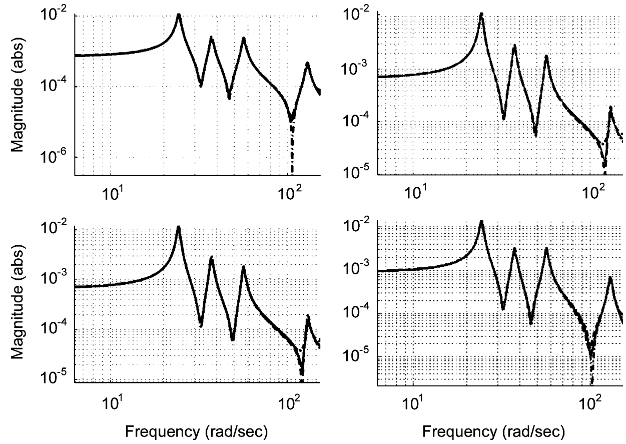

Fig. 1. Reduction of deformable mirror model. solid—original model, dashed—Hankel model reduction, dashed—optimization based method (QCO).

have close zero frequency response. The model is truncated to 20 states by means of the described quasi-convex optimization technique (QCO method), Hankel model reduction.

We implement QCO method on the frequency grid with 84 samples with tolerance in bisection procedure 10^{-6}. The optimization together with calculating frequency samples took 74 seconds and the resulting approximation error is $2.9 \cdot 10^{-5}$. Hankel model reduction took around 20 minutes providing the error $7.98 \cdot 10^{-5}$. Results, see in the Fig. 1. For the given frequency interval QCO provided a better model than Hankel reduction. However, in general we do not expect QCO approximations to be better than Hankel reduction approximations. This example shows, that for large/medium scale systems we win sufficiently in time and do not really lose in approximation quality.

VII. Conclusion

In this technical note we have discussed multi-input-multi-output extension of [5], where convex optimization is used to search for low order models. We have shown that the same approximation gap bound for MIMO extension stands as in SISO methods. The method can be very useful for large scale systems, since it is rational fit algorithm with stability guarantee and relatively low computational complexity.

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Set-Membership Filtering for Discrete-Time Systems With Nonlinear Equality Constraints

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***Abstract*—In this technical note, the problem of set-membership filtering is considered for discrete-time systems with nonlinear equality constraint between their state variables. The nonlinear equality constraint is first linearized and transformed into a state linear equality constraint with two uncertain quantities related to linearizing truncation error and base point error. S-procedure method is then applied to merge all inequalities into one inequality and the solution to the unconstrained set-membership filtering problem is provided. The set-membership filter with state constraint is finally derived from projecting the unconstrained set-membership filter onto the constrained surface by using Finsler's Lemma. A time-varying linear matrix inequality optimization based approach is proposed to design the set-membership filter with nonlinear equality constraint. A recursive algorithm is developed for computing the state estimate ellipsoid that guarantees to contain the true state. An illustrative example is provided to demonstrate the effectiveness of the proposed set-membership filtering with nonlinear equality constraint.**

***Index Terms*—Nonlinear equality constraint, state constraint, state estimate ellipsoid, set-membership filtering, time-varying linear matrix inequality (LMI) optimization based approach.**

I. Introduction

Filtering technique has been playing an important role in target tracking, image processing, signal processing and control engineering [2]. Most filtering approaches require the system noises including process noise and measurement noise in a stochastic framework and then provide a *probabilistic state estimation* [31], [33]–[35]. The probabilistic nature of the estimates leads to the use of mean and variance to describe the state spreads (distributions). These spreads

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cannot guarantee that the state is included in some region, because they are not hard bounds. However, in many real-world applications, such as, target tracking and attack, system guidance and navigation, they need 100% confidence to be estimated [15]. This has motivated to develop an *ellipsoidal state estimation* [10]. The idea of the ellipsoidal state estimation is to provide a set of state estimates in state space which always contains the true state of the system by assuming *hard bounds* instead of stochastic descriptions on the system noises [3], [22]. The actual estimate is a set in state space rather than a single vector. These methods are therefore known as set-membership or set-valued state estimation (filtering) [3], [14], [22], [30]. We prefer to adopt the name set-membership filtering in this technical note as it is easy to distinguish between a set estimation and a point estimate in the stochastic framework.

