Chapter 1

Introduction

The goal of the research project is to extend novel two and three dimensional implementations of linear elastic, small strain deformation algorithms using Finite Volume (FV) discretisation techniques [43, 42, 4], in order to model non-linear material behaviour, such as elasto-plastic and elasto-visco-plastic deformation. The novelty of the original deformation algorithms is their ease of coupling with Computational Fluid Dynamics (CFD) procedures based upon FV discretisation techniques [42, 3]. This was achieved using generically similar discretisation, formulation and solution techniques for both Computational Solid Mechanics (CSM) and CFD procedures [42, 3]. The ultimate aim of this research is the modelling of multi-physics problems, such as the shape-casting of metals, within a completely integrated numerical framework [26, 25].

A review of numerical discretisation methods for CSM problems involving material non-linearity is presented, with passing reference to the applicability of the methods discussed to CFD problems. Specifically, the various classes of Finite Volume Methods (FVM) are then described in more detail, with particular reference to the methods employed in this research project. Finally, a brief outline of the remaining chapters of the thesis is included.

1.1 Review of numerical discretisation methods

Historically, it is accepted that the broad field of continuum physics is conventionally limited by the extreme behaviour of either solid or fluid continua.

Over the last three decades the Finite Element Method (FEM) has firmly established itself as the pioneering approach for CSM, especially with regard to solid body stress analysis [107, 108, 72, 86]. Contemporarily, the FVM, which originated from Finite Difference Methods (FDM) associated with a control volume [101, 74, 52], has similarly established itself within the CFD community.

The following section describes these trends in more detail by studying the above mentioned numerical techniques separately, with particular regard to CSM problems involving material non-linearity. Additionally, the Boundary Element Method (BEM) is described as, potentially, a further alternative.

1.1.1 Finite Difference Methods (FDM)

The FDM was widely used in continuum physics well before the advent of computers, particularly in such fields as solid mechanics where applications date back to the turn of the century. Indeed Timoshenko and Goodier [94] credit the first application of the FDM to the solution of elastic problems by Runge in 1908, who applied the method to torsional problems.

Conversely, the FDM has had limited use in CSM since the advent of computers, especially with regard to problems involving non-linear materials. This was mostly attributable to the early domination of the extremely efficient FEM in this field as developed from the early 1960's onwards. At that time some interest was directed at the FDM, but for mainly linear elastic analysis of two and three dimensional continua, beam, plate and shell problems [94, 38, 45].

As described by Fenner [38], two distinct types of governing equations dominate in quasi-static CSM problems, namely, second-order harmonic and fourth-order biharmonic types, where the unknown, which is usually a displacement or stress function, is defined in terms of the relevant coordinates. With regard to stress functions, for planar and axisymmetric problems a single Airy stress function can be introduced, while for three dimensional problems the three Clerk Maxwell stress functions can be introduced [38].

The problems are discretised by the classical FDM in conjunction with a Taylor series approximation. The diagonally dominant system of algebraic equations thus formed, are solved either, directly, by suitable elimination techniques such as the tri-diagonal matrix solver or, iteratively, by techniques such as Successive Over Relaxation (SOR) schemes [38, 10, 45]. These early techniques had limited success with CSM problems compared with other methods such as the FEM. There were a number of reasons for this, the most important being the difficulty of applying these techniques to irregular geometries in a simple fashion [42, 38].

The FDM received a renewed interest when associated with a control volume. This was a major influence in the field of CFD [74, 52, 75]. The approach allowed the numerical analyst a simple interpretation of the method when applied to a physical situation. The method enforces conservation of the dependent variable over the designated control volume as described by Patankar et al [74, 52]. This was a ground breaking step in discretisation methods, as the method originally had the appearance of a FDM but employed some of the typical conventions of a FEM.

