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GRADED MESH REFINEMENT AND ERROR ESTIMATES FOR FINITE ELEMENT SOLUTIONS OF ELLIPTIC BOUNDARY VALUE PROBLEMS IN NON-SMOOTH DOMAINS

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Graded Mesh Refinement and Error Estimates for Finite Element Solutions of Elliptic Boundary Value Problems in Non-Smooth Domains

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Abstract. This paper is concerned with the effective numerical treatment of elliptic boundary value problems when the solutions contain singularities. The paper deals first with the theory of problems of this type in the context of weighted Sobolev spaces and covers problems in domains with conical vertices and non-intersecting edges, as well as polyhedral domains with Lipschitz boundaries. Finite element schemes on graded meshes for second order problems in polygonal/polyhedral domains are then proposed for problems with the above singularities. These schemes exhibit optimal convergence rates with decreasing mesh size. Finally, we describe numerical experiments which demonstrate the efficiency of our technique in terms of "actual" errors for specific (finite) mesh sizes in addition to the asymptotic rates of convergence.

Key Words. Elliptic boundary value problem, singularities, finite element method, mesh grading.

AMS(MOS) subject classification. 65N30

1 Introduction

This paper is concerned with the effective numerical treatment of elliptic boundary value problems in domains with vertices and edges when singularities of the solution are present. It is well known that standard numerical techniques lose accuracy as a result of the non-smooth boundary, which causes the regularity of the solutions of the problems to be low in comparison with that of the solutions of smooth problems. As a result many specially adapted numerical methods have been developed and there is an extensive literature for this field, see for example [2, 3, 4, 5, 6, 12, 16, 20, 34, 36, 39, 44, 46, 48].

In Section 2, a fairly general class of linear elliptic boundary value problems is considered, in which the differential operators can be of high order and variable coefficients can appear. The regularity of the weak (variational) solution is studied for two- or three-dimensional domains with corners and edges. The framework for the regularity investigations is the theory of Kondrat'ev [22] and Maz'ya and Plamenevskiĭ [29]. The present paper shows that this framework is well suited to and sufficient for deriving error estimates for the finite element method.

In Section 3, finite element schemes on locally graded meshes for a large class of second order boundary value problems in polygonal/polyhedral domains are described. The grading is determined mainly by the regularity results of Section 2. Graded meshes have been studied by many authors for finite element methods, see [20, 34, 39] for the Poisson problem in polygonal domains and [4, 18, 19] for three-dimensional domains with edges, and also for boundary element methods, see for example [36]. We give

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here a generalization of the finite element results to polyhedral domains and to more general differential operators and prove asymptotic error estimates which exhibit optimal convergence rates with decreasing mesh size. — Finite element error estimates in the energy norm are usually proved via Céa's lemma by estimating the interpolation error, see for example [13], and we adopt this approach in Subsection 3.3. However, if the solution is not smooth enough, so that pointwise values are not well defined, we have to use an alternative approximation operator. Because this case can arise, we give in Subsection 3.4 an error estimate using the approximation operator first introduced in [45].

Finally, we show with two numerical experiments how the theory can be applied. Examples are given for the Lamé system in a two-dimensional L-shaped domain, and for the Poisson equation in a three-dimensional polyhedral domain with three 270°-edges meeting in one corner (Fichera corner). Both examples demonstrate the efficiency of our technique in terms of "actual" errors for specific (finite) mesh sizes in addition to the asymptotic rates of convergence. Note that other numerical tests for treating the Poisson equation in a three-dimensional domain have been documented in [4, 5].

2 Regularity results

2.1 The boundary value problems

We consider the following linear elliptic boundary value problems

$$A(x,D_x)u(x) = f(x) \qquad \text{in } \Omega \tag{2.1}$$
$$B_j(x,D_x)u(x) = 0, \quad j=1,\dots,m, \qquad \text{on } \partial\Omega\setminus M \tag{2.2}$$

with

$$\begin{aligned} A(x,D_x)u(x) &:= \sum_{|\gamma|,|\psi|\le m} (-1)^{|\gamma|} D_x^\gamma (a_{\gamma\psi}(x) D_x^\psi) u(x) \\ &:= \sum_{|\alpha|\le 2m} a_\alpha(x) D_x^\alpha u(x) \end{aligned} \tag{2.3}$$
$$B_j(x,D_x)u(x) := \sum_{|\alpha|\le m_j} b_{j,\alpha}(x) D_x^\alpha u(x), \quad j=1,\dots,m, \tag{2.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n=2,3)$ with conical points (for example polygons in $\mathbb{R}^2$), with non-intersecting edges (for example rotationally symmetric domains in $\mathbb{R}^3$) or with corners of polyhedral type (polyhedrons in $\mathbb{R}^3$). We assume that the coefficients of A are smooth and real, and that the coefficients of B_j $(j=1,\dots,m)$ are piecewise smooth and real; this last condition means that the type of boundary condition may change. In this case we denote by $B_{ij}(x,D_x)$ the restriction of B_j to the pieces $\partial\Omega^i\subset\partial\Omega$, $\left(\bigcup_{i=1}^I \overline{\partial\Omega^i}=\partial\Omega,\ \partial\Omega^i\cap\partial\Omega^k=\emptyset \text{ for } i\ne k\right)$. We denote by M the set of singular boundary points, which consists of corner-points, edges, and points (lines) at which the type of the boundary condition changes.

Note that we have restricted problem (2.1–2.2) to the case of homogeneous boundary conditions. This is important as in the weak formulation the solution and the test functions lie in the same space; thus it makes the analysis simpler.

In order to treat problem (2.1–2.2), we derive its weak form. Due to the fact that for non-smooth domains special Green's formulae hold [20, Theorem 1.5.3.11., p. 61] [40, Lemma 1, p. 568] which include additional terms generated by the set M, we introduce the set $C_M^\infty(\Omega) := \{u\in C^\infty(\overline{\Omega}) : \operatorname{supp} u\cap M=\emptyset\}$ and define a space V as the closure of $\{u\in C_M^\infty(\Omega) : B_{ij}(x,D_x)u=0 \text{ on } \partial\Omega^i \text{ for all } i,j \text{ with } \operatorname{ord}(B_{ij})\le m-1\}$ in $W^{m,2}(\Omega)$.

With V^* being the dual space of V, the weak formulation is: Find a solution $u \in V$ such that for a given $f \in V^*$

$$
\begin{aligned}
a(u,v) &= \langle f, v\rangle \quad \text{for all } v \in V, \\
a(u,v) &:= \sum_{|\gamma|,|\psi|\le m} \int_\Omega a_{\gamma\psi}(x) D^\psi u D^\gamma v \, dx + \\
&\quad + \sum_{i=1}^{I} \sum_{j=1}^{m} \int_{\partial\Omega^i} G_{ij} u F_{ij} v \, d\Gamma, \\
\langle f, v\rangle &:= \int_\Omega f v \, dx,
\end{aligned}
\tag{2.5}
$$

where the boundary operators G_{ij} and F_{ij} are appropriate normal boundary systems (for the definition of normal boundary systems see for example [49, §14]) which are generated by the essential boundary operators $B_{ij}(x, D_x)$ on $\partial\Omega^i$.

We also assume that the coefficients in (2.3) and (2.4) are such that the weak problem (2.5) has a uniquely determined solution $u \in V$, or more precisely, that the assumptions of the Lax-Milgram theorem hold:

$$|a(u,v)| \le c_1 \|u; W^{m,2}(\Omega)\| \, \|v; W^{m,2}(\Omega)\| \quad \text{for all } u, v \in V, \tag{2.6}$$

$$a(u,u) \ge c_2 \|u; W^{m,2}(\Omega)\|^2 \quad \text{for all } u \in V. \tag{2.7}$$

2.2 Statement of the regularity problem

The regularity theory for elliptic boundary value problems in non-smooth domains with corners and edges is well developed, especially in the framework of weighted Sobolev spaces. Specifically, boundary value problems in domains with conical points are handled in [22], in domains with non-intersecting edges in [23, 29, 31], and in polyhedral domains in [15, 30, 37]. The field is treated in [20] in standard Sobolev spaces.

We formulate here regularity results for solutions of the general weak problem (2.5) in the following weighted Sobolev spaces: Let Ω be a bounded domain with the set M of singular boundary points. The space $V^{k,p}(\Omega, \beta)$ is the closure of the set $C_M^\infty(\Omega)$ with respect to the norm

$$\|u; V^{k,p}(\Omega,\beta)\| = \left(\sum_{|\alpha|\le k} \int_\Omega r^{p(\beta-k+|\alpha|)} |D^\alpha u|^p dx \right)^{1/p}, \tag{2.8}$$

where $r = r(x) = \operatorname{dist}(x, M)$ and β is a real number. We remark that $V^{k,p}(\Omega,\beta) \subset V^{k-1,p}(\Omega, \beta - 1)$.

Let $u \in V$ be the weak solution of (2.5) for $f \in L_2(\Omega)$. We consider now the regularity problem, for which β the solution u is contained in the space $V^{2m,2}(\Omega,\beta)$.

