Appendix A

Standard Formulae

A.1 Divergence Theorem (Gauss' Theorem)

A closed region Ω is bounded by a simple closed surface Γ. If the vector field $\mathbf{F}$ and its divergence are defined throughout τ, then

$$\oint_{\Gamma} \mathbf{F}.\mathbf{d\Gamma} = \oint_{\Gamma} \mathbf{F}.\mathbf{n}\, d\Gamma = \int_{\Omega} \mathrm{div}\mathbf{F}\, d\Omega. \tag{A.1}$$

where $\mathbf{n}$ is the outward normal to the surface [14].

A.2 Green's First Theorem

Let the scalar fields ϕ and ψ, together with $\nabla^2\phi$ and $\nabla^2\psi$, be defined throughout a closed region Ω, bounded by a simple closed surface Γ. Then, Green's first theorem is that [14]

$$\oint_{\Gamma} \phi \frac{\partial\psi}{\partial n} d\Gamma = \int_{\Omega} (\phi\nabla^2\psi + \mathrm{grad}\phi.\mathrm{grad}\psi)\, d\Omega. \tag{A.2}$$

Here, $\partial/\partial n$ denotes the directional derivative along the outward normal to Γ and

$$\frac{\partial\phi}{\partial n} = \mathrm{grad}\,\phi.\mathbf{n}.$$

By defining the vector field $\mathbf{f} = \operatorname{grad}\phi$ such that $\nabla^2\phi = \nabla.\mathbf{f}$, Green's first theorem is redefined as

$$\oint_\Gamma \phi\mathbf{f}.\mathbf{n}\, d\Gamma = \int_\Omega \phi(\nabla.\mathbf{f})\, d\Omega + \int_\Omega \nabla\phi.\mathbf{f}\, d\Omega.$$

A.3 Stokes's Theorem (in the plane)

Let $\phi(x,y)$ and $\psi(x,y)$ be defined and have continuous first derivatives throughout a closed region Ω in the xy-plane. Let Ω be bound by the closed curve Γ described in the anti-clockwise sense. Then Stokes's theorem is that [14]

$$\oint_\Gamma (\phi dx + \psi dy) = \iint_\Omega \left(\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial y}\right) dx\, dy. \tag{A.3}$$

A.4 Error Function

The error function is defined as

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} \mathrm{d}\xi$$

and has the properties $\mathrm{Erf}(0) = 0$ and $\mathrm{Erf}(\infty) = 1$ [17].

A.5 Kronecker delta

The Kronecker delta is defined by [14]

$$\delta_{ij} = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}.$$

Appendix B

Shape Functions

The following shape or basis functions and their associated derivatives are defined in the local coordinates s, t and u.

B.1 Constant Strain Triangular Elements

Shape functions,

$$\begin{aligned} N_1(s,t) &= \tfrac{1+2s}{3}, \\ N_2(s,t) &= \tfrac{1-s+\sqrt{3}t}{3}, \\ N_3(s,t) &= \tfrac{1-s-\sqrt{3}t}{3}. \end{aligned} \tag{B.1}$$

Local derivatives,

$$\begin{aligned} \tfrac{\partial N_1}{\partial s} &= \tfrac{2}{3}, & \tfrac{\partial N_1}{\partial t} &= 0, \\ \tfrac{\partial N_2}{\partial s} &= -\tfrac{1}{3}, & \tfrac{\partial N_2}{\partial t} &= \tfrac{1}{\sqrt{3}}, \\ \tfrac{\partial N_3}{\partial s} &= -\tfrac{1}{3}, & \tfrac{\partial N_3}{\partial t} &= -\tfrac{1}{\sqrt{3}}. \end{aligned} \tag{B.2}$$