This was the initial step in the creation of a new concept of discretisation under the heading Finite Volume Methods (FVM) and early credit for the naming convention, Finite Volume Method, in the context of CFD can be attributed to Jameson [57], though the origin of the discretisation approach can be traced much earlier. McDonald [66] proposed a novel Finite Area Method, applicable to two dimensional CFD applications and even earlier, in 1967, Winslow [101] applied a novel FDM, using a nonuniform triangular mesh, to the numerical solution of magnetostatic problems. The nonuniform mesh consisted of linear triangular elements with regular topology. Winslow illustrated the equivalence of this novel FDM, which was associated with a vertex based control volume, and a standard Rayleigh-Ritz variational approach [101]. Indeed, as the Rayleigh-Ritz variational approach is equivalent to the Bubnov-Galerkin weighted residual approach, by virtue of Green's theorem [30], this is an early indication of the direct equivalence of a FVM and a FEM with regard to linear elements. This equivalence will be commented upon in more detail in the following sections.

The success of the control volume – finite difference methods in CFD is widely reported [74, 52, 75], particularly with regard to the leading commercial CFD software packages [19, 18, 41]. The inherent satisfaction of the principle of conservation and the extremely high efficiency with respect to non-linear iterative procedures on a structured mesh [74, 75] has caused the FVM to be the dominant method employed in CFD applications.

As the original development of a discretisation technique using a FDM associated with a control volume was restricted to a structured mesh, additional discretisation methods were developed associating a control volume with an unstructured mesh for the solution of CFD problems [74]. A number of researchers developed such methods initially described as control-volume-based finite-element methods [7, 6, 8, 57]. Standard CFD problems were modelled on unstructured meshes using this discretisation technique, such as conduction, convection-diffusion, laminar fluid flow and laminar forced and natural convection [83, 5]. This technique is now well established for the modelling of CFD problems and has been analysed quite extensively by Morton et al with regard to accuracy when compared to the traditional FEM for a variety of CFD problems [69, 54].

This vertex based approach to numerical discretisation as described by Patankar and Schneider for CFD applications [5, 83], provides the unstructured discretisation method as implemented by Fryer et al [43, 42, 4] for the solid body stress analysis of linear elastic materials, upon which the present research is based. The discretisation technique has also been described generally and analytically compared against the standard FEM with regard to linear elastic, structural mechanics problems [71]. The method will be fully described in the context of a FVM in the following section.

More recently, the control volume FDM has been extended to unstructured meshes for CFD problems [20] and the above mentioned CFD problems have been successfully modelled on unstructured meshes [20, 22], particularly in connection with solidification processes [20]. This cell centred technique is currently being applied to more complex fluid flow situations involving such phenomena as swirl with highly irregular mesh geometries [24].

A similar research trend has developed in the field of CSM. Initially, with Hattel et al investigating the applicability of the control volume FDM for thermo-elastic and thermo-elasto-plastic problems on structured meshes [48, 31] and lately Demirdzic et al have had some success extending these discretisation methods to unstructured meshes, though this work has so far been limited to linear elastic materials [32]

A summary of the most recent discretisation techniques associated with the FVM is presented in section 1.2.

1.1.2 Finite Element Methods (FEM)

The FEM has been applied extensively to the field of solid mechanics, since the advent of numerical computation in the early 1960's. This is historically illustrated if we focus on the modelling of small strain, non-linear material behaviour, particularly elasto-plastic and elasto-visco-plastic deformation.

Some of the earliest applications of the FEM to an elasto-plastic constitutive relationship were performed in the 1960's by Marcal and King, 1967 [65], Yamada et al, 1968 [103] and Zienkiewicz et al, 1969 [109]. During the 1970's Zienkiewicz et al applied the FEM to an elasto-visco-plastic constitutive relationship [105, 23]. This work utilised the well known Perzyna model to describe the visco-plastic strain rate [76, 77]. The Perzyna model is also adopted in this research and a detailed description is provided in the next chapter, though it should be noted that the FVM described in this work can be generally applied to other non-linear material models.

The introduction of the Finite Element (FE) discretisation technique to problems involving material non-linearity was a natural extension of the FE discretisation approach as previously applied to a linear elastic constitutive relationship. Solid mechanics problems often involve irregular geometries and the FE discretisation approach is well suited to unstructured meshes, for such reasons the FEM is well established as the ruling discretisation technique for solid mechanics problems. Detailed accounts are available in standard texts [107, 30, 38].

