Chapter 2

Material Non-linearity

In this chapter an overview of material non-linearity with regard to solid mechanics is presented. Initially, a general description of the constitutive relationships associated with material non-linearity in solid mechanics is presented. Then a brief description of the most common cases of material non-linearity is given, with specific material examples included where appropriate. This description will involve a comparison of rate dependent and rate independent material non-linearity. The concept of a yield surface will be introduced and a number of examples will be described with regard to their applicability to particular classes of material. Finally, a detailed description of the well known Perzyna model as utilised in this research to describe an elasto-visco-plastic constitutive relationship is presented.

2.1 Classification of Material Non-linearity

Material non-linearities occur in solid mechanics when the relationship between stress and strain, otherwise known as the constitutive relationship of the material, is no longer linear. The direct proportionality of stress and strain can no longer be assumed, as it is in the simple linear elastic case.

The constitutive relationship may now be a function of the combined or individual stress, strain or strain rate and may also be path dependent with regard to the load history. The variation of the constitutive relationship also causes the stiffness of the structure or component consisting of the non-linear material to vary also. Thus the stiffness of the structure or component may vary as a function of the combined or individual load level and load history [51].

To describe a particular case of non-linear material behaviour in solid mechanics a suitable model must be adopted. Non-linear material models describe the macroscopic behaviour of the material, hence they are approximations to the real behaviour of the material as the real behaviour is also related to micro-mechanical effects within the material. For example, the plastic behaviour of metals is related to dislocations and slip planes within the crystal lattice [50, 59, 34]. These defects are assumed to be randomly distributed throughout the material such that a degree of homogeneity can be assumed by the model at a macroscopic level. This allows a uniform macroscopic approximation of the discrete microscopic behaviour of the material over a suitably large volume [50].

It is possible to classify non-linear material behaviour in solid mechanics into two categories, rate independent and rate dependent [51]. Some of the most important cases are described for each category in the following sub-sections.

2.1.1 Rate Independent Material Non-linearity

The cases of material non-linearity described under this category are assumed to be independent of time. This is an immediate approximation as all materials are dependent to some degree upon the rate at which the load is applied [105]. The rate dependence for some materials under specific loading conditions is such that it can be neglected, without reasonable loss of accuracy.

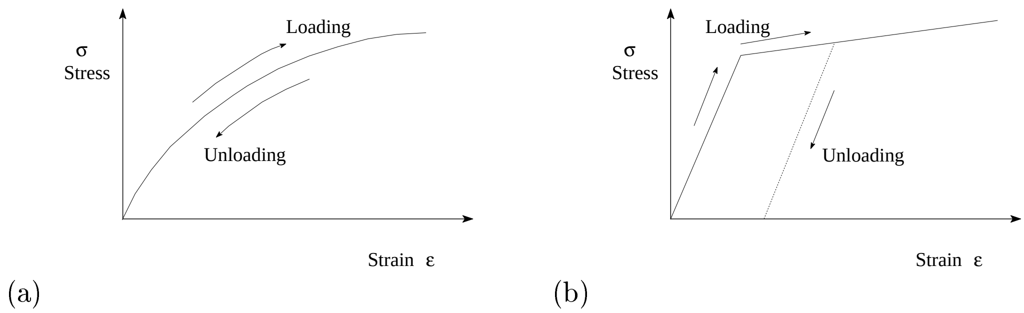

Figure 2.1: Non-linear stress-strain relationships. (a) Non-linear elasticity and (b) elasto-plasticity.

2.1.1.1 Non-linear Elasticity

A case of material non-linearity in solid mechanics for which rate independence is assumed is non-linear elastic behaviour, where the stress is not linearly related to the strain. In this case the deformation is recoverable and no energy is lost from the system. A particular case is the hyper-elastic behaviour of materials such as rubber, where the stresses are a function of a strain dependent constitutive relationship [108, 72]. A simple non-linear elastic relationship is illustrated in Figure 2.1(a), which indicates the conservative nature of the non-linear stress-strain relationship, as it follows the same path through loading and unloading.