B.2 Bilinear Quadrilateral Elements

Shape functions,

$$\begin{aligned} N_1(s,t) &= \tfrac{1}{4}(1+s)(1+t), & N_2(s,t) &= \tfrac{1}{4}(1-s)(1+t), \\ N_3(s,t) &= \tfrac{1}{4}(1-s)(1-t), & N_4(s,t) &= \tfrac{1}{4}(1+s)(1-t). \end{aligned} \tag{B.3}$$

Local derivatives,

$$\begin{array}{rclrcl} \frac{\partial N_1}{\partial s} &=& \frac{1}{4}(1+t), & \frac{\partial N_1}{\partial t} &=& \frac{1}{4}(1+s), \\ \frac{\partial N_2}{\partial s} &=& -\frac{1}{4}(1+t), & \frac{\partial N_2}{\partial t} &=& \frac{1}{4}(1-s), \\ \frac{\partial N_3}{\partial s} &=& -\frac{1}{4}(1-t), & \frac{\partial N_3}{\partial t} &=& -\frac{1}{4}(1-s), \\ \frac{\partial N_4}{\partial s} &=& \frac{1}{4}(1-t), & \frac{\partial N_4}{\partial t} &=& -\frac{1}{4}(1+s). \end{array} \tag{B.4}$$

B.3 Linear Tetrahedral Elements

Shape functions,

$$\begin{array}{rclrcl} N_1(s,t,u) &=& \frac{1}{4}+\frac{2}{3}s-\frac{1}{3\sqrt{2}}u, & N_2(s,t,u) &=& \frac{1}{4}-\frac{1}{3}s+\frac{2\sqrt{3}}{6}t-\frac{1}{3\sqrt{2}}u, \\ N_3(s,t,u) &=& \frac{1}{4}-\frac{1}{3}s-\frac{2\sqrt{3}}{6}t-\frac{1}{3\sqrt{2}}u, & N_4(s,t,u) &=& \frac{1}{4}+\frac{1}{\sqrt{2}}u. \end{array} \tag{B.5}$$

Local derivatives,

$$\begin{array}{rclrclrcl} \frac{\partial N_1}{\partial s} &=& \frac{2}{3}, & \frac{\partial N_1}{\partial t} &=& 0, & \frac{\partial N_1}{\partial u} &=& -\frac{1}{3\sqrt{2}}, \\ \frac{\partial N_2}{\partial s} &=& -\frac{1}{3}, & \frac{\partial N_2}{\partial t} &=& \frac{2\sqrt{3}}{6}, & \frac{\partial N_2}{\partial u} &=& -\frac{1}{3\sqrt{2}}, \\ \frac{\partial N_3}{\partial s} &=& -\frac{1}{3}, & \frac{\partial N_3}{\partial t} &=& -\frac{2\sqrt{3}}{6}, & \frac{\partial N_3}{\partial u} &=& -\frac{1}{3\sqrt{2}}, \\ \frac{\partial N_4}{\partial s} &=& 0, & \frac{\partial N_4}{\partial t} &=& 0, & \frac{\partial N_4}{\partial u} &=& \frac{1}{\sqrt{2}}. \end{array} \tag{B.6}$$

B.4 Bilinear Pentahedral Elements

Shape functions,

$$\begin{array}{rclrcl} N_1(s,t,u) &=& \frac{1}{6}(1+2s)(1-u), & N_2(s,t,u) &=& \frac{1}{6}(1-s+\sqrt{3}t)(1-u), \\ N_3(s,t,u) &=& \frac{1}{6}(1-s-\sqrt{3}t)(1-u), & N_4(s,t,u) &=& \frac{1}{6}(1+2s)(1+u), \\ N_5(s,t,u) &=& \frac{1}{6}(1-s+\sqrt{3}t)(1+u), & N_6(s,t,u) &=& \frac{1}{6}(1-s-\sqrt{3}t)(1+u). \end{array} \tag{B.7}$$