Let us emphasize the crucial point. We start with a solution $u \in V$ from a standard Sobolev space. But the space $V \subset W^{m,2}(\Omega)$ does not belong automatically to the scale of weighted Sobolev spaces (2.8). Therefore we demand that

$$u \in V \cap V^{m,2}(\Omega, 0) \tag{2.9}$$

when $f \in L_2(\Omega)$. In [41, Property (R)] it is shown that (2.9) is satisfied for a large class of problems including Dirichlet problems and mixed boundary value problems. — Assumption (2.9) can be omitted if M does not contain edges.

The investigations in the papers mentioned above show that the distribution of the eigenvalues of a parameter dependent boundary value problem is crucial to the regularity of the solution. One can get this parameter dependent boundary value problem by considering the principal parts of A and B_j in (2.1–2.4) with frozen coefficients at points of M, using spherical coordinates, followed by a Mellin transform with respect to r.

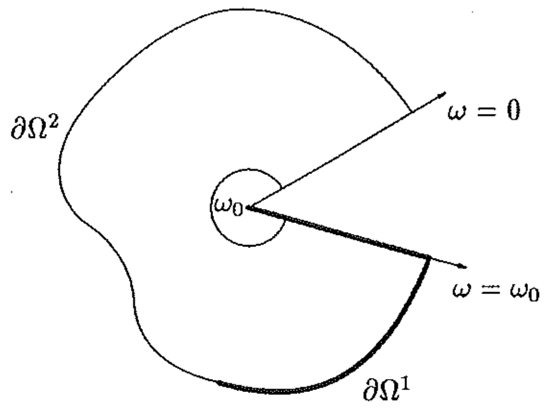


Figure 2.1: Sample domain.

2.3 Domains with conical points

Let us illustrate this approach first for a domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) which has only one conical point O on its boundary. For simplicity we assume that there is a ball-neighbourhood of O, for which Ω coincides with the cone $K = \{(r, \omega) : 0 < r < \infty, \omega \in G\}$. Here we further assume that G, the intersection of Ω with the surface S^{n-1} of the ball-neighbourhood, is a smooth domain.

We consider a special boundary value problem in K, which is generated by the principal parts of A and B_j with frozen coefficients in O:

$$\begin{aligned} A_0(O, D_x)u(x) &:= \sum_{|\alpha|=2m} a_\alpha(O) D^\alpha u(x) &= f(x) \quad \text{in } K, \\ B_{0,j}(O, D_x)u(x) &:= \sum_{|\alpha|=m_j} b_{j,\alpha}(O) D^\alpha u(x) &= g_j(x) \quad \text{on } \partial K, \end{aligned}$$

(j=1,...,m). Introducing spherical coordinates (r, ω) and using the Mellin transform

$$\hat{u}(\alpha, \omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-\alpha-1} u(r, \omega) dr$$

we obtain a boundary value problem with the parameter α:

$$\begin{aligned} L(\omega, D_\omega, \alpha)\hat{u}(\alpha, \omega) &= \hat{F}(\alpha, \omega) \quad \text{for } \omega \in G, \\ M_j(\omega, D_\omega, \alpha)\hat{u}(\alpha, \omega) &= \hat{G}j(\alpha, \omega) \quad \text{for } \omega \in \partial G,\ j = 1, \ldots, m, \end{aligned} \tag{2.10}$$

where $F = r^{2m} f$ and $G_j = r^{m_j} g_j$.

The distribution of the eigenvalues α (those complex numbers α_0 for which non-trivial solutions $\hat{u}$ of (2.10) for $\hat{F} = 0$ and $\hat{G}_j = 0$ ($j = 1, \ldots, m$) exist) in a certain strip in the complex plane determines the regularity. The following theorem was proved in [22] and can also be found in [27, 41].

Theorem 2.1 *Let Ω be a bounded domain with one conical boundary point O. The weak solution u of (2.5) with the right hand side $f \in L_2(\Omega)$ is contained in $V^{2m,2}(\Omega, m - H_0 + \varepsilon)$:*

$$\|u; V^{2m,2}(\Omega, m - H_0 + \varepsilon)\| \leq C\|f; L_2(\Omega)\|, \tag{2.11}$$

where $H_0 = \mathrm{Re}(\alpha_0) - (-\frac{n}{2} + m)$. Here, α_0 is such an eigenvalue of problem (2.10), that the strip $-\frac{n}{2} + m < \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha_0)$ is free of eigenvalues, and $\varepsilon > 0$ is an arbitrarily small real number.

Example 2.1 Let Ω be a plane domain with only one corner point O with the angle ω_0. We consider the mixed boundary value problems (see Figure 2.1)

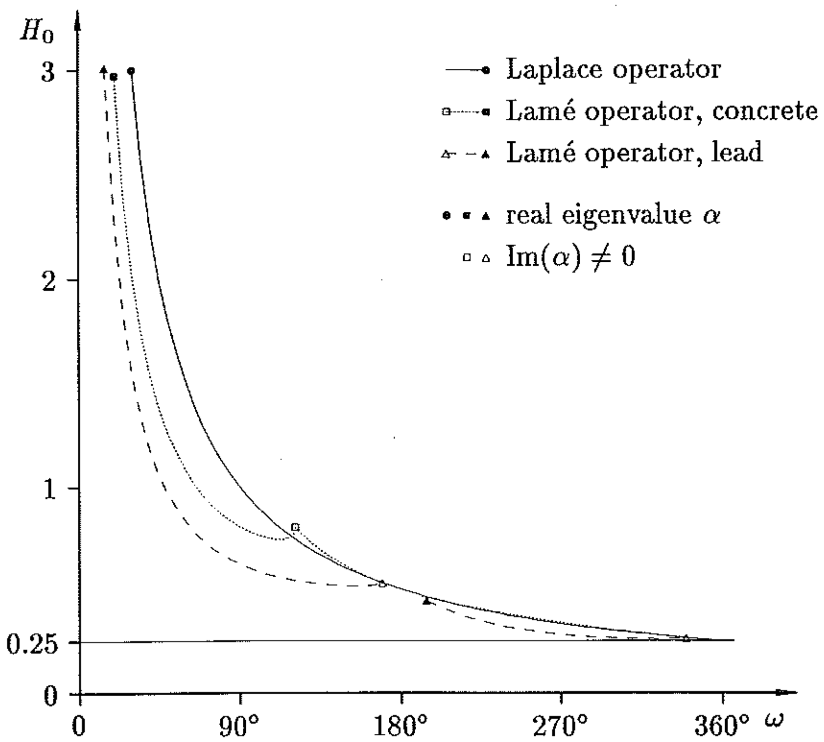


Figure 2.2: Diagram of $H_0(\omega_0)$ for the two-dimensional example.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega^1, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega^2, \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} -Lu &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega^1, \\ S[u] \cdot n &= 0 \quad \text{on } \partial\Omega^2, \end{aligned} \tag{2.13}$$

where L is the Lamé operator defined by

$$Lu := \tilde{\mu}\Delta u + (\tilde{\lambda} + \tilde{\mu})\,\mathrm{grad}\,\mathrm{div} u \tag{2.14}$$

with the Lamé coefficients $\tilde{\lambda}$ and $\tilde{\mu}$, $S[u]$ denotes the stress tensor with Cartesian components

$$[S(u(x))]_{ij} := \tilde{\mu}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \delta_{ij}\tilde{\lambda}\nabla \cdot u(x).$$

Here, u_i is the i-th component of u, δ_{ij} is the Kronecker delta, and n denotes the outward normal to $\partial\Omega$ at the point x.

We have $H_0 = \frac{\pi}{2\omega_0}$ for problem (2.12) and get for problem (2.13) that the eigenvalues α_0 are the zeros of the transcendental equation

$$\sin^2 \alpha\omega_0 = \frac{-\alpha^2(\tilde{\lambda} + \tilde{\mu})^2 \sin^2 \omega_0 + (\tilde{\lambda} + 2\tilde{\mu})^2}{(\tilde{\lambda} + \tilde{\mu})(\tilde{\lambda} + 3\tilde{\mu})}$$

[42] and $H_0 = H_0(\omega_0)$ can be calculated, see Theorem 2.1. Figure 2.2 shows the graphs of $H_0 = H_0(\omega_0)$ for problem (2.12), and (2.13) for the material constants $\tilde{\sigma} := \frac{\tilde{\lambda}+\tilde{\mu}}{\tilde{\mu}} = 10$ (lead) and $\tilde{\sigma} = 1.51515$ (concrete).