2.1.1.2 Plasticity

Another case of material non-linearity which can be assumed to be rate independent for particular materials under specific conditions is plasticity. Plasticity describes non-linear material behaviour where the material deforms permanently due to the application of a loading condition. Some materials exhibit rigid-plastic or to be more specific almost rigid-plastic behaviour when large deformations occur, where the elastic strains are negligible when compared to the plastic strains [56]. Most engineering materials in solid mechanics exhibit elasto-plastic behaviour, in either case a transition to plastic behaviour must occur at some point. This transition occurs when the stress level in the loaded structure or

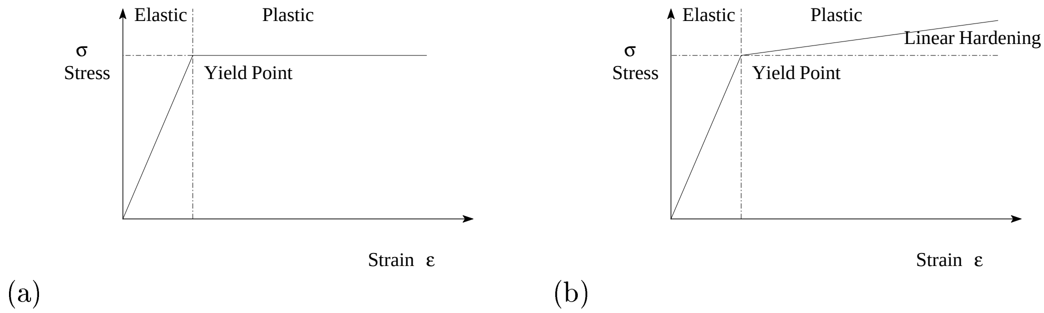

Figure 2.2: Plastic material behaviour. (a) Elastic, perfectly plastic and (b) elastic, linear work-hardening.

component exceeds the yield point stress level [72, 51]. A detailed description of yield criteria for particular classes of material will be presented in the next section.

For an elasto-plastic case the material behaves elastically below the yield point and any straining which occurs is recoverable. Typical elasto-plastic stress-strain relationships are illustrated in Figure 2.2(a) for an elastic, perfectly plastic material and in Figure 2.2(b) for an elastic, linear work-hardening material [59, 34]. Work-hardening or equivalently strain-hardening will be described in more detail in the following section.

As plastic strains are not recoverable and a problem exhibiting plastic strains is non-conservative, the problem is path dependent and the solution relies upon the load history of the problem. This is analogous to the laws governing reversible processes in classical thermodynamics, where a process is irreversible if it exhibits hysteresis [2]. This is the case for an elasto-plastic problem. When a load is applied which raises the stress level beyond the yield condition, and causes plastic deformation an initial path is followed, but when the load condition is reversed an alternative path is followed. This behaviour is illustrated in Figure 2.1(b) for an elastic, linear work-hardening material. The problem is initially loaded elastically until the yield point is reached and then deforms plastically. When the problem is unloaded it returns elastically to a permanently deformed state [108]. The problem is non-conservative as the plastic strains are associated with heat generation, for a complete thermo-mechanical analysis the heat loss must be included as a source term in the thermal analysis, which will satisfy conservation of energy. However, in many cases the total strains are small, much less than 1%, so the heat loss can be neglected in the thermal analysis without any significant loss of accuracy [27, 49].

2.1.2 Rate Dependent Material Non-linearity

The cases of material non-linearity described in this category are time-dependent. This is true for a large number of materials under specific conditions, where the rate dependency of the material can no longer be neglected. An example is the behaviour of metals at elevated temperatures [34].

2.1.2.1 Visco-elasticity

An example of material non-linearity in continuum mechanics for which rate dependence is evident is visco-elastic behaviour. This behaviour is evident in materials undergoing forming processes, where the material has a tendency to recoil over time after a deformation has been imposed. The material is associated with a shape memory, which consists of the full history of the viscous strain development. Some materials exhibit what is described as a fading shape memory when the deformation is imposed over longer periods of time. This causes the tendency to recoil to diminish resulting in permanent deformation [78].

There are a variety of forming or processing situations in which visco-elastic effects have to be modelled. Examples are flowing material processes which include polymer extrusion and melt fibre drawing [78], in these examples only the viscous terms are modelled. The inertial effects have been neglected, as a very low Reynolds number ($Re << 1$) is associated with the flowing material. When modelling material flows with a higher Reynolds number inertial terms can no longer be neglected and are included in the governing equations. Darwish et al have applied a cell-centred FVM discretization technique on staggered grids to problems involving such flows [28]. A full discussion of visco-elastic behaviour and the associated constitutive relationships is not presented in this thesis, only a brief reference is made in the context of rate dependent, elastic, material behaviour and the finite volume method. However, complete descriptions are available in the following references [108, 72].