A great deal of study has been completed into a variety of weighted residual criteria associated with FE discretisation techniques [107, 30]. This work has compared many such criteria, the most salient being collocation, least squares and Bubnov-Galerkin. It is well known that the Bubnov-Galerkin weighted residual approach is accepted as the optimum weighted residual method within the FE community [107, 71]. The fundamental reason being the self-adjoint nature of linear elastic, small strain problems, which is inherently satisfied by the Bubnov-Galerkin residual method.

For small strain problems involving associative, non-linear material behaviour, such as the elasto-plastic or elasto-visco-plastic deformation of metals with a von-Mises or Tresca yield criteria the problem can remain self-adjoint [70]. For this reason the Bubnov-Galerkin weighted residual FEM has generally been applied to these problems [65, 103, 109, 70].

At this point it is also interesting to note the equivalence of the sub-domain collocation, weighted residual, approach and the FVM from a mathematical description of the weighting functions. However, from a conceptual view point the FVM differs as it is developed directly from the principle of conservation over an elemental volume as opposed to an abstract mathematical technique [52, 107, 71]. This point will be described in detail in the following section, but it is immediately obvious from this equivalence of the FVM and the sub-domain collocation method that symmetry is no longer enforced with regard to the choice of weighting functions, thus providing an argument for the inferiority of the FVM when applied to self-adjoint problems [106, 71]. As the self-adjoint nature of the problem depends upon the type of material non-linearity encountered with a specific problem, part of the research described here, was to establish the effect of this possible asymmetry with reference to the FVM when applied to problems involving material non-linearity.

1.1.3 Boundary Element Methods (BEM)

The Boundary Element (BE) discretisation technique, in addition to the FEM, has also been employed in a wide variety of problems in CSM [11]. However, it is generally accepted that the BEM is more suitable to specific types of problems as described below, when compared to other discretisation techniques such as the FEM or the FVM.

The BEM is generally suitable for problems involving homogeneous and linear elastic materials, requiring a high accuracy of boundary stresses and a low ratio of boundary surface to volume with regard to problem geometry [11]. Alternatively, the FEM or potentially the FVM are more suitable for problems where the material is non-homogeneous and exhibits non-linear material behaviour, where boundary stresses are not of primary importance and where geometrically there exists a high ratio of boundary surface to volume [11]. As the object of this research is to model problems such as the shape-casting and quenching of metals, which exhibit material non-linearity and where, geometrically, there exists a high ratio of boundary surface to volume, the FVM presents itself, potentially, as a more suitable discretisation technique when compared to the BEM, for the applications studied in this research.

Though it is generally accepted that the BEM is not directly suitable for problems involving material non-linearity, Heinlein et al have successfully applied the BEM to the complete analysis of temperature fields and stresses during solidification processes [49]. Initially, this implementation was restricted to a one dimensional approach, but further work was suggested to extend the techniques to a two dimensional model.

Difficulties will arise in applying the BEM generally, as the method essentially involves the application of the analytically obtained fundamental solution as a weighting function in the formulation of the overall system of equations [104, 49]. Obviously, for non-linear problems involving two and three dimensions, the practicality of this method is severely limited.

In this research, it was not possible to compare and analyse the BEM in detail with other discretisation techniques, but in closure it should be noted that a number of researchers have investigated the coupling of the BEM with the FEM at an iterative level for problems involving material non-linearity [11]. Additionally, with particular regard to extrusion or forming processes, it is possible to adopt a staggered or stepped coupling approach where in this instance the momentum and constitutive equations are solved by different discretisation techniques, while the velocity field may be computed using the BEM, the stress field may be computed by the FEM or any other potentially suitable discretisation technique [78].

1.2 Finite Volume Methods

As originally stated by Hirsch [52], the FVM is the name given to the technique by which the integral formulation of the conservation laws are discretised directly in physical space. This definition illustrates the conceptual approach to a physical problem which is particular to the FV discretisation technique.