Local derivatives,

$$\begin{array}{rclrclrcl} \frac{\partial N_1}{\partial s} &=& \frac{1}{3}(1-u), & \frac{\partial N_1}{\partial t} &=& 0, & \frac{\partial N_1}{\partial u} &=& -\frac{1}{6}(1+2s), \\ \frac{\partial N_2}{\partial s} &=& -\frac{1}{6}(1-u), & \frac{\partial N_2}{\partial t} &=& \frac{\sqrt{3}}{6}(1-u), & \frac{\partial N_2}{\partial u} &=& -\frac{1}{6}(1-s+\sqrt{3}t), \\ \frac{\partial N_3}{\partial s} &=& -\frac{1}{6}(1-u), & \frac{\partial N_3}{\partial t} &=& -\frac{\sqrt{3}}{6}(1-u), & \frac{\partial N_3}{\partial u} &=& -\frac{1}{6}(1-s-\sqrt{3}t), \\ \frac{\partial N_4}{\partial s} &=& \frac{1}{3}(1+u), & \frac{\partial N_4}{\partial t} &=& 0, & \frac{\partial N_4}{\partial u} &=& \frac{1}{6}(1+2s), \\ \frac{\partial N_5}{\partial s} &=& -\frac{1}{6}(1+u), & \frac{\partial N_5}{\partial t} &=& \frac{\sqrt{3}}{6}(1+u), & \frac{\partial N_5}{\partial u} &=& \frac{1}{6}(1-s+\sqrt{3}t), \\ \frac{\partial N_6}{\partial s} &=& -\frac{1}{6}(1+u), & \frac{\partial N_6}{\partial t} &=& -\frac{\sqrt{3}}{6}(1+u), & \frac{\partial N_6}{\partial u} &=& \frac{1}{6}(1-s-\sqrt{3}t), \end{array} \tag{B.8}$$

B.5 Trilinear Hexahedral Elements

Shape functions,

$$\begin{aligned}
N_1(s,t,u) &= \tfrac{1}{8}(1+s)(1+t)(1+u), & N_2(s,t,u) &= \tfrac{1}{8}(1-s)(1+t)(1+u),\\
N_3(s,t,u) &= \tfrac{1}{8}(1-s)(1-t)(1+u), & N_4(s,t,u) &= \tfrac{1}{8}(1+s)(1-t)(1+u),\\
N_5(s,t,u) &= \tfrac{1}{8}(1+s)(1+t)(1-u), & N_6(s,t,u) &= \tfrac{1}{8}(1-s)(1+t)(1-u),\\
N_7(s,t,u) &= \tfrac{1}{8}(1-s)(1-t)(1-u), & N_8(s,t,u) &= \tfrac{1}{8}(1+s)(1-t)(1-u).
\end{aligned} \tag{B.9}$$