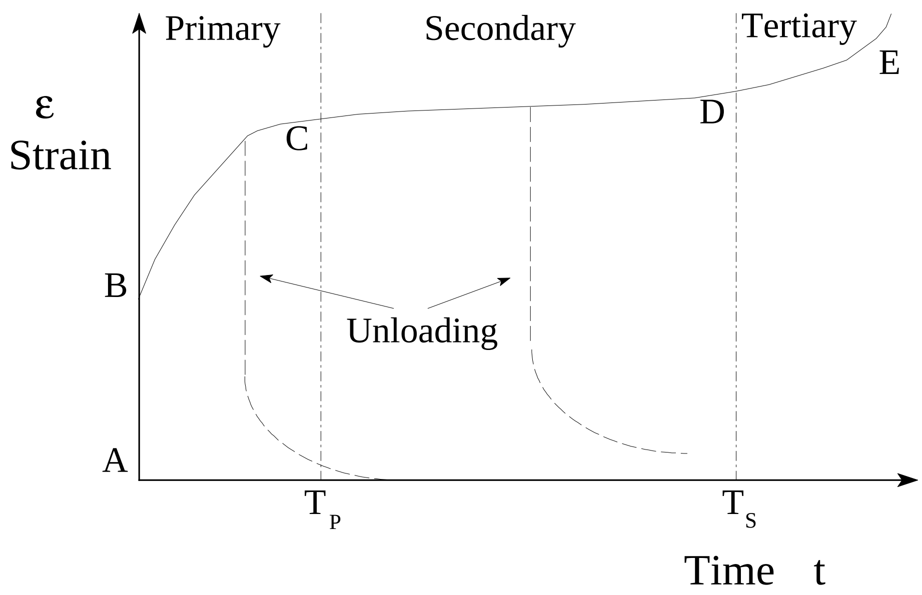

Figure 2.3: Uniaxial strain–time curve at constant stress.

2.1.2.2 Visco-plasticity

Another example of material non-linearity which includes rate dependence is visco-plastic behaviour. This behaviour always results in a permanent deformation of the material and possesses the yield criterion as described for rate independent plasticity [72, 108]. Materials exhibiting visco-plastic behaviour are assumed to be rate independent below the yield point and rate dependent when the yield point is exceeded [72, 108, 51]. In the research presented here visco-plastic behaviour is described using the Perzyna model, which will be discussed in detail in the final section of this chapter.

2.1.2.3 Creep and Stress Relaxation

Another example of material non-linearity which includes rate dependence is the phenomenon of creep. This is simply described by the strain–time relationship at constant stress as illustrated in Figure 2.3 [72, 73]. The creep strain develops after an instantaneous elastic deformation along the line $\overline{AB}$. Initially, a primary creep condition occurs during which time the strain rate is decelerating. This is a relatively short lived condition and is described by the curve $\overline{BC}$. If during this period of time the material is unloaded complete recovery will occur via an instantaneous elastic recovery, followed by a visco-elastic recovery. As indicated by the dashed unloading curve in the primary region. When the load is applied beyond the primary creep region ie. longer than time T_p, then a secondary creep condition occurs which has a constant strain rate associated with it as indicated by the line $\overline{CD}$. If the material is unloaded in this region a permanent deformation or visco-plastic effect is also included as illustrated by the dashed unloading curve in the secondary region. Finally, a tertiary creep condition occurs after time T_S, this is again a relatively short lived condition during which time the strain rate accelerates as indicated by the curve $\overline{DE}$. This condition eventually results in the failure of the material at point E on the curve. For these reasons the tertiary creep condition is usually of less interest than the primary or secondary conditions in the modelling of deformation. Nearly all materials exhibit creep to some degree, a particular case when creep phenomena cannot be neglected is metals at high temperatures, typically over 50% of their melting temperature [34].

Another rate dependent phenomenon that is associated with creep is stress relaxation. This occurs when a constant stress is applied to a material over a period of time during which the material can no longer deform and the internal stresses decrease [34].