In the previous section, though the FVM was not explicitly described passing reference was made when necessary. From these references it should be noted that the FVM can be viewed in two ways, it may be considered a FDM associated with the conservation principle applied over a control volume or as the sub-domain collocation technique as developed from the standard FEM discretisation approach. As the importance and the application of the FVM has increased it has emerged as a discretisation technique in its own right, this emergence has been formally described by a number of authors Hirsch [52], Oñate et al [71, 54, 106] and Selim [84] to mention but a few.

In the following section the FV discretisation technique is examined in detail. The FVM has been developed recently, with regard to other discretisation techniques, and for some applications it is still under development as indicated by the research presented here. For these reasons the FVM is not yet as rigorously or formally defined as say the FEM. In this section a current overview of the FVM is provided, which attempts to expand on the naming conventions as used by many authors. The classification is independent of which particular field of continuum physics the FVM discretisation technique is applied to, though specific reference will be made to how the techniques have been applied within the fields of CSM or CFD when appropriate.

The FVM is generally divided under two headings, the cell-centred and the cell-vertex. It is possible to describe all cases of the FVM within either of these two categories. Firstly this section describes the cell-centred FVM, then secondly the cell-vertex FVM and finally concentrates specifically on the Control Volume – Unstructured Mesh (CV-UM) vertex based FVM [43, 42, 4], where comparisons are made with the Bubnov-Galerkin FEM.

1.2.1 Cell-centred FVM

The cell-centred FVM is traditionally associated with CFD discretisation techniques. In these cases suitable values of the dependent variable are stored at the cell centres, the control volume over which the conservation principle is applied is usually over the mesh cell and no overlapping of the control volumes can occur. A definition of the cell-centred FVM has been described by Hirsch [52], which simply states:

> When, for instance, the average value of the variable over the cell is associated with the central point of the cell, a *cell-centred* FVM is defined.

The implementation of a cell-centred FV discretisation technique on a two dimensional structured mesh is illustrated in Figure 1.1, where the control volume over a mesh cell $\overline{ABCD}$ is designated by $\Omega_{i,j}$. In this case a simple structure is illustrated by the orientation of the neighbouring control volumes, which for a simple two dimensional case involve the subscripts i and j for the rows and columns, respectively. The concept may be simply extended to three dimensions. Additionally, more complex structured meshes are possible employing variable cell sizes and, alternatively, curvilinear coordinate systems. However, the topology remains consistent throughout the mesh and generally structured meshes are employed.

Over the last few years a considerable amount of research has been performed by Hattel et al [48, 47, 46] to model thermo-mechanical effects in casting processes using the cell-

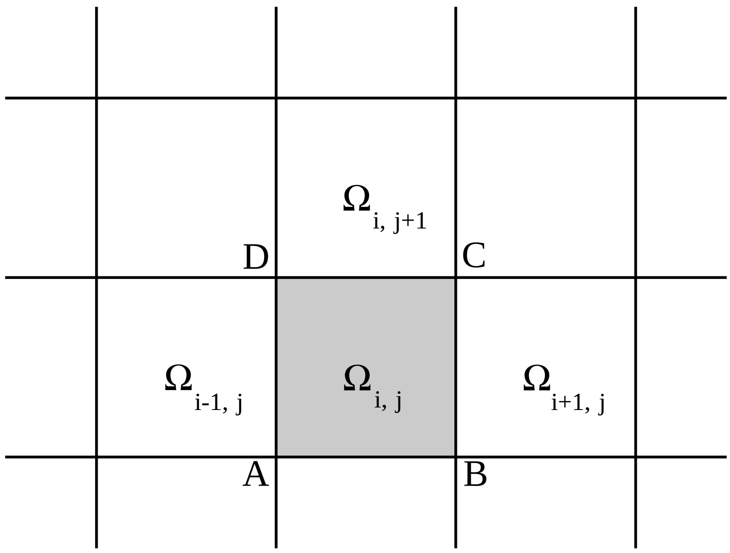

Figure 1.1: Cell-centred FVM applied to a structured mesh

centred FVM. When employing a cell-centred technique a decoupling phenomenum can occur between the displacement and stress fields. A more detailed discussion of this phenomenum is provided in Chapter 3. However, a brief outline of the techniques employed to address this problem is included here. In this particular cell-centred approach a novel technique is utilised to overcome this problem, where a collection of staggered grids, and hence control volumes over which the conservation principle is applied, are associated with each dependent variable, in this case displacement components [48, 47, 46]. Though this technique is commonly incorporated in the field of CFD [74, 19] it is new to the field of CSM. This research has had initial success in stress analysis involving thermal and mechanical loading conditions, but is so far limited to linear elastic material behaviour on structured meshes [48, 47, 46].