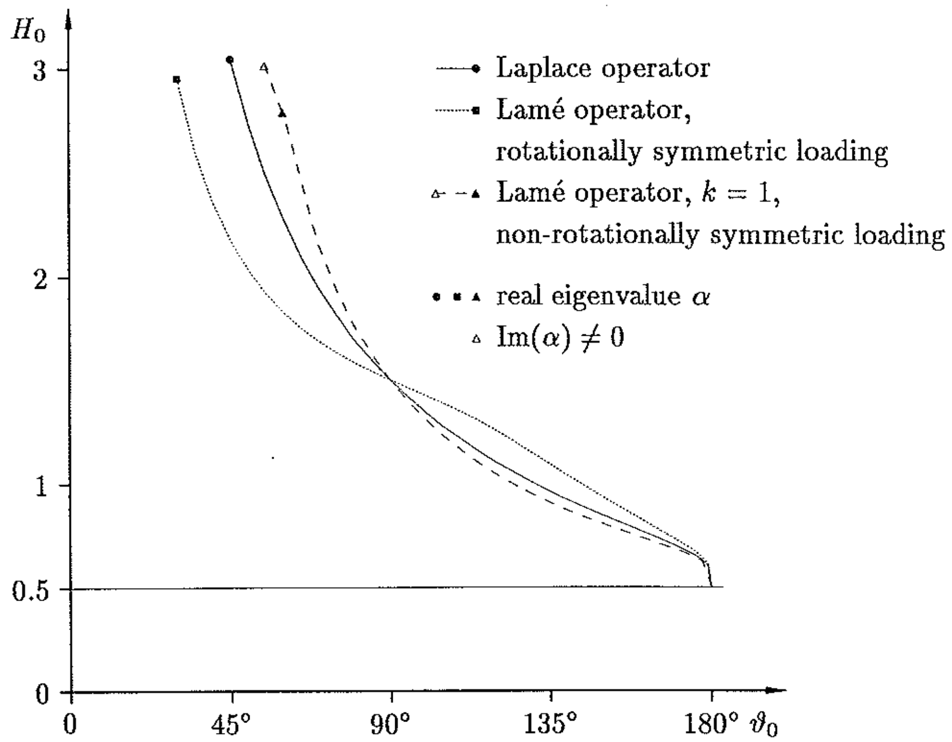


Figure 2.3: Diagram of $H_0(\vartheta_0)$ for the three-dimensional example.

Example 2.2 Let Ω be a three-dimensional domain with a conical point O which coincides in a neighbourhood of O with a circular cone

$$K := \{(r, \varphi, \vartheta) : 0 < r < \infty, 0 \leq \varphi < 2\pi, 0 \leq \vartheta < \vartheta_0\}$$

and consider the boundary value problems

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} -Lu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.16}$$

where L is again the Lamé operator (2.14).

The values of $H_0 = H_0(\vartheta_0)$ are calculated for problem (2.15) for example in [11] and for problem (2.16) for the Poisson ratio $\nu = 0.3$ in [9, 10] under rotationally symmetric forces. If the loading is not rotationally symmetric the singularity can be more significant [43]. Figure 2.3 shows the graphs of $H_0(\vartheta_0)$ in these three cases.

Remark 2.1 A very general estimate for H_0 is proved in [24] for the first and second boundary value problems for strongly elliptic systems of the order $2m$; namely that $H_0 \geq \frac{1}{2}$ in these cases.

2.4 Domains with edges

We now consider a domain $\Omega \subset \mathbb{R}^3$ with a single edge M. Assume there is a neighbourhood of each point of M in which Ω is diffeomorphic to a three-dimensional dihedral angle $D := K \times \mathbb{R}$, where $K := \{(r, \omega) : 0 < r < \infty, 0 < \omega < \omega_0\}$. Let z_0 be a point of M and D_{z_0} the corresponding dihedral angle. For simplicity assume that Ω coincides in a ball-neighbourhood of z_0 with the dihedral angle D_{z_0}. We take in this neighbourhood the Cartesian coordinate system

$$x = (y, z) = (y_1, y_2, z), \quad y \perp z, \quad y \in K, \quad z \in \mathbb{R},$$

and consider the special boundary value problem

$$\begin{array}{rclcll} A_0(z_0,D_x)u(x) & = & \sum\limits_{\substack{|\alpha|=2m\\ \alpha_1+\alpha_2=\alpha}} a_\alpha(z_0)D_y^{\alpha_1}D_z^{\alpha_2}u(x) & = & f(x) & \text{in } D_{z_0} \\ B_{0j}^{\pm}(z_0,D_x)u(x) & = & \sum\limits_{\substack{|\alpha|=m_j\\ \alpha_1+\alpha_2=\alpha}} b_{j,\alpha}^{\pm}(z_0)D_y^{\alpha_1}D_z^{\alpha_2}u(x) & = & g_j^{\pm}(x) & \text{on } \Gamma_{z_0}^{\pm}, \\ & & & & j=1,\ldots,m, & \end{array}$$

where $\Gamma_{z_0}^{\pm}$ are the faces of D_{z_0}.

After the real Fourier transformation with respect to z and the normalization of the corresponding parameter [27, 29] we get a two-dimensional boundary value problem

$$\begin{array}{rcll} A_0(z_0,D_y,\theta)u & = & f & \text{in } K_{z_0} \\ B_{0j}^{\pm}(z_0,D_x,\theta)u & = & g_j^{\pm} & \text{on } \partial K_{z_0}^{\pm},\ j=1,\ldots,m. \end{array} \tag{2.17}$$

with $\theta = \pm 1$. Using polar coordinates (r_{z_0},ω), $r_{z_0} = |y - z_0|$, we again consider the principal parts of (2.17)

$$\begin{array}{rcll} A_0(z_0,D_y,0) & = & r_{z_0}^{-2m}L(\omega,D_\omega,r_{z_0}D_{r_{z_0}}) & \\ B_{0j}^{\pm}(z_0,D_x,0) & = & r_{z_0}^{-m_j^{\pm}}M_j^{\pm}(\omega,D_\omega,r_{z_0}D_{r_{z_0}}) & j=1,\ldots,m. \end{array}$$

After the Mellin transform with respect to r_{z_0} we get the operator pencil

$$\mathfrak{A}_0(z_0,\alpha) := \{L(\omega,D_\omega,\alpha); M_j^{\pm}(\omega,D_\omega,\alpha)\}_{j=1,\ldots,m}. \tag{2.18}$$

The distribution of the eigenvalues of $\mathfrak{A}(z_0,\alpha)$ in a certain strip of the complex plane yields regularity results.

Theorem 2.2 *Let Ω be a bounded domain with the edge $M \subset \partial\Omega$. Assume the weak solution of (2.5) with the right hand side $f \in L_2(\Omega)$ is contained in $V \cap V^{m,2}(\Omega,0)$, see condition (2.9). Then u belongs to $V^{2m,2}(\Omega, m - H_0 + \varepsilon)$:*

$$\|u; V^{2m,2}(\Omega, m - H_0 + \varepsilon)\| \le C\|f; L_2(\Omega)\|, \tag{2.19}$$

where $H_0 = \mathrm{Re}(\alpha_0) - (-1+m)$. Here, α_0 is the element from $\{\alpha_0(z_0)\}_{z_0 \in M}$ with the smallest real part and $\alpha_0(z_0)$ is such an eigenvalue of the operator (2.18), that the strip $-1+m < \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha_0(z_0))$ is free of eigenvalues. Again $\varepsilon > 0$ is an arbitrarily small real number [29, 40].

2.5 Polyhedral domains

Finally we consider polyhedral domains in $\mathbb{R}^3$. In this case we have to distinguish between corner and edge singularities. The corner singularities can be handled analogously to the conical points, and we have only to notice that the domain G in (2.10), the intersection of Ω with the surface of the ball-neighbourhood considered, now has corner points.

Let us introduce the leading eigenvalues α_0 associated with the corner problems and the leading eigenvalues associated with the edge problems. Thus we define $\alpha_0(O_i)$ for every corner point O_i as that eigenvalue of the modified boundary value problem (2.10) for which the strip $-\frac{3}{2} + m < \mathrm{Re}(\alpha) < \mathrm{Re}(\alpha_0(O_i))$ is free of eigenvalues. Then, $\alpha_0 \in \{\alpha_0(O_i)\}_i$ is the eigenvalue $\alpha_0(O_{i_0})$ with the smallest real part. For every edge E_j we consider the eigenvalues $\alpha_0(E_j)$ which are defined in Theorem 2.2 and take the $\beta_0 \in \{\alpha_0(E_j)\}_j$ with the smallest real part. Finally, let

$$H_0 := \min\{\mathrm{Re}(\alpha_0) - (-\frac{3}{2} + m), \mathrm{Re}(\beta_0) - (-1+m)\}. \tag{2.20}$$

Then the following result holds [30, 32].

Theorem 2.3 *Let Ω be a polyhedron in $\mathbb{R}^3$. The weak solution u of (2.5) with the right hand side $f \in L_2(\Omega)$ for which (2.9) holds is contained in $V^{2m,2}(\Omega, m - H_0 + \varepsilon)$:*

$$\|u; V^{2m,2}(\Omega, m - H_0 + \varepsilon)\| \leq C\|f; L_2(\Omega)\|, \qquad (2.21)$$

where H_0 is given by (2.20) and $\varepsilon > 0$ is an arbitrarily small real number.

Remark 2.2 It is proved in [26] that $H_0 \geq \frac{1}{2}$ for the Dirichlet problem for strongly elliptic systems in polyhedral domains.

Remark 2.3 It is possible to obtain a more precise estimate for H_0 than that given in Remark 2.2 using the results for a circular cone (compare Example 2.2) for the Dirichlet problem for the Poisson equation (2.15) or for the Lamé equation system (2.16), where f is a non-rotationally symmetric force.

There is a relationship between the size of the eigenvalues and the size of domains in form of a monotonicity principle. It was proved in [17, 28] for the Laplacian: For the region G (arising from the polyhedral cone) the dominant α_0 is not less than that for the rotationally symmetric part of the unit ball with the same surface area as G. Note that the eigenvalues α_0 are real.

