\|\bar{\theta}_{m+1}\right\|_1^2 \le C\big(\|\partial_t\theta_1\|_0^2 + \|\theta_0\|_1^2 + \|\theta_1\|_1^2 + h^{2r} + k^4\big) + Ck\sum_{i=0}^{m-1}\Big(\|\partial_t\theta_{i+1}\|_0^2 + \left\|\bar{\theta}_{i+1}\right\|_1^2\Big).$$

Gronwall's discrete lemma now implies that for $1 \le m \le N-1$:

$$\|\partial_t\theta_{m+1}\|_0^2 + \left\|\frac{\theta_{m+1}+\theta_m}{2}\right\|_1^2 \le C\big(\|\partial_t\theta_1\|_0^2 + \|\theta_0\|_1^2 + \|\theta_1\|_1^2 + h^{2r} + k^4\big),$$

and, finally, we take square roots using (4.6) to prove the theorem. ∎

Note that Theorem 4.1 yields an order $O(h^r)$ superconvergent estimate for the spatial discretization of θ, this is consistent with estimates given for the wave equation (based on the Ritz projection) and the semidiscrete results given in [13]. We now move on to the error bound for the fully discrete approximation:

THEOREM 4.2. *Assume that Theorem 4.1 holds, then for $k \in (0, k_0]$ there exists a positive constant, C, depending upon u, but independent of h and k, such that*

$$\max_{1\le m\le N-1}\left\{\left\|u'(t_{m+\frac{1}{2}}) - \partial_t U_{m+1}\right\|_0 + h\left\|u(t_{m+\frac{1}{2}}) - \frac{U_{m+1}+U_m}{2}\right\|_1\right\} \le C\left(\|\partial_t\theta_1\|_0 + \|\theta_0\|_1 + \|\theta_1\|_1 + h^r + k^2\right).$$

Proof. This is straightforward, note first the Taylor series results,

$$\left\|w'(t_{m+\frac{1}{2}}) - \partial_t w(t_{m+1})\right\|_0 \le Ck^2 \sup_{t\in\mathcal{I}} \|w'''(t)\|_0,$$

$$\left\|u(t_{m+\frac{1}{2}}) - \frac{u(t_{m+1})+u(t_m)}{2}\right\|_1 \le Ck^2 \sup_{t\in\mathcal{I}} \|u''(t)\|_1.$$

Then we see that

$$\begin{aligned}\left\|u'(t_{m+\frac{1}{2}}) - \partial_t U_{m+1}\right\|_0 &= \left\|u'(t_{m+\frac{1}{2}}) - w'(t_{m+\frac{1}{2}}) + w'(t_{m+\frac{1}{2}}) - \partial_t w(t_{m+1}) - \partial_t\theta_{m+1}\right\|_0 \\ &\le \left\|D_t(u-w)(t_{m+\frac{1}{2}})\right\|_0 + \left\|w'(t_{m+\frac{1}{2}}) - \partial_t w(t_{m+1})\right\|_0 \\ &\quad + \|\partial_t\theta_{m+1}\|_0 \\ &\le Ch^r \sup_{t\in\mathcal{I}} \|u(t)\|_{1,r} + Ck^2 \sup_{t\in\mathcal{I}} \|u(t)\|_{3,r} + \|\partial_t\theta_{m+1}\|_0,\end{aligned}$$

and,

$$\begin{aligned}\left\|u(t_{m+\frac{1}{2}}) - \frac{U_{m+1}+U_m}{2}\right\|_1 &= \left\|u(t_{m+\frac{1}{2}}) - \frac{u(t_{m+1})+u(t_m)}{2} + \frac{u(t_{m+1})+u(t_m)}{2}\right. \\ &\qquad - \frac{w(t_{m+1})+w(t_m)}{2} + \frac{w(t_{m+1})+w(t_m)}{2} \\ &\qquad \left. - \frac{U_{m+1}+U_m}{2}\right\|_1 \\ &\le Ck^2 \sup_{t\in\mathcal{I}} \|u''(t)\|_1 + \left\|\frac{\theta_{m+1}+\theta_m}{2}\right\|_1 \\ &\quad + \left\|\frac{u(t_{m+1})-w(t_{m+1})}{2}\right\|_1 + \left\|\frac{u(t_m)-w(t_m)}{2}\right\|_1 \\ &\le C(k^2+h^{r-1}) \sup_{t\in\mathcal{I}} \|u(t)\|_{2,r} + \left\|\frac{\theta_{m+1}+\theta_m}{2}\right\|_1.\end{aligned}$$

Therefore, we conclude that for $1 \le n \le N-1$,

$$\begin{aligned}\left\| u'(t_{n+\frac{1}{2}}) - \partial_t U_{n+1} \right\|_0 + h \left\| u(t_{n+\frac{1}{2}}) - \frac{U_{n+1}+U_n}{2} \right\|_1 \\ \le \|\partial_t \theta_{n+1}\|_0 + h \left\| \frac{\theta_{n+1}+\theta_n}{2} \right\|_1 + C(h^r + k^2) \\ \le C(\|\partial_t\theta_1\|_0 + \|\theta_0\|_1 + \|\theta_1\|_1 + h^r + k^2),\end{aligned}$$

as required. ∎

5. Closure. We have seen that not just the rate of convergence, but the actual convergence itself depends upon the way the initial conditions are represented in the discrete scheme. Dupont in [16] proposes a method of obtaining suitable starting values, U_0 and U_1, for the wave equation and here, in brief outline, we propose an adaptation of that method to our problem.

Firstly we choose $U_0 = w(0)$, that is we solve the elliptic problem:

$$a\big(U_0, v\big) = a\big(u(0), v\big), \tag{5.1}$$

and then define an $O(k^3)$ approximation to $u(t)$ as $\tilde{U}(t)$, where

$$\tilde{U}(t) := u(0) + tu'(0) + \frac{t^2}{2}u''(0) \quad \text{for } t \le k, \tag{5.2}$$

and $u''(0) = f(0) - Au(0)$. Now we project $\tilde{U}(t)$ into $\mathcal{V}^h$ via (3.1) and obtain $\hat{U}(t)$ through

$$a\big(\hat{U}(t), v\big) + \int_0^t b\big(t, s; \hat{U}(s), v\big)\ ds = a\big(\tilde{U}(t), v\big) + \int_0^t b\big(t, s; \tilde{U}(s), v\big)\ ds. \tag{5.3}$$

In principle the right hand side of (5.3) is completely determined, thus (5.3) is a Volterra equation of the second kind for $\hat{U}(t)$. To remain consistent with the approach detailed in Section 2 we replace the left hand integral with the trapezoidal rule approximation and obtain the fully discrete approximation, U_1, to $u(k)$ via

$$a\big(U_1, v\big) + \frac{k}{2}\Big(b\big(k, k; U_1, v\big) + b\big(k, 0; U_0, v\big)\Big) = a\big(\tilde{U}(t), v\big) + \int_0^t b\big(t, s; \tilde{U}(s), v\big)\ ds, \tag{5.4}$$

where the error is again of order $O(k^3)$. As with the method suggested by Dupont we also have two problems to solve in order to obtain the initial conditions; a Galerkin elliptic problem for U_0 and a Volterra problem for U_1, the latter of which has a complicated right hand side.

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