Additionally, Ivankovic et al have modelled the thermo-mechanical effects associated with the Rapid Crack Propagation (RCP) in polymer pipes using a cell-centred FVM. This implementation does not utilise a staggered approach as described above and stores all variables at the cell centres. A higher order term is included in the displacement gradient approximation, which requires next nearest neighbour information [31, 55, 35]. However, this higher order scheme will suffer from the usual difficulties when applied to an unstructured mesh. The implementation has included non-linear material effects, but is again restricted to a structured mesh [31, 55, 35].

A further difficulty which arises when using a cell-centred scheme is obtaining the required accuracy of the variables, such as displacement or stress, at the boundary of the problem domain. This requires suitably accurate interpolation and extrapolation of the variable from the cell faces to the cell centres and vice versa [46, 31]. A more detailed discussion of these problems is provided in Chapter 3.

At present, the cell-centred FV technique as described above has been mainly applied to a structured mesh, but recently the technique has been extended to unstructured meshes for CFD applications. These applications include thermally convective and conductive solidification processes by Chow et al [20, 22] and complex swirling flows by Croft et al [24]. The technique has also been extended to unstructured meshes for CSM applications, these include linear elastic thermo-mechanical behaviour by Demirdzic et al [32].

An example of the cell-centred FVM applied to a two dimensional unstructured mesh is illustrated in Figure 1.2(a). The control volume Ω_1 is described over the mesh cell $\overline{ABC}$ and similarly the control volume Ω_2 is described over the mesh cell $\overline{ACDE}$. The concept may again be simply extended to three dimensions, but is described here in two dimensions for simplicity. From the numbering of the surrounding control volumes it is self evident that there is no orientation or structure to the mesh as previously described.

1.2.2 Cell-vertex FVM

The cell-vertex FVM describes techniques as originally applied to unstructured meshes, where variables are typically stored at the vertices of the mesh cell and the control volume over which the principle of conservation is applied is vertex based and may include a variety of contributions from surrounding mesh elements. A variety of cell-vertex techniques as described by Hirsch [52] may be utilised so long as they meet the defined criterion:

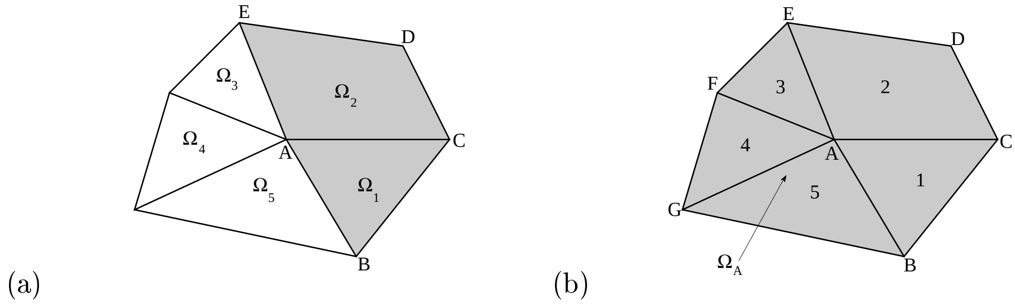

Figure 1.2: FVM applied to an unstructured mesh. (a) Cell-centred and (b) Cell-vertex.

When the variables are attached to the mesh points, that is, to the cell vertices, a *cell-vertex* FVM is defined.

An illustrative example of the cell-vertex FVM applied to a two dimensional unstructured mesh is given by Figure 1.2(b). The control volume Ω_A is based around the vertex at point A and includes contributions from the five surrounding mesh cells. The complete control volume over which the conservation principle is applied is described by the polygon $\overline{BCDEFG}$.