For the Lamé equation system the following result holds [25]: Let $K_i = (0, \infty) \times G_i$ $(i = 1, 2)$ be cones in $\mathbb{R}^3$, where $G_i \subset S^2$ can have corners. Then for the eigenvalues $\alpha_k(G_i)$ holds $\mathrm{Re}(\alpha_k(G_2)) \geq \mathrm{Re}(\alpha_k(G_1))$ if $G_1 \supset G_2$ and if $\alpha_k \in (-\frac{1}{2}, \Lambda_{\tilde{\sigma}}(G))$, where $\tilde{\sigma} = \frac{\tilde{\lambda}+\tilde{\mu}}{\tilde{\mu}}$ and $\Lambda_{\tilde{\sigma}}(G) \geq 1$ is some real number.

3 Finite element methods

3.1 Graded partitions

Let Ω be a polygonal domain in $\mathbb{R}^2$ or a polyhedral domain in $\mathbb{R}^3$. We consider a family of partitions $\mathcal{T}_h$ of $\overline{\Omega}$ with the usual regularity properties:

(a) $\overline{\Omega} = \bigcup_{\Omega_e \in \mathcal{T}_h} \overline{\Omega}_e$, where $\overline{\Omega}_e$ are polygons in $\mathbb{R}^2$ (triangles or quadrilaterals) or polyhedra in $\mathbb{R}^3$ (for example tetrahedra or bricks),

(b) $\Omega_{e_1} \cap \Omega_{e_2} = \emptyset$ for $e_1 \neq e_2$,

(c) any edge (for $n = 2, 3$) or face (for $n = 3$) of Ω_{e_1} is either a subset of $\partial\Omega$ or an edge or face of another Ω_{e_2}.

Denote by h_e the diameter of Ω_e and by ϱ_e the diameter of the largest inner ball of Ω_e, then we assume that there is a constant σ independent of $\mathcal{T}_h$ with

(d) $\frac{h_e}{\varrho_e} < \sigma$ for all e with $\Omega_e \in \mathcal{T}_h$.

The quotient h_e/ϱ_e is called aspect ratio of the element. — Further denote $h = \max_{\Omega_e \in \mathcal{T}_h} h_e$.

In order to treat the singularities of the solution near the irregular part M of the boundary, we assume that the partition $\mathcal{T}_h$ is graded in the following way:

$$\text{(e)} \quad \begin{array}{lll} \text{if } \overline{\Omega}_e \cap M \neq \emptyset & \text{then} & \underline{C}_1 h^{1/\mu} \leq h_e \leq \overline{C}_1 h^{1/\mu}, \\ \text{if } \overline{\Omega}_e \cap M = \emptyset & \text{then} & \underline{C}_2 h r_e^{1-\mu} \leq h_e \leq \overline{C}_2 h r_e^{1-\mu}, \end{array}$$

where $r_e = \mathrm{dist}(\Omega_e, M)$, and $\mu \in (0, 1]$ is a parameter to control the grading. Note that for $\mu = 1$ an unrefined partition is produced.

Such refinements were studied for the Poisson problem in polygonal domains in [20, 34, 39] and in three-dimensional domains with edges in [4]. Approximation results can also be found in [18, 19]. We give here a generalization of these results to polyhedral domains and to more general differential operators.

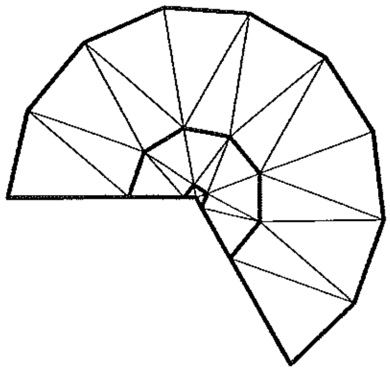


Figure 3.1: Construction of graded meshes using layers.

Remark 3.1 The number of elements Ω_e with $\overline{\Omega}_e \cap M \neq \emptyset$ is bounded by a quantity of the order $Ch_e^{-\dim(M)} = Ch^{-\dim(M)/\mu}$. On the other hand the number of elements Ω_e with $\overline{\Omega}_e \cap M = \emptyset$ is bounded by Ch^{-n} for $\mu > \frac{\dim(M)}{n}$. This can be shown by the following calculation:

$$\begin{aligned}
\sum_{e:\overline{\Omega}_e \cap M = \emptyset} 1 &= \sum_{e:\overline{\Omega}_e \cap M = \emptyset} C_e h_e^{-n} \int_{\Omega_e} 1 d\Omega \\
&\leq Ch^{-n} \sum_{e:\overline{\Omega}_e \cap M = \emptyset} r_e^{-n(1-\mu)} \int_{\Omega_e} 1 d\Omega \\
&\leq Ch^{-n} \sum_{e:\overline{\Omega}_e \cap M = \emptyset} \int_{\Omega_e} r^{-n(1-\mu)} d\Omega \\
&= Ch^{-n} \int_{\Omega} r^{-n(1-\mu)} d\Omega.
\end{aligned}$$

The integral is bounded iff $-n(1-\mu) > -n + \dim(M)$. That means, that the number of elements does not increase asymptotically in comparison with a non-refined mesh, if $\dim(M) = 0$ (that means, we have only corners, no edges) or $\mu > \frac{1}{3}$ ($\dim(M) = 1$, $n = 3$).

For $\mu \leq \frac{1}{3}$ and $\dim(M) = 1$ one gets by a similar calculation a number of elements of the order $h^{-3} \ln h$ for $\mu = \frac{1}{3}$ and $h^{-1/\mu}$ for $\mu < \frac{1}{3}$.

We discuss now possibilities for the construction of graded partitions and start with the two-dimensional case. Consider a polygonal domain Ω with a corner point O, at which a singularity occurs. Following [34] we introduce in a neighbourhood $D := \{x \in \Omega : \operatorname{dist}(x, O) < b\}$ of some radius b the N layers $d_i := \{x \in \Omega : r_{i-1} < \operatorname{dist}(x, O) \leq r_i\}$ ($i = 1, \ldots, N$) with $r_i := b(\frac{i}{N})^{1/\mu}$ ($i = 0, \ldots, N$), which are approximately partitioned into triangles of mesh size $h_i := r_i - r_{i-1}$ ($i = 1, \ldots, N$); see Figure 3.1 for $N = 3$, $\mu = 0.4$.

For such partitions one can easily calculate [2], that

$$\frac{1}{N} b^\mu r_i^{1-\mu} \leq h_i \leq \frac{1}{\mu N} b^\mu r_i^{1-\mu}, \quad i = 1, \ldots, N, \tag{3.1}$$

$$h_{i-1} \leq h_i \leq (2^{1/\mu} - 1) h_{i-1}, \quad i = 2, \ldots, N, \tag{3.2}$$

$$r_{i-1} \leq r_i \leq 2^{1/\mu} r_{i-1}, \quad i = 2, \ldots, N. \tag{3.3}$$

With N being of the order h^{-1} the desired property (e) is fulfilled. — Note that the second relation (3.2) leads to adjacent elements that are of comparable size and that condition (d) is fulfilled though $\sigma = \sigma(\mu)$ may become large when μ is small. However elements that are remote from each other are not of comparable size. For example we have $h_1 = \mathcal{O}(h_N^{1/\mu})$.

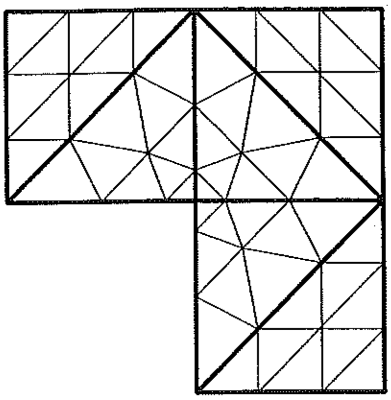


Figure 3.2: Construction of graded meshes using an initial triangulation.

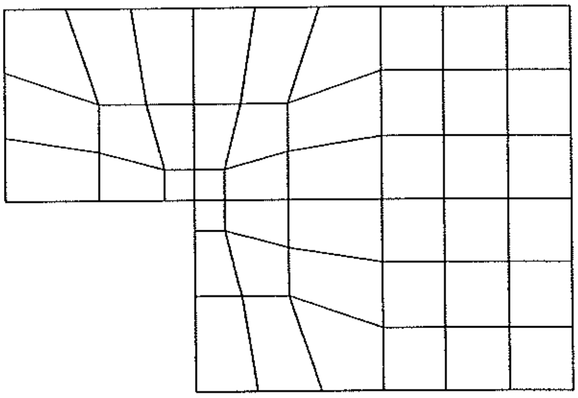


Figure 3.3: Graded meshes with quadrilaterals.