Local derivatives,

$$\begin{aligned}
\tfrac{\partial N_1}{\partial s} &= \tfrac{1}{8}(1+t)(1+u), & \tfrac{\partial N_1}{\partial t} &= \tfrac{1}{8}(1+s)(1+u),\\
\tfrac{\partial N_2}{\partial s} &= -\tfrac{1}{8}(1+t)(1+u), & \tfrac{\partial N_2}{\partial t} &= \tfrac{1}{8}(1-s)(1+u),\\
\tfrac{\partial N_3}{\partial s} &= -\tfrac{1}{8}(1-t)(1+u), & \tfrac{\partial N_3}{\partial t} &= -\tfrac{1}{8}(1-s)(1+u),\\
\tfrac{\partial N_4}{\partial s} &= \tfrac{1}{8}(1-t)(1+u), & \tfrac{\partial N_4}{\partial t} &= -\tfrac{1}{8}(1+s)(1+u),\\
\tfrac{\partial N_5}{\partial s} &= \tfrac{1}{8}(1+t)(1-u), & \tfrac{\partial N_5}{\partial t} &= \tfrac{1}{8}(1+s)(1-u),\\
\tfrac{\partial N_6}{\partial s} &= -\tfrac{1}{8}(1+t)(1-u), & \tfrac{\partial N_6}{\partial t} &= \tfrac{1}{8}(1-s)(1-u),\\
\tfrac{\partial N_7}{\partial s} &= -\tfrac{1}{8}(1-t)(1-u), & \tfrac{\partial N_7}{\partial t} &= -\tfrac{1}{8}(1-s)(1-u),\\
\tfrac{\partial N_8}{\partial s} &= \tfrac{1}{8}(1-t)(1-u), & \tfrac{\partial N_8}{\partial t} &= -\tfrac{1}{8}(1+s)(1-u),\\
\tfrac{\partial N_1}{\partial u} &= \tfrac{1}{8}(1+s)(1+t),\\
\tfrac{\partial N_2}{\partial u} &= \tfrac{1}{8}(1-s)(1+t),\\
\tfrac{\partial N_3}{\partial u} &= \tfrac{1}{8}(1-s)(1-t),\\
\tfrac{\partial N_4}{\partial u} &= \tfrac{1}{8}(1+s)(1-t),\\
\tfrac{\partial N_5}{\partial u} &= -\tfrac{1}{8}(1+s)(1+t),\\
\tfrac{\partial N_6}{\partial u} &= -\tfrac{1}{8}(1-s)(1+t),\\
\tfrac{\partial N_7}{\partial u} &= -\tfrac{1}{8}(1-s)(1-t),\\
\tfrac{\partial N_8}{\partial u} &= -\tfrac{1}{8}(1+s)(1-t).
\end{aligned} \tag{B.10}$$

Appendix C

Local-global transformation

Assuming that x_i, y_i and z_i are the global coordinates at a local node i defined in the local (s, t, u) coordinate system, the coordinate transformation is simply described by

$$x(s,t,u) = \sum_{i=1}^{n} N_i(s,t,u)x_i,$$
$$y(s,t,u) = \sum_{i=1}^{n} N_i(s,t,u)y_i,$$
$$z(s,t,u) = \sum_{i=1}^{n} N_i(s,t,u)z_i.$$

Where n is the number of nodes associated with the element under consideration. Obviously in the two dimensional instance the z and u coordinates are neglected.

Similarly for any variable ϕ_i described at the nodes, the variation within the element can be described by the same shape functions employed above when the element is isoparametric.

$$\phi(s,t,u) = \sum_{i=1}^{n} N_i\phi_i.$$

Additionally, the partial derivatives of the variable with respect to the local coordinates can be represented as follows:

$$\frac{\partial \phi(s,t,u)}{\partial s} = \sum_{i=1}^{n} \frac{\partial N_i(s,t,u)}{\partial s}\phi_i,$$
$$\frac{\partial \phi(s,t,u)}{\partial t} = \sum_{i=1}^{n} \frac{\partial N_i(s,t,u)}{\partial t}\phi_i,$$
$$\frac{\partial \phi(s,t,u)}{\partial u} = \sum_{i=1}^{n} \frac{\partial N_i(s,t,u)}{\partial u}\phi_i.$$

To map the local derivatives to global derivatives the following standard transformation is employed:

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \\ \frac{\partial N_i}{\partial u} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \\ \frac{\partial N_i}{\partial u} \end{bmatrix}, \tag{C.1}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian matrix associated with a mesh element. As x, y and z are explicitly given by the relations C.1, the Jacobian can be written explicitly in terms of the local coordinates. Hence, the Jacobian can be defined in terms of the shape functions defining the coordinate transformation as follows:

$$\mathbf{J} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial s} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial s} y_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial s} z_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial t} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial t} y_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial t} z_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial u} x_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial u} y_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial u} z_i \end{bmatrix}. \tag{C.2}$$

Additionally in a typical FEM, to transform the variables and the region with respect to which the integration is performed involves the determinant of the Jacobian matrix. Hence, a volume element is transformed as follows:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \det \mathbf{J}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u.$$

Appendix D

Two dimensional approximations

In the following sections the elasticity matrices associated with FVM and the FEM for two dimensional approximations are are illustrated, where E is the Young's modulus and ν is the Poisson's ratio. Additionally, the differential and normal operator matrices are stated.