This method has been extensively applied in the field of CFD for a variety of problems as described earlier in the previous section under the heading of control-volume finite-element methods, and has been rigorously compared with traditional FEM discretisation techniques with regard to order of accuracy for standard CFD problems [69]. Additionally, Chow et al have compared a cell-vertex and a cell-centred FVM when applied to thermally conductive solidification processes [20, 22].

In closure of this sub-section it should be noted that a number of researchers have described various implementations of the cell-vertex FVM on a structured mesh for CFD applications. These include a general overview of current techniques by Hirsch [52], and specific techniques described by Denton [33] and McDonald [66].

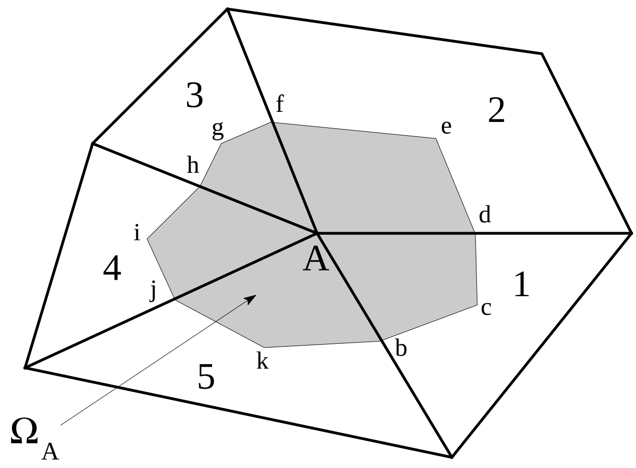

Figure 1.3: CV-UM vertex based FVM applied to an unstructured mesh

More recently, the cell-vertex method has been applied to problems concerning CSM. The previous work upon which the present research is based involved a CV-UM vertex based FVM in the analysis of two and three dimensional linear elastic problems [43, 42, 4]. Also, Oñate et al have broadly investigated the accuracy of the cell-vertex FVM when compared against the Bubnov-Galerkin FEM for standard CSM problems in one and two dimensions, again involving linear elastic material behaviour [71, 106]. The conclusions of these comparisons are discussed in the following section.

1.2.3 Control Volume-Unstructured Mesh vertex based FVM

At this point, the CV-UM vertex based FVM is introduced, with reference to the specific characteristics of the method within the cell-vertex category. Then the method is compared with the standard Bubnov-Galerkin FEM to identify the fundamental differences of the two discretisation techniques.

A comprehensive definition of the above two categories of the FVM has been compiled by Hirsch [52]. In this definition, the CV-UM vertex based FVM is described as being a particular case of the cell-vertex FVM. Other authors such as Oñate et al [71, 106] and Selim [84] describe this particular FVM separately from the general cell-vertex method, thus proposing a third category where the dependent variable is located at the vertices and the control volumes are centred around the vertices, but the control volumes do not overlap.

This suggests that the CV-UM vertex based FVM is unique from cell-vertex methods as it does not allow overlapping control volumes, where as cell-vertex methods always have overlapping control volumes over which the conservation principle is applied. Examples of overlapping control volumes with regard to cell-vertex schemes have been described by Oñate et al [71, 106] in the context of CSM and Hirsch [52] in the context of CFD.

In this research, the CV-UM vertex based scheme is regarded as a particular case of the cell-vertex FVM. This may be justified by the fact that all the key attributes of the scheme satisfy the above description, as originally defined by Hirsch [52], for the cell-vertex FVM.

The CV-UM vertex based FVM is illustrated in Figure 1.3 with regard to a two dimensional mesh for simplicity, though the concept will apply generally to a three dimensional unstructured mesh. The complete control volume over which the conservation principle is applied is circumscribed by the polygon $\overline{bcdefghijk}$. The polygon is defined by the mid-points of the mesh cell sides b, d, f, h, j and the centres of the mesh cells c, e, g, i, k. The control volume is based around a vertex or node, in this case A, and has contributions from the five surrounding elements.