For an easier construction of the mesh outside the neighbourhood D one can approximate the arcs by polygons. According to [39] this construction can be described as follows. We consider a rough initial triangulation of Ω into elements of size $\mathcal{O}(1)$. Now each triangle is divided into N^2 elements: If the corner O is not a vertex of the triangle it is subdivided into N^2 congruent elements. For triangles near the corner we put the nodes graded towards the corner, in the sense that their barycentric coordinate $\tilde{b}$ with respect to the side opposite to the corner O is chosen as $1 - \left(\frac{i}{N}\right)^{1/\mu}$ instead of $1 - \frac{i}{N}$ $(i = 1, \ldots, N)$, see Figure 3.2 for $N = 3$. — Though the aspect ratio σ of the elements depends also on the initial triangulation, the condition (d) is always fulfilled.

The same technique can be applied for constructing meshes with quadrilateral elements, see Figure 3.3, and also for graded meshes near corners in three-dimensional domains.

For three-dimensional domains with edges it is a natural idea to reproduce the two-dimensional graded meshes constructed in a plane perpendicular to the edge. But because of $h_1 = \mathcal{O}(h_N^{1/\mu})$ this yields elongated elements, which do not fulfil property (d), see Figure 3.4. We remark that these so called anisotropic meshes have also been successfully used for the approximation of solutions of boundary value problems in domains with edges, see [3], but for error estimates the data was assumed to be smoother than that here.

In order to fulfil assumption (d) in graded meshes near edges, one can define in analogy with the two-dimensional case N layers around the edge with diameter h_i $(i = 1, \ldots, N)$, h_i as introduced above. Then one starts with filling the inner layer with tetrahedra of diameter h_1, and continues with filling the subsequent layers with tetrahedra using the boundary nodes of the previous layer. In this strategy one exploits again relation (3.2). — A more practical way of constructing the partitions seems to be

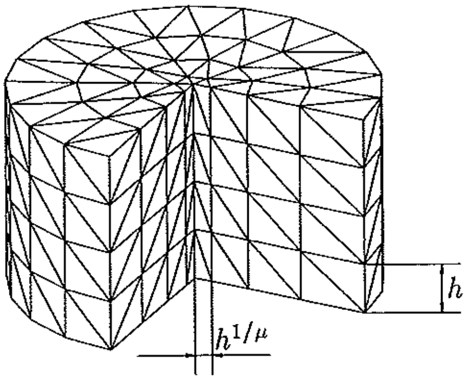


Figure 3.4: Anisotropic, graded mesh near an edge.

the following which is adopted from a posteriori mesh refinement techniques: Start with an initial triangulation ($\overline{\Omega} = \bigcup \overline{\Omega}_e$) and divide all elements Ω_e into 2^n smaller ones until (e) is fulfilled with suitable constants $\underline{C}_1$, $\overline{C}_1$, $\underline{C}_2$, and $\overline{C}_2$ (for example $\underline{C}_1 = b^{1-1/\mu}$, $\overline{C}_1 = 2\underline{C}_1$, $\underline{C}_2 = \frac{1}{\mu} b^{\mu}$, $\overline{C}_1 = 2\underline{C}_2$, compare (3.1), with some modification of $\underline{C}_2$ for elements with r_e close to b). Then divide all elements with irregular nodes in order to fulfil (c), compare Figure 4.2.

Finally we remark that it seems to be possible to construct the desired graded partition also using a mesh density function on a background mesh, compare [35, 38], but the authors have no experience with such codes yet.

3.2 Finite element methods on graded partitions

In the following we want to consider a finite-dimensional space S_h of piecewise polynomials with the following properties:

Conformity, that means $S_h \subset V$, with V given in (2.5),

Transformability, in the sense that the elements shall belong to an affine or isoparametric family,

Approximability, in the sense that $\lim_{h \to 0} \inf_{v_h \in S_h} \|u - v_h; W^{1,2}(\Omega)\| = 0$.

For $m = 1$ (differential operators of second order) these conditions are fulfilled for example by Langrangian C^0-elements. But it is difficult to construct elements for the cases $m > 1$ with these properties (see [13, Chapter 7] for $m = 2$), and many different elements with different advantages and disadvantages are in use. That is why we restrict our consideration to second order problems.

The finite element solution $P_h u$ of problem (2.5) is defined by

$$a(P_h u, v_h) = \langle f, v_h \rangle \quad \text{for all } v \in S_h. \tag{3.4}$$

Because $a(.,.)$ fulfils (2.6–2.7), Ceá's lemma implies

$$\|u - P_h u; W^{1,2}(\Omega)\| \le C \inf_{u_h \in S_h} \|u - u_h; W^{1,2}(\Omega)\|. \tag{3.5}$$

When u is smooth enough so that the interpolant $I_h u$ of u in S_h is well defined, the interpolation error in $W^{1,2}(\Omega)$ gives a bound for the finite element error. This case is studied in Subsection 3.3. Otherwise we have to consider another approximation operator. In Subsection 3.4 we apply the operator which was introduced in [45].

Lemma 3.1 *The condition number of the stiffness matrix A which is related to problem (3.4) is of order h^{-2} if $\mu > \frac{n-2}{n}$ and of order $h^{1-(1+\epsilon)/\mu}$ otherwise. These bounds are sharp.*

The upper bound for the condition number is proved for the two-dimensional case in [34] and for the case of non-intersecting edges in [4]. The proof extends easily to the more general case of polyhedrons included here and is omitted. An example that shows that the estimate is sharp is given for the three-dimensional case in [4], but it is also valid in the two-dimensional case.

Remark 3.2 In the proof of Lemma 3.1 the special geometry of the mesh (especially condition (e)) is exploited. Another approach for investigating the condition number is demonstrated in [8] by using scaled basis functions. It leads to a condition number of order h^{-2} in three dimensions but only to $h^{-2}|\ln h|$ for $\mu < 1$ in two dimensions.

The advantage consists of the applicability to a wider class of refined finite element meshes including the meshes employed here for any $\mu \in (0,1]$. Consequently, the use of scaled basis functions can be recommended for problems with edges and $\mu \leq \frac{1}{3}$ in order to improve the condition number. In other words, the scaling then works like a preconditioner.

Note, that the result presented in Lemma 3.1 is valid for the usual basis functions with a maximum norm equal to one.

3.3 Interpolation error estimates

In this subsection we consider the interpolation function $I_h u \in S_h \subset V$ by demanding

$$I_h u = u \quad \text{at all nodes of } \mathcal{T}_h. \tag{3.6}$$

That means we assume $u \in C(\overline{\Omega})$, but for the proof of the local interpolation error estimate given below we need the more restrictive condition $u \in W^{n/2+\varepsilon,2}(\Omega)$, $\varepsilon > 0$ arbitrarily small. Note that the embedding $V^{2,2}(\Omega, 1 - H_0 + \varepsilon) \hookrightarrow W^{n/2+\varepsilon,2}(\Omega)$ [27] holds, if

$$H_0 > \frac{n}{2} - 1, \tag{3.7}$$

which is a restriction for problems with edges.

The advantage of interpolation is that it produces an approximation error estimate locally in each element, and these have been studied for example in [13]. For our purposes we state only the result that for $u \in W^{2,2}(\Omega_e)$ the relation

$$\|u - I_h u; W^{1,2}(\Omega_e)\| \leq C h_e \|\nabla_2 u; L_2(\Omega_e)\| \tag{3.8}$$

holds, ∇_ℓ denotes the vector of all partial derivatives of order ℓ.

Theorem 3.2 *Let Ω be a polygonal domain in $\mathbb{R}^2$ or a polyhedral domain in $\mathbb{R}^3$ with the set M of singular boundary points. Let $\mathcal{T}_h$ be a family of graded partitions of $\overline{\Omega}$ as defined in Subsection 3.1, and H_0 the number given in Theorem 2.1 or 2.3. For $H_0 > \frac{n}{2} - 1$, the finite element error can be estimated by*

$$\|u - P_h u; W^{1,2}(\Omega)\| \leq C h^{\alpha} \|f; L_2(\Omega)\| \tag{3.9}$$

with

$$\alpha = \begin{cases} 1 & \text{for } \mu < H_0, \\ \dfrac{H_0 - \varepsilon}{\mu} & \text{for } \mu \geq H_0, \end{cases} \tag{3.10}$$

$\varepsilon > 0$ arbitrarily small.

Proof It follows from (3.5) that

$$\begin{aligned} \|u - P_h u; W^{1,2}(\Omega)\|^2 &\leq C\|u - I_h u; W^{1,2}(\Omega)\|^2 \\ &= C \sum_{\Omega_e \in \mathcal{T}_h} \|u - I_h u; W^{1,2}(\Omega_e)\|^2. \end{aligned} \tag{3.11}$$

If $\overline{\Omega}_e \cap M = \emptyset$ it follows from the interior regularity results [1] that $u \in W^{2,2}(\Omega_e)$. Using (3.8) we get

$$\begin{aligned} \|u - I_h u; W^{1,2}(\Omega_e)\|^2 &\le Ch_e^2 \|\nabla_2 u; L_2(\Omega_e)\|^2 \\ &\le Ch_e^2 r_e^{-2\beta} \int_{\Omega_e} r_e^{2\beta} |\nabla_2 u|^2 dx \\ &\le Ch_e^2 r_e^{-2\beta} \|u; V^{2,2}(\Omega_e, \beta)\|^2 \end{aligned} \quad (3.12)$$

with $\beta = 1 - H_0 + \varepsilon$. Using (e) we have

$$h_e r_e^{-\beta} \le Chr_e^{1-\mu-\beta} \le Ch \quad \text{for } 1 - \mu - \beta \ge 0, \quad (3.13)$$

that means $\mu \le H_0 - \varepsilon$. Because $\varepsilon > 0$ is arbitrarily small this condition reduces to $\mu < H_0$. For $\mu \ge H_0$ we can only estimate (using $h_e \le Cr_e$)

$$\begin{aligned} h_e^1 r_e^{-\beta} = h_e^\alpha h_e^{1-\alpha} r_e^{-\beta} &\le Ch^\alpha r_e^{\alpha(1-\mu)} r_e^{1-\alpha} r_e^{-\beta} \\ = Ch^\alpha r_e^{1-\alpha\mu-\beta} &= Ch^\alpha r_e^{H_0-\varepsilon-\alpha\mu} = Ch^\alpha \end{aligned} \quad (3.14)$$

for $\alpha = \dfrac{H_0 - \varepsilon}{\mu}$. For larger α, the term $r_e^{H_0-\varepsilon-\alpha\mu}$ becomes unbounded for r_e tending to 0.