Set-membership filtering (ellipsoidal state estimation) is more suited than probabilistic state estimation to be applied in the following two cases. The first case is for the systems in which the bounds on system inputs and observation errors are known. This case exists in many systems. For example, for a vehicle tracking system, we always know its maximum acceleration although we do not know exactly how much it is when the vehicle is running. A bound can be applied to the acceleration as the bound on system inputs. Moreover, observation errors can be also viewed as belonging to some bound due to quantization errors and measurement errors. The second case is for the filtering performance requirement which is used to check whether the future state, subject to uncertainty, can definitely be brought into a specified desirable region. There are numerous potential applications for this. In order to avoid an obstacle, for example, a robot must make sure of where it is by estimating its position in presence of modelling and observation errors. Another example is that a vehicle is required 100% confidence not to enter into the collision area. Due to its practical significance, the problem of set-membership filtering has been extensively studied(see, for instance, [6], [8], [10], [13], [17]–[20], [23] and the references therein).

In addition to the filtering performance requirement, some physical systems possess an additional equality constraint *between some state variables*. For example, in vehicle tracking, the equality constraint can arise from the vehicle position when the vehicle is travelling on a known road (straight line or curve). Such tracking problem can be regarded as a filtering problem incorporating a state constraint with the road network information from digital maps [11], [24]. The filtering problems with state constraints have been studied within the Kalman filter framework. There have been several approaches to address this problem, which can be classified into augmented measurement and projection approaches. The augmented measurement approach is to treat the state constraints as additional fictitious or pseudo measurements in perfect forms (i.e., no measurement noise) [1], [5], [28]. This approach is simple and intuitive, but the incorporation of state constraints as perfect measurements brings the possibility of numerical problems and increases the dimensionality of the problem [26]. The projection approach is first to obtain an unconstrained Kalman filter solution and then project the unconstrained state estimate onto the constrained surface [9], [25], [26], [32]. The approach overcomes the numerical and dimensional problems. The key point of this approach is to find an appropriate projection method. To the best of our knowledge, the problem of set-membership filtering incorporating state constraints has not been addressed, which motivates this work.

In this technical note, we are concerned with the filtering problems with nonlinear equality constraint within the set-membership filter framework. The nonlinear equality constraint is first linearized and transformed into a linear equality constraint with two uncertain quantities related to linearizing truncation error and base point error. S-procedure method is then applied to merge all inequalities into one inequality and the solution to the unconstrained set-membership filtering problem is obtained. The set-membership filter with state constraint is finally derived from projecting the unconstrained set-membership filter onto the constrained surface by using Finsler's Lemma. The solution to the problem of set-membership filtering with nonlinear equality constraint is obtained by solving a time-varying linear matrix inequality (LMI). A recursive algorithm is developed for computing the state estimate ellipsoid that guarantees to contain the true state.

The remainder of this technical note is organized as follows. The set-membership filtering problem with nonlinear equality constraint is formulated for discrete-time systems in Section II. A set-membership filter with nonlinear equality constraints is designed in Section III for determining a state estimation ellipsoid where the true state resides. An illustrative example is provided in Section IV to demonstrate the effectiveness of our method. Conclusions are drawn in Section V.

Notation: The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). The superscript T stands for matrix transposition. The notation trace (P) denotes the trace of P.

II. Problem Formulation

Consider the following discrete time-varying system:

$$x_{k+1} = A_k x_k + F_k u_k + B_k w_k \tag{1}$$
$$y_k = C_k x_k + D_k v_k \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state; $u_k \in \mathbb{R}^l$ is the known deterministic input; $y_k \in \mathbb{R}^m$ is the measurement output; A_k, B_k, C_k, D_k and F_k are known time-varying matrices with appropriate dimensions; $w_k \in \mathbb{R}^r$ is the process noise and $v_k \in \mathbb{R}^p$ is the measurement noise, which is assumed to be confined to specified ellipsoidal sets

$$\mathcal{W}_k = \{w_k : w_k^T Q_k^{-1} w_k \leq 1\} \tag{3}$$
$$\mathcal{V}_k = \{v_k : v_k^T R_k^{-1} v_k \leq 1\} \tag{4}$$

where $Q_k = Q_k^T > 0$ and $R_k = R_k^T > 0$ are known matrices with compatible dimensions; the initial state x_0 belongs to a given ellipsoid

$$(x_0 - \hat{x}_0)^T P_0^{-1} (x_0 - \hat{x}_0) \leq 1 \tag{5}$$

where $\hat{x}_0$ is an estimate of x_0 which is assumed to be given, and $P_0 = P_0^T > 0$ is a known matrix.