From this description of rate dependent material non-linearity, the equivalence of secondary creep when permanent deformation occurs and visco-plasticity is indicated. This is further illustrated by reference to the theory behind the models governing creep in metals. Where the equivalence of an associated form of visco-plasticity using the von-Mises yield criterion and the widely used Norton-Soderberg creep law for metals can be derived analytically [108, 72]. From the illustrated equivalence of the two phenomena on a macroscopic scale, the similarity of the micro-mechanical behaviour governing rate dependent plasticity and creep is indicated. This is generally accepted, as experimentally it is very difficult to distinguish between the two phenomena and this has instigated the development of unified models which offer smooth transitions between different material behaviour [105, 51]. The unified modelling approach will be discussed in more detail at the end of this chapter.

2.2 Mathematical Theory of Plasticity

In this section a brief theoretical description of materials which exhibit plasticity is presented, with particular regard to metals. The nature of yield criteria will be discussed with attention focused on the von-Mises yield criterion. Finally the phenomenon of work-hardening will be described. This section provides the basic rate independent plasticity theory required for a description of the elasto-visco-plastic constitutive relationship as described using the Perzyna model in the final section of this chapter.

2.2.1 Overview of Yield Criteria

When considering the phenomena of plasticity and the associated yield criterion, the nature of the material under consideration is very important. For example, experimental tests in tension, compression and torsion on a number of metals including copper and steel have indicated that hydrostatic pressure has negligible effect on the yield point and flow stress level [59]. This is not the case when considering other classes of materials such as ceramics. For these classes of materials the yield point and flow stress level are generally dependent upon the hydrostatic pressure. For example, experimental tests in compression and tension on sandstone and marble under hydrostatic pressure indicated that materials which are brittle at atmospheric pressures deformed in a manner typical of ductile materials at high pressures [59]. Thus, when considering metals which are ductile independently of the hydrostatic pressure and hence the volumetric component of stress the yield criterion should be a function of the stress component associated with a change of shape only [50, 72, 34].

Another important consideration of the material with regard to the yield criterion is isotropy. If the material is isotropic then the yield criterion should be independent of the orientation of the coordinate system employed. Thus, the yield criterion should be an invariant function of the components of stress in the coordinate system [50, 72, 34].

There is no theoretical method of deriving a relationship between the stress components in order to correlate yielding for a three dimensional state of stress with yielding in a uniaxial test. At present yielding criteria are essentially empirical relationships. However, a yield criterion must agree with the material behaviour as observed experimentally [50, 34].

At present there are two established yield criteria for isotropic ductile metals. These are the von-Mises and Tresca yield criteria [50, 34]. These two criteria have been widely applied to problems involving metals. For most metals the von-Mises criterion is in better agreement with experimental data than the Tresca criterion [50, 72]. For this reason the von-Mises yield criterion is utilised in this research, though the techniques will apply generally to any suitable yield criterion. The von-Mises yield criterion is described in detail in the following section.

2.2.2 The von-Mises Yield Criterion

Initially, some basic concepts from the mathematical theory of plasticity as required for the definition of the yield criterion will be defined and finally the yield criterion will be defined with regard to these concepts. When describing the von-Mises yield criterion it is useful to consider a general three dimensional stress state σ_{ij} at a point in static equilibrium as described using the following Cartesian tensor notation:

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}. \tag{2.1}$$

In the mathematical theory of plasticity it is meaningful to consider direct stress relative to the mean direct stress

$$\overline{\sigma} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{\sigma ii}{3}. \tag{2.2}$$

The mean direct stress can be regarded as a hydrostatic stress which acts equally in all directions and is therefore related to volumetric changes only [34, 38].

This allows the stress to be decomposed into a deviatoric stress s_{ij} and a volumetric stress $\delta_{ij}\overline{\sigma}$, utilizing the standard Kronecker delta δ_{ij} as defined in Appendix A, such that