From (3.12–3.14) we can conclude

$$\|u - I_h u; W^{1,2}(\Omega_e)\|^2 \le Ch^{2\alpha} \|u; V^{2,2}(\Omega_e, 1 - H_0 + \varepsilon)\|^2 \quad (3.15)$$

with α from (3.10).

If $\overline{\Omega}_e \cap M \ne \emptyset$ we do not have the relation $u \in W^{2,2}(\Omega_e)$. We split in the following way and estimate both terms:

$$\|u - I_h u; W^{1,2}(\Omega_e)\|^2 \le 2\|u; W^{1,2}(\Omega_e)\|^2 + 2\|I_h u; W^{1,2}(\Omega_e)\|^2. \quad (3.16)$$

Because of $r < h_e$ in Ω_e we have

$$\begin{aligned} \|u; W^{1,2}(\Omega_e)\|^2 &= \sum_{\ell=0}^{1} \|r^{-\ell+2-\beta} r^{\beta-2+\ell} \nabla_\ell u; L_2(\Omega_e)\|^2 \\ &\le \sum_{\ell=0}^{1} h_e^{2(-\ell+2-\beta)} \|r^{\beta-2+\ell} \nabla_\ell u; L_2(\Omega_e)\|^2 \\ &\le h_e^{2(1-\beta)} \sum_{\ell=0}^{1} \|r^{\beta-2+\ell} \nabla_\ell u; L_2(\Omega_e)\|^2 \\ &\le h_e^{2(1-\beta)} \|u; V^{2,2}(\Omega_e, \beta)\|^2. \end{aligned} \quad (3.17)$$

For the estimation of the norm of $I_h u$ we use the inverse inequality and on the reference element Ω_0 the embedding $V^{2m,2}(\Omega_0, \beta) \hookrightarrow C(\overline{\Omega}_0)$, which holds for $\beta \ge 0$:

$$\begin{aligned} \|I_h u; W^{1,2}(\Omega_e)\| &\le Ch_e^{-1} \|I_h u; L_2(\Omega_e)\| \\ &= Ch_e^{-1} h_e^{-n/2} \|I_h u; L_2(\Omega_0)\| \\ &\le Ch_e^{-1-n/2} \|I_h u; C(\overline{\Omega}_0)\| \\ &\le Ch_e^{-1-n/2} \|u; C(\overline{\Omega}_0)\| \\ &\le Ch_e^{-1-n/2} \|u; V^{2,2}(\Omega_0, \beta)\| \\ &\le Ch_e^{-1-n/2} h_e^{2-\beta+n/2} \|u; V^{2,2}(\Omega_e, \beta)\| \\ &\le Ch_e^{1-\beta} \|u; V^{2,2}(\Omega_e, \beta)\|. \end{aligned} \quad (3.18)$$

From (3.16–3.18), $\beta = 1 - H_0 + \varepsilon$ and $h_e = h^{1/\mu}$ we conclude

$$\|u - I_h u; W^{1,2}(\Omega_e)\| \le Ch^{(H_0-\varepsilon)/\mu} \|u; V^{2,2}(\Omega_e, 1 - H_0 + \varepsilon)\| \quad (3.19)$$

Together with (3.11) and (3.15), as well as (2.11) and (2.21) this finishes the proof. □

Remark 3.3 The restriction to polygonal/polyhedral domains is not essential in the application of graded finite element meshes for solving partial differential equations with singular solutions. Because the analytical behaviour of the solution of such boundary value problems can also be formulated in terms of weighted Sobolev spaces, it can be conjectured that similar graded partitions will also lead to optimal convergence results, because the new difficulty which arises is not due to the singularities.

The new difficulty is the treatment of the curved parts of the boundary, which can be done in a non-conforming way by placing the boundary nodes of the approximating domain Ω_h on the boundary $\partial\Omega$ (see for example [50]) or in a conforming way by approximating the part $\partial\Omega^1 \subset \partial\Omega$ with the essential boundary condition by a part $\partial\Omega_h^1 \subset \partial\Omega_h$ with $\partial\Omega_h^1 \subset \overline{\Omega}$ and approximating the part $\partial\Omega^2 \subset \partial\Omega_h$ with the natural boundary condition by a part $\partial\Omega_h^2 \not\subset \Omega$ (see for example [34]). In both cases additional error terms have to be estimated.

The conforming way which is not always applicable (for example it cannot be done for vector functions with given boundary conditions of different type in the components on some part of the boundary), is investigated for Poisson problems in three-dimensional domains with edges (without corners) for essential boundary conditions in [2] and for natural boundary conditions in [4].

3.4 Relaxation of an assumption

If the solution u from (2.5) is not contained in $W^{s,2}(\Omega)$ for some $s > \frac{n}{2}$, then the pointwise values of u are not well defined, and the interpolation operator introduced in (3.6) cannot be employed without modification. But other approximation operators can be constructed by replacing the nodal values of u by the nodal values of one or more continuous functions v (or v_i) which are close to u in some sense. Such operators were studied by different authors including [14, 34, 45, 47]. In the following we want to use the operator described by Scott and Zhang [45], and we have to restrict the consideration to simplicial elements.

In S_h we consider the nodal basis $\{\phi_j\}_{j=1}^J$ of functions $\phi_j \in S_h$ with $\phi_j(x_i) = \delta_{ij}$ $(i, j = 1, \ldots, J)$, where x_i are the nodal points of our finite element mesh, J is their number, and δ_{ij} is the Kronecker delta. Let an approximation $\Pi_h u \in S_h$ be defined by

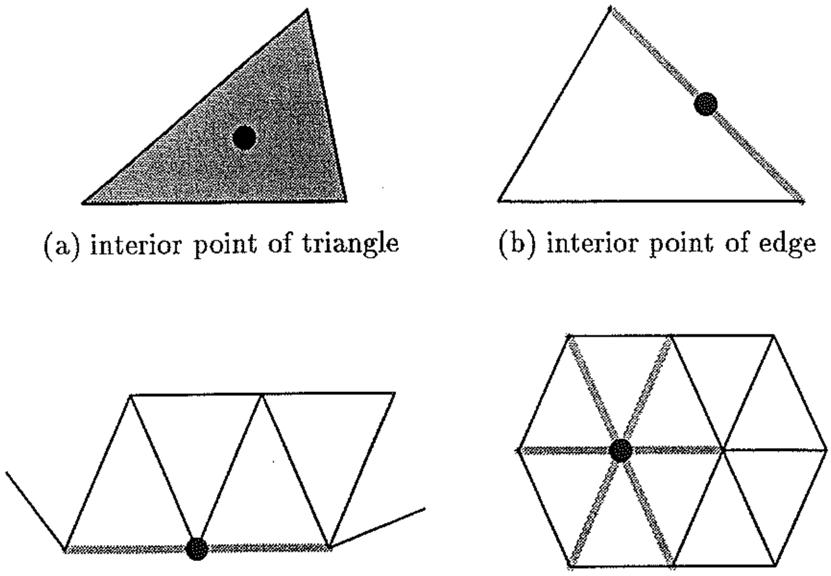
$$(\Pi_h u)(x) := \sum_{i=1}^{J} v_i(x_i) \cdot \phi_i(x), \tag{3.20}$$

where $v_i \in C(\overline{\sigma}_i)$ is the L_2-projection (see Remark 3.4) of u in $S_h|_{\sigma_i}$. The subdomains σ_i $(i = 1, \ldots, J)$ with $x_i \in \overline{\sigma}_i$ are chosen by the following rules (see also Figure 3.5):

- If x_i is an interior point of some n-simplex $\Omega_{e_0} \subset \mathcal{T}_h$ then $\sigma_i := \Omega_{e_0}$.
- Otherwise x_i is boundary point of one or more n-simplices Ω_e and σ_i is chosen as some face ς (which is a $(n-1)$-simplex) of one of these elements $\Omega_e \subset \mathcal{T}_h$:
  - If there is an ς so that x_i is an interior point of ς, then σ_i is uniquely determined by $\sigma_i := \varsigma$.
  - If not, then σ_i is taken as one of the faces with $x_i \in \overline{\varsigma}$, but with the restriction that $\sigma_i \subset \partial\Omega$ if $x_i \in \partial\Omega$.