When comparing the FV discretisation technique with the Bubnov-Galerkin FE technique, it is possible to illustrate the fundamental differences between the two techniques by describing each technique with regard to the associated weighting function W.

The weighting functions for a cell-vertex FVM, a Bubnov-Galerkin FEM and a CV-UM vertex based FVM are illustrated in Figure 1.4. The two dimensional mesh in the $x - y$ plane is drawn inclined to the plane of the page to illustrate in three dimensions the variation

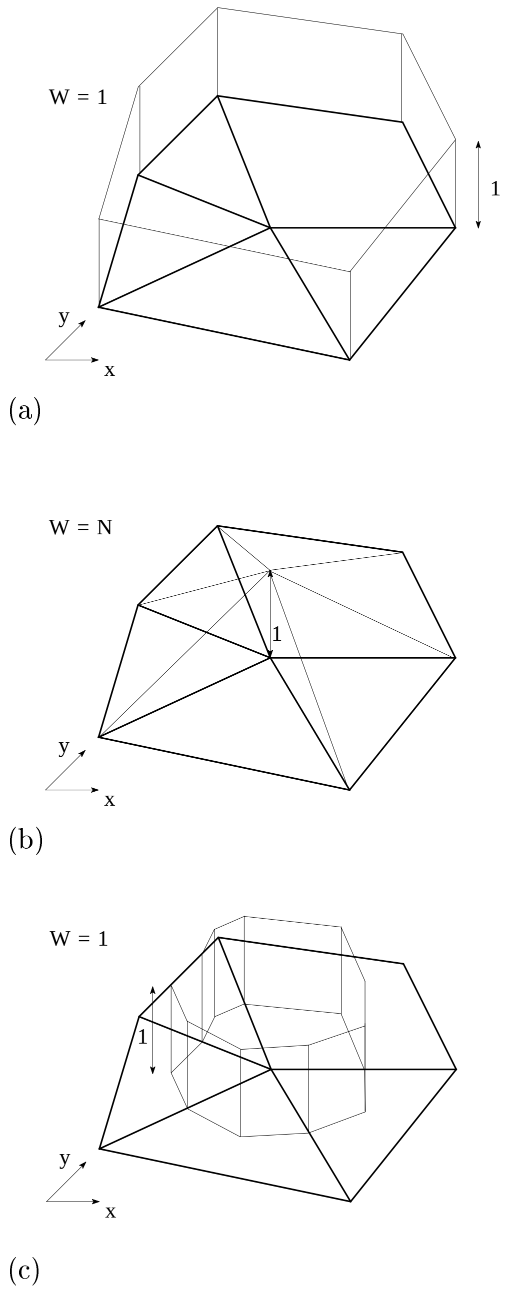

Figure 1.4: Weighting function W. (a) Cell-vertex FVM and (b) Bubnov-Galerkin FEM and (c) CV-UM vertex based FVM.

of the weighting functions over the mesh.

From the description of the weighting functions as utilised in the FV techniques illustrated in Figure 1.4(a) and Figure 1.4(c), the essential equivalence of the sub-domain collocation technique as derived from FE theory and the FV technique is now clearly apparent. This equivalence was initially mentioned in the previous section and has been noted by a number of authors [52, 71]. It should be noted that the existence of a number of possible alternatives for the FVM as described in this section, warrants an independent analysis of the FVM as a discretisation technique in its own right.

Oñate et al have originally analysed the techniques described in Figure 1.4 for one and two dimensional linear elastic problems. In the results they presented, the CV-UM vertex based FVM was described as a 'cell-centred' FVM, this description can be misleading as the term cell-centred FVM generally refers to the other category of the FVM, where variables are stored at the centre of the mesh cells, as defined above by Hirsch [52]. To avoid confusion, the 'cell-centred' FVM technique as described by Oñate et al will be referred to as a CV-UM vertex based scheme in this discussion, as the two techniques are exactly the same [71, 106].