D.1 Plane stress

The augmented elasticity matrix for the plane stress approximation is

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{D.1}$$

The redundant row and column allows the plane stress and strain elasticity matrices to be treated similarly in computational terms.

D.2 Plane strain and Axisymmetry

The elasticity matrix for the plane strain and axisymmetric approximation is

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \tag{D.2}$$

D.3 Differential and normal operators

The general differential $\mathbf{L}$ and normal $\mathbf{R}$ operators are also augmented, as the out of plane contributions are neglected in the construction of the internal and external force terms. They are defined for the plane stress and strain approximations as follows:

$$\mathbf{R} = \begin{bmatrix} n_x & 0 \\ 0 & n_y \\ 0 & 0 \\ n_y & n_x \end{bmatrix}, \tag{D.3}$$

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}. \tag{D.4}$$

The differential operator for the axisymmetric approximation is defined as follows:

$$\mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}. \tag{D.5}$$

Appendix E

Constraint equations

The simplest use of constraint equations is the slaves to to master coupling of unknown variables, where n slave unknowns are directly equivalent to a master unknown variable.

Consider the set of linear simultaneous equations in the unknown u_j:

$$\sum_{j=1}^{L} K_{kj} u_j = f_k \qquad (1 \leq k \leq L). \tag{E.1}$$

The slave unknown u_i can be related to the master unknown as follows:

$$u_i - u_m = 0. \tag{E.2}$$

Rearranging equation E.1 as follows:

$$K_{ki} u_i + \sum_{j=1}^{L, j \neq i} K_{kj} u_j = f_k \qquad (1 \leq k \leq L) \tag{E.3}$$

and multiplying equation E.2 by K_{ki}

$$K_{ki} u_i - K_{ki} u_m = 0$$

and subtracting from equation E.3 gives

$$\sum_{j=1}^{L, j \neq i} K_{kj} u_j + K_{ki} u_m = f_k \qquad (1 \leq k \leq L).$$

For the unknown u_i explicitly

$$\sum_{j=1}^{L,j\neq i} K_{ij}u_j + K_{ii}u_m = f_i \qquad (k=i)$$

and adopting the standard Lagrange multiplier technique

$$\sum_{j=1}^{L,j\neq i} K_{kj}u_j + K_{ki}u_m - f_k + \lambda_k \left(\sum_{j=1}^{L,j\neq i} K_{ij}u_j + K_{ii}u_m - f_i \right) = 0 \tag{E.4}$$

for $1 \leq k \leq L$, where the Lagrange multiplier is

$$\lambda_k = \frac{\partial u_i}{\partial u_k}$$

and in this case

$$\begin{aligned} \lambda_k &= 1, \qquad \text{if } k = m, \\ \lambda_k &= 0, \qquad \text{if } k \neq m. \end{aligned}$$

Hence, if $k = m$

$$\begin{aligned} \sum_{j=1}^{L,j\neq i} K_{mj}u_j + K_{mi}u_m - f_m + \sum_{j=1}^{L,j\neq i} K_{ij}u_j + K_{mi}u_m - f_i &= 0, \\ \sum_{j=1}^{L,j\neq i} \left(K_{mj} + K_{ij}\right)u_j + \left(K_{mi} + K_{ii}\right)u_m &= f_m + f_i, \end{aligned}$$

which is equivalent to adding the linear equation for the slave unknown to equation for the master unknown and adding the coefficient of the slave unknown to the coefficient of the masterunknown.

If $k \neq m$

$$\sum_{j=1}^{L,j\neq i} K_{kj}u_j + K_{ki}u_m = f_k,$$

which is equivalent to adding the coefficient of the slave unknown to the coefficient of the master unknown.

Essentially, these operations are equaivalent to reducing the linear system of equations by the number of slave unknowns.