Remark 3.4 The $L_2(\sigma_i)$-projection v_i of u in $S_h|_{\sigma_i}$ is defined by

$$\|u - v_i; L_2(\sigma_i)\| = \min_{v \in S_h|_{\sigma_i}} \|u - v; L_2(\sigma_i)\| \tag{3.21}$$



(a) interior point of triangle (b) interior point of edge

(c) vertex at boundary (2 possibilities for σ_i) (d) vertex within the domain (here: 6 possibilities for σ_i)

Figure 3.5: Choise of σ_i in dependence on x_i

and can be determined in the following way: Denote by n_0 the dimension of $S_h|_{\sigma_i}$ and let $\{\phi_{ij}\}_{j=1}^{n_0}$ be the nodal basis for σ_i with $\phi_{ij}(x_k) = \delta_{jk}$ $(j,k = 1,\ldots,n_0)$, and $\{\psi_{ij}\}_{j=1}^{n_0}$ its $L_2(\sigma_i)$-dual basis:

$$\int_{\sigma_i} \psi_{ij}(x)\phi_{ik}(x)dx = \delta_{jk} \quad (j,k = 1,\ldots,n_0).$$

Then we get as a standard property of a projection

$$v_i(x) = \sum_{j=1}^{n_0} \int_{\sigma_i} u(\xi)\psi_{ij}(\xi)d\xi \cdot \phi_{ij}(x).$$

Note that this formula simplifies for $x = x_i$:

$$v_i(x_i) = \int_{\sigma_i} u(\xi)\psi_{ii}(\xi)d\xi.$$

Note further that (though it is originally defined by (3.21) for $u \in L_2(\sigma_i)$) this approach can be extented to functions $u \in L_1(\sigma_i)$ because the polynomial functions ψ_{ij} are from $L_\infty(\sigma_i)$ so that the integral is finite. That means, the approximation operator $\Pi_h : W^{k,p}(\Omega) \to S_h$ can be defined for

$$k \geq 1 \quad \text{for } p = 1, \qquad k > \frac{1}{p} \quad \text{otherwise.} \tag{3.22}$$

The restrictions on k and p in (3.22) follow from a trace theorem and guarantee that $u|_{\sigma_i} \in L_1(\sigma_i)$ also for $(n-1)$-dimensional σ_i, but this is no restriction for our application.

Remark 3.5 The approximation operator Π_h does not only preserve homogeneous Dirichlet conditions but also inhomogeneous conditions $u = g$ on $\partial\Omega$ (at least in the sense of $L_1(\partial\Omega)$) if $g \in S_h|_{\partial\Omega}$.

Denote by $\Delta_e := \text{int}\left(\bigcup\{\overline{\Omega}_i : \overline{\Omega}_i \cap \overline{\Omega}_e \neq \emptyset, \Omega_i \in \mathcal{T}_h\}\right)$ the patch of elements around Ω_e and note that $\sigma_i \subset \Delta_e$ for all i with $x_i \in \overline{\Omega}_e$. If $k \in I\!N$ and $p \in [1,\infty]$ fulfil (3.22)

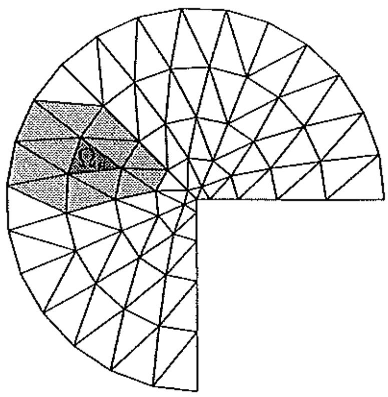


Figure 3.6: Illustration of Δ_e

then under the assumption (d) the following local approximation property holds for $u \in W^{k,p}(\Delta_e)$ [45]:

$$\|\nabla_\ell(u - \Pi_h u); L_p(\Omega_e)\| \le C h_e^{k-\ell} \|\nabla_k u; L_p(\Delta_e)\|, \quad 0 \le \ell \le k \le d+1. \tag{3.23}$$

Here, d is the polynomial degree of the shape functions and h_e is the diameter of Ω_e as introduced above.

This estimate allows for a finite element error estimate similar to Theorem 3.2 but without the restriction (3.7).

Theorem 3.3 *Theorem 3.2 holds without the assumption $H_0 > \frac{n}{2} - 1$.*

Proof The first part of the proof is similar to that of Theorem 3.2, but we consider Π_h instead of I_h:

$$\begin{aligned} \|u - P_h u; W^{1,2}(\Omega)\|^2 &\le C\|u - \Pi_h u; W^{1,2}(\Omega)\|^2 \\ &= C \sum_{\Omega_e \in \mathcal{T}_h} \|u - \Pi_h u; W^{1,2}(\Omega_e)\|^2. \end{aligned} \tag{3.24}$$

For all elements Ω_e with $\overline{\Delta}_e \cap M = \emptyset$ we can use (3.23) with $\ell = 0, 1$, $k = 1$, $p = 2$ and get

$$\begin{aligned} \|u - \Pi_h u; W^{1,2}(\Omega_e)\|^2 &\le C h_e^2 \|\nabla_2 u; L_2(\Delta_e)\|^2 \\ &\le C h_e^2 r_e^{-2\beta} \|u; V^{2,2}(\Delta_e, \beta)\|^2. \end{aligned} \tag{3.25}$$

Here we have used the fact that there is a constant C such that $r_e \le C\mathrm{dist}(\Delta_e, M)$ holds, which follows from

$$r_e \le \mathrm{dist}(\Delta_e, M) + h_{e'} = \mathrm{dist}(\Delta_e, M) + C'h(\mathrm{dist}(\Delta_e, M))^{1-\mu}$$

for sufficiently small h, see Figure 3.6 for an illustration. In the same way as in the proof of Theorem 3.2 we conclude from (3.25)

$$\|u - \Pi_h u; W^{1,2}(\Omega_e)\|^2 \le C h^{2\alpha} \|u; V^{2,2}(\Delta_e, \beta)\|^2 \tag{3.26}$$

The second part of the proof is even simpler than in the previous proof, because (3.23) holds also for $k = 1$. For elements Ω_e with $\overline{\Delta}_e \cap M \ne \emptyset$ we derive

$$\|u - \Pi_h u; W^{1,2}(\Omega_e)\| \le C\|\nabla u; L_2(\Delta_e)\|. \tag{3.27}$$

Because adjacent elements are of comparable size (see for example (3.2)), and the diameter of elements touching the set of irregular boundary points M is of the order $h^{1/\mu}$, we have $r \le Ch^{1/\mu}$ for all points in Δ_e. This leads in the same way as in (3.17) to

$$\|\nabla u; L_2(\Delta_e)\| \le C h^{(1-\beta)/\mu} \|u; V^{2,2}(\Delta_e, \beta)\| \tag{3.28}$$

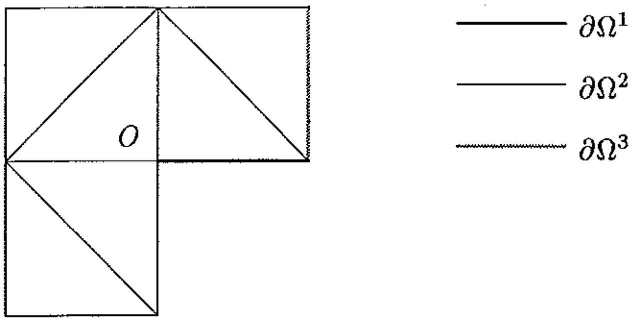


Figure 4.1: The domain of the test problem with the initial mesh.

		$\mu = 1.0$		$\mu = 0.75$		$\mu = 0.5$		$\mu = 0.4$		$\mu = 0.3$	
i	nodes	$\eta/10^3$	α	$\eta/10^3$	α	$\eta/10^3$	α	$\eta/10^3$	α	$\eta/10^3$	α
0	8	2.5321	-0.120	2.5321	-0.133	2.5321	-0.150	2.5321	-0.173	2.5321	-0.259
1	21	2.7524	0.124	2.7772	0.223	2.8093	0.321	2.8550	0.324	3.0302	0.274
2	65	2.5259	0.291	2.3787	0.404	2.2493	0.552	2.2806	0.592	2.5064	0.589
3	225	2.0650	0.342	1.7980	0.466	1.5340	0.652	1.5131	0.725	1.6663	0.771
4	833	1.6288	0.497	1.3019	0.629	0.9761	0.858	0.9157	0.937	0.9766	0.969
5	3201	1.1542		0.8417		0.5425		0.4781		0.4989	

Table 4.1: Estimated error η in the energy norm for various mesh sizes and gradings and the derived approximation order α

From (3.27) and (3.28) we conclude with $\beta = 1 - H_0 + \varepsilon$ and $\frac{H_0+\varepsilon}{\mu} \geq \alpha$, that (3.26) also holds in the case $\overline{\Delta}_e \cap M \neq \emptyset$. Due to (3.24) and the fact that only a finite number (independent of h) of patches Δ_e overlap, the theorem is proved. □

4 Test examples

4.1 Lamé system in a two-dimensional domain

We consider the Lamé system $Lu = 1$ with L from (2.14) in a two-dimensional L-shaped domain, together with boundary conditions

$$\begin{aligned} u_1 &= u_2 = 0 && \text{on } \partial\Omega^1, \\ T_1[u] &= T_2[u] = 0 && \text{on } \partial\Omega^2, \\ u_1 &= T_2[u] = 0 && \text{on } \partial\Omega^3, \end{aligned}$$

see Example 2.1 for the notation and Figure 4.1 for an illustration. Additionally we let $T_1[u]$, $T_2[u]$ be the components of the normal stress: $(T_1[u], T_2[u])^T := S[u] \cdot n$. The Lamé coefficients are those of concrete, namely $\tilde{\lambda} = 2.20$, $\tilde{\mu} = 4.27$. The boundary is chosen such that we have only one singularity of the solution in the strip $(0,1)$, namely near point O with $H_0 \approx 0.34$, see Figure 2.2. Note that the boundary condition on $\partial\Omega^3$ is typical for an axis of symmetry.