The results presented by Oñate et al [71] indicate the superiority of the CV-UM vertex based FVM described in Figure 1.4(c) to the cell-vertex FVM as described in Figure 1.4(a). Also, the complete equivalence of the CV-UM vertex based FVM and the Bubnov-Galerkin FEM for one dimensional problems and two dimensional problems involving linear Constant Strain Triangular (CST) elements, in static elastic analysis is indicated. For higher order elements in two dimensions such as Bilinear Quadrilateral (BLQ) elements, the CV-UM vertex based FVM and the Bubnov-Galerkin FEM are not exactly equivalent, but this inequality is within an acceptable numerical tolerance and does not immediately indicate the superiority or inferiority of one method compared to the other [71].

For the cell-vertex FVM and the Bubnov-Galerkin FEM techniques as described in Figure 1.4(a) and Figure 1.4(b), respectively, the overlapping of the control volumes based around the vertices of the mesh is indicated. There are no overlapping control volumes in the CV-UM vertex based FVM and the prescribed control volumes obviously enforce conservation at a more local level than the other two methods, as illustrated in Figure 1.4(c). From this conceptual viewpoint it is possible to interpret the greater accuracy of the CV-UM vertex based FVM when compared against alternative cell-vertex FVM.

Fryer et al [43, 42] have compared the two methods for a number of standard two dimensional linear elastic problems with a variety of thermal and mechanical loading conditions on meshes consisting of BLQ elements. From these results the equivalence of the CV-UM vertex based FVM and the Bubnov-Galerkin FEM with regard to solution accuracy is indicated.

Bailey and Cross [4] have extended this work to three dimensions and have compared the two methods when applied to linear elastic problems involving thermal and mechanical load conditions on meshes consisting of Trilinear Hexahedral (TLH) elements. Again, the equivalence of the two techniques with regard to solution accuracy is indicated.

1.3 Overview of the thesis

In this section a brief overview of the remaining thesis is given. From this outline a general understanding of the direction and content of the research undertaken in this project is available.

In Chapter two, material non-linearity is described generally within the context of solid mechanics. The background theory to an elasto-visco-plastic constitutive relationship is presented, followed by a specific description of the Perzyna model as utilised in the research presented here.

In Chapter three, the governing and constitutive equations associated with material non-linearity are described, with specific regard to the elasto-visco-plastic constitutive relationship as described by the Perzyna model. The equations will then be discretised using the CV-UM vertex based FVM, and compared to a standard Bubnov-Galerkin FEM. Additionally, the possible iterative techniques available for the solution of the non-linear problem are discussed. The techniques are described as possible algorithms within a FORTRAN 77 software framework.

In Chapter four, the CV-UM vertex based FVM is theoretically analysed and compared with the Bubnov-Galerkin FEM. The direct equivalences and the basic differences of the two techniques are described and discussed at an elemental level, in two and three dimensions.

In Chapter five, the discretisation and solution techniques described in chapter three are applied to a variety of non-linear material problems in the field of CSM. These applications include a simple uniaxial problem involving strain hardening, for which an analytical solution is available. Then a pressurized thick cylinder exhibiting an ideal plastic behaviour is modelled with a plane strain approximation assumed, a reference solution is available. A perforated tensile strip with strain hardening is also modelled with a plane stress approximation, for which experimental results are available. Finally, a fully three dimensional analysis of a hollow spherical vessel undergoing internal pressure is performed, an analytical solution is available.

In Chapter six, a general discussion of the coupling of heat transfer and non-linear solid mechanical problems is presented, with reference to the merits of a variety of coupling techniques. A specific treatment of coupled problems within a FV framework will be described, as implemented and utilised in the modelling of an infinite steel plate in two and three dimensions. The implementation of constraint boundary conditions as required for this problem is also described. Finally an infinite steel slab undergoing solidification by heat conduction only is modelled. Classical analytical solutions to these thermo-mechanical problems are available from Weiner and Boley [98], who used the well known thermal analysis of Carlsaw and Jaeger [17].

In Chapter seven, the main physical processes associated with the shape-casting of metals are described. A complete description of the modelling approach will be given. Including a discussion of the required internal and external boundary conditions. Realistic, complex geometries will be modelled in three dimensions, including a 3D test bar problem.

In Chapter eight, the conclusions and suggestions for future work relating to the research presented in this thesis will be given.