$$\sigma_{ij} = s_{ij} + \delta_{ij}\overline{\sigma}. \tag{2.3}$$

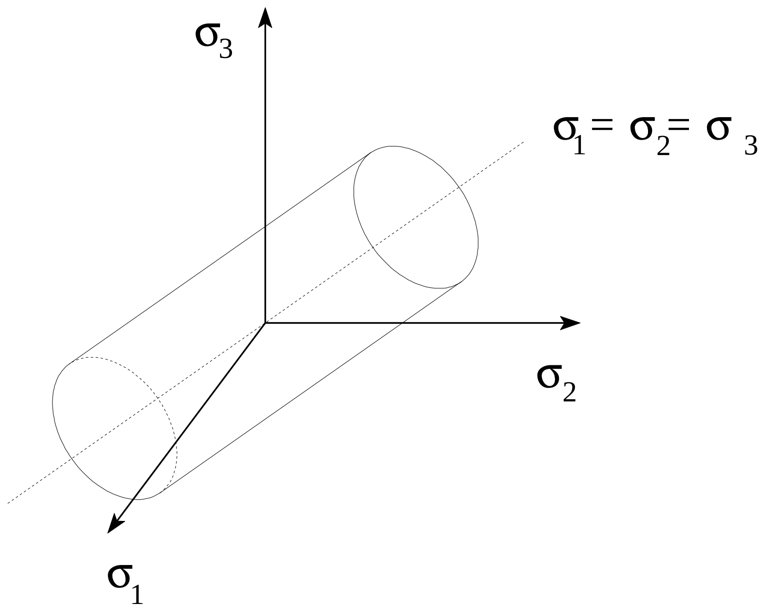

Figure 2.4: Yield surface in principal stress space.

The normal stress relative to the mean normal stress is then described by the deviatoric stress

$$\begin{aligned} s_{ij} &= \sigma_{ij} - \delta_{ij}\overline{\sigma}, \\ s_{ij} &= \begin{pmatrix} \sigma_{xx} - \overline{\sigma} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \overline{\sigma} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \overline{\sigma} \end{pmatrix}, \end{aligned} \tag{2.4}$$

which is associated with a change of shape only. As described earlier, most metals can be considered to be independent of hydrostatic pressure with regard to the yield point. Thus the yield criterion can be assumed to be dependent on the deviatoric stress only [50, 72, 34].

Any three dimensional stress state as described by the symmetrical tensors in equations (2.1) or (2.4) can be resolved quite simply to three principal stresses acting at a point [34, 38]. The principal stresses are the three roots of the cubic characteristic polynomial obtained during the resolution. The coefficients of the characteristic polynomial are invariant functions and may be expressed in terms of the stress state [38].

Thus it is possible to define the von-Mises yield criterion for isotropic metals as follows. When the second invariant of the deviatoric stress tensor J_2 reaches a critical value yielding occurs. From the considerations of this theory for uniaxial tension it is possible to define the von-Mises yield criterion in terms of the effective stress [72, 34]

$$\begin{aligned} \sigma_{eff} &= \sqrt{3}\{J_2\}^{\frac{1}{2}}, \\ \sigma_{eff} &= \sqrt{3}\{\frac{1}{2}s_{ij}s_{ij}\}^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

The yield criterion can now be defined as the point when the effective stress reaches a critical value Y. The critical value is obtained by experimental tests in uniaxial tension on the specific material. Hence, a yield function F can now be defined for the von-Mises yield criterion as follows:

$$F = \sigma_{eff} - Y. \tag{2.6}$$

The von-Mises yield criterion can be illustrated graphically when equation (2.5) is plotted in relation to the three dimensional space defined by the principal stresses. This results in a cylindrical surface of radius $Y\sqrt{2/3}$ aligned along the axis as illustrated in Figure 2.4. The axis of the cylinder is the hydrostatic component of the stress and any stress state that exists inside the cylinder remains elastic [34, 72, 50].

A number of physical interpretations have been suggested for the criterion, which was originally proposed by von-Mises in 1913 because of its mathematical simplicity. The earliest is that proposed by Hencky in 1924 which implies that yielding begins when the recoverable elastic energy of distortion reaches a critical value. The distortion energy is that part of total strain energy per unit volume which is associated with a change of shape as opposed to a change in volume [34]. A complete description of the physical interpretations available for the von-Mises criterion is available in the standard texts [50, 34, 72].

2.2.3 Strain-hardening of Materials

An important factor governing the plastic behaviour of a material is the phenomenon of strain-hardening. Also referred to as work-hardening in the associated literature [59]. In simple terms, the phenomenon occurs during the plastic deformation of metals at a micro-mechanical level due to the generation and the changing interaction between dislocations as the degree of deformation increases. Basically, the larger the number of dislocations produced, the larger their interaction and hence the larger the stresses required for the yielding of the metal [50]. Temperature is an important consideration when describing strain-hardening.