The problem was solved with mesh sizes $h_i = 2^{-i}$ ($i = 0, \ldots, 5$; for $i = 0$ see the mesh in Figure 4.1) and grading parameters $\mu = 1.0$, 0.75, 0.50, 0.40, and 0.30. The energy norm of the finite element error was estimated with an error estimator of residual type [7, 33]. The norms, together with the resulting approximation order α, are arranged in Table 4.1.

The experiment shows that the theoretical approximation order can be verified in practical calculations with realistic mesh sizes in the range of $\frac{1}{16}$ and $\frac{1}{32}$, which correspond to 833 and 3201 nodes, respectively. For $\mu > H_0$ the experimental convergence

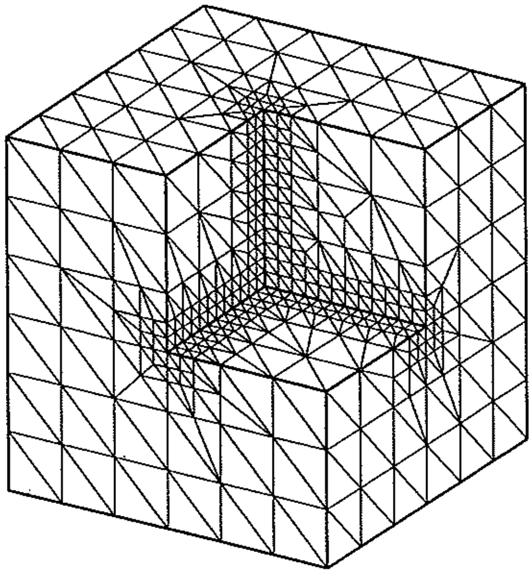


Figure 4.2: Fichera corner

order is better than the theoretically predicted one. An explanation is that the solution consists of a singular and a regular part: $u = u_s + u_r$; that means the approximation error can be estimated by

$$\|u - u_h; W^{1,2}\| \le \|u_s - u_{sh}; W^{1,2}\| + \|u_r - u_{rh}; W^{1,2}\| \le C_1 h^\alpha + C_2 h,$$

for α see Theorem 3.2. Only for sufficiently small h (depending on C_1 and C_2) will the first part of this sum dominate. It was impossible to undertake further tests with smaller mesh sizes due to the limitations of the computer which was used.

Note that for the effect of the mesh grading to be observed (based upon varying μ and a constant number of unknowns), the mesh size has to be sufficiently small. In our example, the error decreases when μ is reduced, provided that $h \le \frac{1}{8}$ and $\mu \ge 0.4$. For $\mu = 0.3$ the error is larger than with $\mu = 0.4$ but because of the higher approximation order one can assume that this effect disappears for smaller mesh sizes. Such effects have also been observed in other tests, see [2, 5].

4.2 Poisson equation in a three-dimensional domain

We consider the Poisson equation with a specific right hand side, together with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= r^{-3/2} (\ln \tfrac{r}{1000})^{-1} && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The domain $\Omega := (-1,1)^3 \setminus [0,1] \times [-1,0] \times [0,1]$ (see Figure 4.2) has three edges with interior angle $\omega_0 = \frac{3}{2}\pi$, which meet in the center of coordinates; we denote by r the distance to this point. Sometimes such a corner is called a Fichera corner and is notoriously difficult to treat. Note that the right-hand side is contained in $L_2(\Omega)$, but not in $L_p(\Omega)$ for $p > 2$.

In order to determine the regularity of the solution, we consider first the corner singularity and use Remark 2.3 and Example 2.2. The intersection of the domain with the surface of the unit ball has the area $\frac{7}{2}\pi$; a rotationally symmetric surface part with the same size has the angle $\vartheta \approx 138.6°$, which yields the lower error estimate $H_0 \ge 0.93$. On the other hand, the edge singularities are described by $H_0 = \frac{\pi}{\omega_0} = \frac{2}{3}$. That means the edge singularities dominate and determine the regularity of the solution.

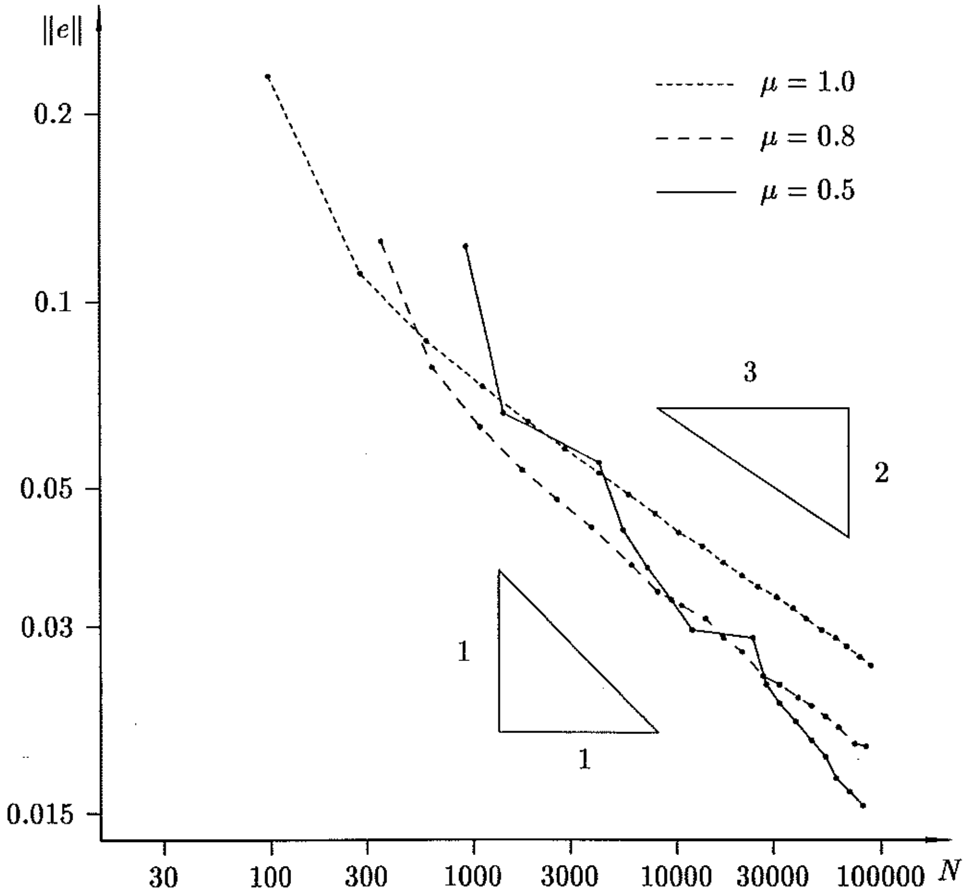


Figure 4.3: Estimated error η in the energy norm for various mesh sizes and gradings

This problem was solved first with ungraded meshes and mesh sizes $h_i = \frac{1}{i}$ $(i = 2, 3, \ldots, 24)$. Then refined meshes with grading parameters $\mu = 0.8$ and $\mu = 0.5$ were constructed using the method of successively dividing the elements until Assumption (e) on page 8 is fulfilled, see the description at the end of Subsection 3.1. Details of the algorithm and the computer program used will be published in a forthcoming paper. — The energy of the finite element error was estimated as in Subsection 4.1 with an error estimator of residual type [7, 33]. The norms are given in form of a diagram in Figure 4.3.

As in the previous example we see that the theoretical approximation order can be verified in the practical calculation. Note that the average mesh size $\bar{h}$ is about $\left(\frac{N}{7}\right)^{-1/3}$ in this example, which means that $\bar{h} = \frac{1}{24}$ corresponds to $N \approx 10^5$ nodes.

In a detailed look at the curves in the diagram we observe a rather smooth gradient for $\mu = 1$ and $\mu = 0.8$ but some exceptional points at the curve for $\mu = 0.5$. These appear when an additional (in comparison to the previous mesh) refinement step is necessary for generating the smallest elements; in these situations the number of nodes nearly doubles, but the error does not decrease by the same amount. The distribution of the nodes seems to be non-optimal in these exceptional cases. Nevertheless we can observe an average approximation order h, even when we consider only these exceptional points.

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