In addition to the dynamic system (1), there exist a nonlinear state constraint in the form of

$$h(x_k) = d_k \tag{6}$$

where $h(\cdot)$ is a nonlinear function and d_k is a known vector.

In this technical note, a filter based on the current measurement is considered for the system (1)–(2) subject to the constraint (6), which is of the form

$$\hat{x}_{k+1} = G_k \hat{x}_k + F_k u_k + L_k y_{k+1} \tag{7}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimate of x_k. G_k and L_k are the filter parameters to be determined.

Our aim is to determine an ellipsoid for the state x_{k+1}, given the measurement information y_{k+1} at the time instant $k+1$ for the process noise $w_k \in \mathcal{W}_k$ and the measurement noise $v_k \in \mathcal{V}_k$ subject to the state constraints (6). In other words, we look for P_{k+1} and $\hat{x}_{k+1}$ such that

$$(x_{k+1} - \hat{x}_{k+1})^T P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \leq 1 \tag{8}$$

subject to $w_k \in \mathcal{W}_k$, $v_k \in \mathcal{V}_k$ and (6). The above filtering problem is referred to as the set-membership filtering problem with state constraints.

Due to an additional constraint on the states, the solution to the set-membership filtering problem become more complex. For the linear state constraint, we can use the method of projecting the unconstrained state estimate onto the constrained surface [26]. However, nonlinear equality constraints are fundamentally different from linear equality constraints. Linearizing nonlinear equality constraints introduces two types of errors: truncation error and base point error [7], [9]. Truncation error arises because of truncating the Taylor series expansion after the first-order term and neglecting the higher order terms. Base point error occurs because of linearizing around the estimated value of the state rather than the true value. This second error may result in convergence problems [7]. When the unconstrained state estimate is projected onto the constrained hyperplane in state space defined by the linearized constraint, the estimate may never converge if the true value of the state is not on this hyperplane. So it is important to consider both truncation error and base point error. In this technical note, we utilize several novel methods to handle the truncation error and base point error. The solution is provided in the next section. Before we end this section, we introduce the following three useful lemmas:

Lemma 1: (S-Procedure) [4], [27]: Let $Y_0(\eta), Y_1(\eta), \cdots, Y_p(\eta)$ be quadratic functions of $\eta \in \mathbb{R}^n$

$$Y_i(\eta) = \eta^T T_i \eta, \quad i = 0, 1, \ldots, p \tag{9}$$

with $T_i = T_i^T$. Then, the implication

$$Y_1(\eta) \leq 0, \ldots, Y_p(\eta) \leq 0 \Longrightarrow Y_0(\eta) \leq 0 \tag{10}$$

holds if there exist $\tau_1, \ldots, \tau_p \geq 0$ such that

$$\eta^T \left(T_0 - \sum_{i=1}^{p} \tau_i T_i \right) \eta \leq 0. \tag{11}$$

Lemma 2: (Schur Complements) [4]: Given constant matrices L_1, L_2, L_3 where $L_1 = L_1^T$ and $L_2 = L_2^T < 0$, then

$$L_1 - L_3^T L_2^{-1} L_3 \leq 0$$

if and only if

$$\begin{bmatrix} L_1 & L_3^T \\ L_3 & L_2 \end{bmatrix} \leq 0$$

or equivalently

$$\begin{bmatrix} L_2 & L_3 \\ L_3^T & L_1 \end{bmatrix} \leq 0.$$

Lemma 3: (Finsler's Lemma) [27]: Let $x \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n \times n}$, and $M \in \mathbb{R}^{m \times n}$ such that $rank(M) = r < n$. The following statements are equivalent:

1) $x^T P x \leq 0, \forall Mx = 0, \ x \neq 0$;
2) $(M^\perp)^T P M^\perp \leq 0$;
3) $\exists \mu \in \mathbb{R} : P - \mu M^T M \leq 0$;
4) $\exists N \in \mathbb{R}^{m \times n} : P + N^T M + M^T N \leq 0$.