Conventionally, materials which become permanently harder during a tensile test at room temperature are said to be cold-worked. This is true for most metals though a few metals such as lead, tin and cadmium only strain-harden permanently below room temperature. If the latter metals are left at room temperature they soften over a period of time, or in other words they self-anneal [59].

When a tensile test has been performed on metals at elevated temperatures, which is conventionally referred to as hot-working, it is experimentally shown that at critical temperatures the hardening phenomenon ceases[59]. This can be attributed to the softening or strain-hardening removal processes such as recrystallization which are thermally activated and cancel out the strain-hardening process [59].

Strain-hardening will be considered in a number of solid mechanics problems in the following chapters. The simplest case is linear strain-hardening, which is actually a reasonable approximation for the cold-working of a number of metals such as the Aluminium alloy 57S [91].

Considering the uniaxial case of linear strain-hardening as described in Figure 2.2(b). It is possible to associate the gradient in the elastic region with the standard Youngs modulus E and the gradient in the plastic region with the elastic/plastic modulus H. From these moduli and the initial yield stress Y_0, it is possible to derive a linear relationship between the yield stress $Y(\epsilon_p)$ and the plastic strain ϵ_p of the form

$$Y(\epsilon_p) = Y_0 + \beta\epsilon_p \tag{2.7}$$

where the hardening function β can be algebraically derived [72, 59] as a function of the elastic and elastic/plastic moduli as follows:

$$\beta = \frac{H}{1 - H/E}. \tag{2.8}$$

It should be noted that the same relationships for the effective stress and effective strains can be obtained directly from the uniaxial experimental data [91, 72], hence enabling the relationships to be applied generally in two and three dimensions.

2.3 Elasto-visco-plasticity

In this section an elasto-visco-plastic constitutive relationship is described using the standard theory of elasticity and the Perzyna model. First, the basic linear elastic constitutive relationship is stated and the associated material properties are described. Then the Perzyna model is described and the constitutive relationship is extended to include visco-plastic and thermal effects. The description will be limited to an associated case with a von-Mises yield criterion. The applicability of the constitutive relationship to a variety of problems involving metals will be indicated.

2.3.1 Linear Elasticity

In this section the linear elastic constitutive relationship is described for a three dimensional cartesian coordinate system. The relationships for the two dimensional plane stress, plane strain and axisymmetric approximations are described as required in the remaining thesis, but the theoretical approach applies generally.

The stress state at a point was defined in equation (2.1). The strain state associated with a stress state at a point is now defined. The strain state at a point is dependent upon the variation of the displacement with regards to the x, y and z coordinates. The displacement at a point can be defined by the three displacement components u, v and w in the x, y and z coordinates, respectively. For infinitesimal strain problems, with strains typically less than 1%, the displacement variation can be assumed to be linear [38].

For metals exhibiting non-linear material behaviour with a limited amount of deformation an infinitesimal strain approximation is possible [103, 105]. In a number of metal forming processes such as rolling and stamping large deformations can occur and a small strain approximation is not suitable. Problems involving large deformations are not considered in this research, though it should be noted that the numerical approach can apply generally to such cases.

2.3.1.1 Tensor Definition

When defining the state of strain at a point it is meaningful to define the strain tensor in terms of the deformation (or displacement) tensor [59, 34]

$$e_{ij} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{2.9}$$

In general the deformation tensor is composed of a strain tensor and a rotation tensor as follows [59, 34]:

$$\begin{aligned} e_{ij} &= \epsilon_{ij} + w_{ij}, \\ e_{ij} &= \frac{1}{2}(e_{ij} + e_{ij}) + \frac{1}{2}(e_{ij} - e_{ij}). \end{aligned} \tag{2.10}$$

Thus from the tensor description the strain is a symmetric second rank tensor of the following form:

$$\epsilon_{ij} = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) & \frac{\partial v}{\partial y} & \frac{1}{2}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}) \\ \frac{1}{2}(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) & \frac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) & \frac{\partial w}{\partial z} \end{pmatrix}. \tag{2.11}$$

The strain tensor defined in equation (2.11) is of the same form as the stress tensor defined in equation (2.1). Therefore, as described by equations (2.2), (2.3) and (2.4) for stress the strain tensor can also be decomposed into a volumetric component, dilation, and a component associated with a change of shape. Also a similar triaxial state of strain can be described in terms of the principal strains, thus allowing allowing an effective strain to be defined. This is particularly useful in strain hardening as described in the previous section, when an effective plastic strain ϵ^p_{eff} is often required and can be defined in terms of the invariant of the strain tensor as follows [72, 34]:

$$\epsilon^p_{eff} = \sqrt{\frac{2}{3}}\{\epsilon^p_{ij}\epsilon^p_{ij}\}^{\frac{1}{2}}. \tag{2.12}$$

The strain tensor component of the deformation tensor is associated constitutively with the stress tensor as follows:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{2.13}$$

where C_{ijkl} is the fourth rank tensor of elastic constants [72, 34]. As the stress and strain tensors are symmetric and the material can be assumed to be isotropic and homogenous, the independent components of the tensor of elastic constants can be reduced significantly. This allows equation (2.13) to be simplified to [34]

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \tag{2.14}$$

where μ and λ are the Lamé constants, which can be defined in terms of the Youngs modulus E and the Poisson ratio ν as

$$\begin{aligned} \mu &= \frac{E}{2(1+\nu)}, \\ \lambda &= \frac{\nu E}{(1+\nu)(1-2\nu)}. \end{aligned}$$

The Lamé constant μ is equivalent to the shear modulus G, there is no direct physical equivalent for the Lamé constant λ. The constitutive relationship can be decomposed into deviatoric and hydrostatic components, respectively, [34]

$$\begin{aligned} s_{ij} &= \frac{E}{1+\nu}\epsilon_{ij}{}' = 2G\epsilon_{ij}{}', \\ \sigma_{ii} &= \frac{E}{1-2\nu}\epsilon_{kk} = 3K\epsilon_{kk}. \end{aligned}$$

where $\epsilon_{ij}{}'$ is the deviatoric strain and K is the bulk modulus.

2.3.1.2 Engineering Definition

When describing the constitutive relationship as employed in engineering problems it is common practice to dispense with tensor notation [38, 107]. The following matrix form of notation is meaningful when describing computational algorithms and will be adopted in the thesis when necessary. It is defined here as a comparison to the succinct and more mathematical tensor notation.

In this case, for an isotropic homogenous material undergoing small strains the linear elastic constitutive relationship is generally defined in a matrix form as follows:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon} \tag{2.15}$$

where the stress $\boldsymbol{\sigma}$ and strain ϵ are represented by vectors of six components for a three dimensional isotropic approximation.

$$\begin{aligned} \boldsymbol{\sigma}^T &= \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{yz} & \tau_{zx} \end{bmatrix}, \\ \boldsymbol{\epsilon}^T &= \begin{bmatrix} \epsilon_x & \epsilon_y & \epsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}, \end{aligned} \tag{2.16}$$

and the elasticity matrix $\mathbf{D}$ is defined in terms of the material properties E and ν as

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{pmatrix} \tag{2.17}$$

This is the standard matrix form of the constitutive relationship for a linear elastic material as described in the standard texts [38, 107].

At this point it should be noted that the shear strain components in the strain vector in equation (2.16) are not equivalent to the shear strains described in the tensor definition of strain in equation (2.11). The former are generally referred to as the engineering shear strains and can be defined as follows using the linear differential operator $\mathbf{L}$ and the displacement vector $\mathbf{u}$ as

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{pmatrix}. \tag{2.18}$$

The engineering shear strains are often described as the total shear deformation as opposed to the average shear deformation as described in the strain tensor in equation (2.11) [38, 59, 34].

2.3.2 Perzyna Model

The elasto-visco-plastic constitutive relationship adopted in this research is based on the original description of the Perzyna model for an associated material as proposed by Perzyna in 1963 [76, 77]. It is also equivalent to the constitutive relationship described by Zienkiewicz and Cormeau [105] which uses a modified Perzyna model to describe associative and non-associative material behaviour.

The visco-plastic strain rate tensor can be defined as follows:

$$\dot{\epsilon}_{ij}^{vp} = \gamma \left\langle \frac{F}{Y} \right\rangle^{N} \frac{\partial Q}{\partial \sigma_{ij}}, \tag{2.19}$$

where γ is the material property fluidity, the yield function F is rendered non-dimensional by the uniaxial yield value Y, the power law is obeyed by raising the dimensionless yield function to a power N and Q is the plastic potential [105]. The operator $\langle . \rangle$ is defined as follows [76]:

$$\langle . \rangle = \begin{cases} 0 & \text{when } . \leq 0 \\ . & \text{when } . > 0 \end{cases} .$$