Remark 1:

1) $M^\perp$ is a basis for the null space of M. That is, all $x \neq 0$ such that $Mx = 0$ is generated by some $z \neq 0$ in the form $x = M^\perp z$.
2) $N = -\mu/2 M^T$ is a solution.

III. Set-Membership Filter Design With Nonlinear Equality Constraints

According to the suggestion by [7], we linearize nonlinear equality constraints (6) about the current estimate by considering truncation error and base point error. The linearized equation can be written as

$$(H_k + \Xi_1 \Delta_1)(x_k - \hat{x}_k) + h(\hat{x}_k) + \Xi_2 \Delta_2 = d_k \tag{12}$$

where the Jacobian matrix H_k is computed by

$$H_k = \left. \frac{\partial h(x)}{\partial x} \right|_{x = \hat{x}_k} \tag{13}$$

Ξ_1 is a known scaling matrix, and Δ_1 is an unknown matrix such that $\|\Delta_1\| \leq 1$. The term $\Xi_1 \Delta_1$ represents the base point error which takes into account the error of linearizing around the estimated value of the state rather than the true value. Ξ_2 is also a known scaling matrix, and Δ_2 is an unknown matrix such that $\|\Delta_2\| \leq 1$. The term $\Xi_2 \Delta_2$ is interpreted as the truncation error due to neglected higher order terms in the Taylor series expansion of the nonlinear (6).

Remark 2: Since the base point error and the truncation error are never exactly known, we introduce two uncertain matrices Δ_1 and Δ_2 incorporating the scaling matrices Ξ_1 and Ξ_2 to describe the base point error and the truncation error. By appropriately choosing the scaling matrices Ξ_1 and Ξ_2, the true state x is guaranteed to reside in the hyperplane in state space defined by the linearized constraint (13). This avoids the convergence problems. Since there are different linearizing errors at different estimation points, thus we use the scaling matrices Ξ_1 and Ξ_2 to cover *all* the linearizing errors by $\Xi_1 \Delta_1$ and $\Xi_2 \Delta_2$. If Ξ_1 and Ξ_2 are too small, then $\Xi_1 \Delta_1$ and $\Xi_2 \Delta_2$ may not cover all the linearizing errors which cause the divergent problem. If Ξ_1 and Ξ_2 are too big, then they will bring the conservativeness, i.e., a bigger estimation ellipsoid. Since we provide the set estimate which contains the true state, we actually know the distance (i.e., E_k) between the state estimate and the true state. If the distance is short, we can choose small Ξ_1 and Ξ_2. If the distance is long, we can choose big Ξ_1 and Ξ_2. For conservativeness, we always choose bigger Ξ_1 and Ξ_2 such that all the errors between the linearized and the original nonlinear constraints are covered in $\Xi_1 \Delta_1$ and $\Xi_2 \Delta_2$. We can use the trial and error procedure to select bigger scaling matrices Ξ_1 and Ξ_2. A interval mathematics can be employed to bound the linearizing errors [21]. A possible method is the scaling matrices Ξ_1 and Ξ_2 by online according to the matrix E_k, since the true state will reside in the set estimates.

Since the (12) involves two uncertain matrices Δ_1 and Δ_2, we need to develop some techniques to handle the state equality constraints with the uncertainties. If $(x_k - \hat{x}_k)^T P_k^{-1} (x_k - \hat{x}_k) \leq 1$, then there exists a z with $\|z\| \leq 1$ such that

$$x_k = \hat{x}_k + E_k z \tag{14}$$

where E_k is a factorization of $P_k = E_k E_k^T$.