In this research the Perzyna model is restricted to associated material behaviour where the yield function is directly equivalent to the plastic potential:

$$F \equiv Q.$$

Thus, the visco-plastic strain rate is simplified as follows:

$$\dot{\epsilon}_{ij}^{vp} = \gamma \left\langle \frac{F}{Y} \right\rangle^{N} \frac{\partial F}{\partial \sigma_{ij}}, \tag{2.20}$$

When the von-Mises yield function as defined in equation (2.6) is substituted in equation (2.20) and is differentiated with regard to the stress tensor the visco-plastic strain rate can be defined in terms of the deviatoric stress tensor as

$$\dot{\epsilon}_{ij}^{vp} = \gamma \left\langle \frac{\sigma_{\mathit{eff}}}{Y} - 1 \right\rangle^{N} \frac{3}{2\sigma_{\mathit{eff}}} s_{ij}. \tag{2.21}$$

Considering a thermo-elasto-visco-plastic constitutive relationship, the total strain rate tensor is comprised of three parts

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^{el} + \dot{\epsilon}_{ij}^{vp} + \dot{\epsilon}_{ij}^{th}, \tag{2.22}$$

which also includes the elastic strain rate $\dot{\epsilon}_{ij}^{el}$ and the thermal strain rate

$$\dot{\epsilon}_{ij}^{th} = \alpha \dot{T} \delta_{ij}, \tag{2.23}$$

where α is the linear Coefficient of Thermal Expansion and $\dot{T}$ is the rate of change of temperature. It should be noted that additional to thermal strain rates it is also possible to consider other self produced strain rates such as those associated with a material transformation, but they are neglected here for simplification.

From equation (2.22) the elastic strain rate can be defined as follows:

$$\dot{\epsilon}_{ij}^{el} = \dot{\epsilon}_{ij} - \dot{\epsilon}_{ij}^{vp} - \dot{\epsilon}_{ij}^{th} \tag{2.24}$$

and if the elastic constitutive relationship described by equation (2.14) is modified to rate form

$$\dot{\sigma}_{ij} = 2\mu \dot{\epsilon}_{ij}^{el} + \lambda \dot{\epsilon}_{kk}^{el} \delta_{ij} \tag{2.25}$$

the non-linear nature of the constitutive relationship with regard to stress is apparent.

In addition to their rate forms, it is also possible to state equations 2.24 and 2.25 in their incremental forms as follows:

$$\begin{aligned} \Delta\epsilon_{ij}^{el} &= \Delta\epsilon_{ij} - \Delta\epsilon_{ij}^{vp} - \Delta\epsilon_{ij}^{th}, \\ \Delta\sigma_{ij} &= 2\mu \Delta\epsilon_{ij}^{el} + \lambda \Delta\epsilon_{kk}^{el} \delta_{ij}. \end{aligned} \tag{2.26}$$

2.3.3 Closure

The Perzyna model, in conjunction with the von-Mises yield criterion, has provided the basis for the constitutive relationship in many applications of the Bubnov-Galerkin FEM to non-linear material problems involving metals [105, 23, 72, 86, 108].

A comprehensive description of the FE implementation is given by Zienkiewicz and Cormeau [105, 108], where a unified approach is described and adopted. This unified approach allows a variety of non-linear material phenomema to be modelled, such as elasto-plasticity, elasto-visco-plasticity and pure creep, by varying the material properties and the model parameters. As discussed in section 2.1.2.3, the Norton-Soderberg creep law for metals can be derived from the associated form of visco-plasticity described here [108, 72]. Numerically, this is achieved by assigning the uniaxial yield stress with a conveniently small value to reflect zero and N to a value in the range $4-7$ which is typical of most metals exhibiting creep behaviour [105]. Thus, the behaviour of metals over an extreme range of temperatures can be modelled.

For these reasons, the Perzyna model and the von-Mises yield criterion have been adopted in a number of FE applications to complex problems such as the shape-casting of metals, where extreme temperature conditions occur causing a variety of non-linear material behaviour [93, 13, 92].

Thus, the Perzyna model is a suitable candidate to introduce non-linear material behaviour into a FV framework to solve multi-physics problems [26, 25]. The 'node-centred' FVM and the Bubnov-Galerkin FE discretization and solution procedures to material non-linearity are theoretically compared with regard to this specific case in the following chapter.