Now we consider the linearized state equality constraint (12). Substituting (14) into (12) yields

$$H_k E_k z + \Xi_1 \Delta_1 E_k z + \Xi_2 \Delta_2 = d_k - h(\hat{x}_k). \tag{15}$$

Denoting

$$\Delta_3 = \Delta_1 E_k z \tag{16}$$

we can rewrite (15) as

$$H_k E_k z + \Xi_1 \Delta_3 + \Xi_2 \Delta_2 = d_k - h(\hat{x}_k). \tag{17}$$

On the other hand, one-step ahead estimation error $x_{k+1} - \hat{x}_{k+1}$ is written as:

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1} &= (I - L_k C_{k+1}) A_k x_k - G_k \hat{x}_k - L_k C_{k+1} F_k u_k \\ &\quad + (I - L_k C_{k+1}) B_k w_k - L_k D_{k+1} v_{k+1}. \end{aligned} \tag{18}$$

By using (14), we have

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1} =&(I - L_k C_{k+1}) A_k \hat{x}_k - G_k \hat{x}_k \\ &- L_k C_{k+1} F_k u_k + (I - L_k C_{k+1}) \\ &\cdot A_k E_k z + (I - L_k C_{k+1}) B_k w_k \\ &- L_k D_{k+1} v_{k+1}. \end{aligned} \tag{19}$$

From (17) and (19), we can see that the common unknown variables are z, w_k, v_{k+1}, Δ_2 and Δ_3. So we define

$$\eta = [\,1 \quad z^T \quad w_k^T \quad v_{k+1}^T \quad \Delta_2^T \quad \Delta_3^T\,]^T. \tag{20}$$

We can write (19) in a compact form

$$x_{k+1} - \hat{x}_{k+1} = \Pi(\hat{x}_k, u_k)\eta \tag{21}$$

where

$$\begin{aligned} \Pi(\hat{x}_k, u_k) =&[(I - L_k C_{k+1}) A_k \hat{x}_k - G_k \hat{x}_k \\ &- L_k C_{k+1} F_k u_k (I - L_k C_{k+1}) A_k E_k \\ &(I - L_k C_{k+1}) B_k \quad - L_k D_{k+1} \quad 0 \quad 0]. \end{aligned} \tag{22}$$

Hence, $(x_{k+1} - \hat{x}_{k+1})^T P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \le 1$ can be written as

$$\eta^T [\Pi^T(\hat{x}_k, u_k) P_{k+1}^{-1} \Pi(\hat{x}_k, u_k) - \mathrm{diag}(1,0,0,0,0,0)]\eta \le 0. \tag{23}$$

Now we can also write (17) in η as

$$\Pi_1(\hat{x}_k)\eta = 0 \tag{24}$$

where

$$\Pi_1(\hat{x}_k) = [\,h(\hat{x}_k) - d_k \quad H_k E_k \quad 0 \quad 0 \quad \Xi_2 \quad \Xi_1\,] \tag{25}$$

and η is defined in (20).

By noticing that $\|\Delta_1\| \le 1$, we can infer from (16) that

$$\Delta_3^T \Delta_3 - z^T E_k^T E_k z \le 0. \tag{26}$$

Thus the unknown variables z, w_k, v_{k+1}, Δ_2 and Δ_3 satisfy the following conditions:

$$\begin{cases} \|z\| \le 1, \\ w_k^T Q_k^{-1} w_k \le 1, \\ v_{k+1}^T R_{k+1}^{-1} v_{k+1} \le 1, \\ \|\Delta_2\| \le 1, \\ \Delta_3^T \Delta_3 - z^T E_k^T E_k z \le 0. \end{cases} \tag{27}$$

We write (27) in η as

$$\begin{cases} \eta^T \mathrm{diag}(-1, I, 0,0,0,0)\eta \le 0, \\ \eta^T \mathrm{diag}(-1, 0, Q_k^{-1}, 0,0,0),\ \eta \le 0, \\ \eta^T \mathrm{diag}(-1,0,0,R_{k+1}^{-1},0,0)\eta \le 0, \\ \eta^T \mathrm{diag}(-1,0,0,0,I,0)\eta \le 0, \\ \eta^T \mathrm{diag}(0, -E_k^T E_k, 0,0,0,I)\eta \le 0. \end{cases} \tag{28}$$

Now we apply S-procedure (Lemma 1) to (23) and (28). According to Lemma 1, the sufficient condition such that the inequalities (28) imply (23) to hold is that there exist nonnegative scalars τ_1, τ_2, τ_3, τ_4 and τ_5 such that

$$\begin{aligned} \eta^T [&\Pi^T(\hat{x}_k, u_k) P_{k+1}^{-1} \Pi(\hat{x}_k, u_k) \\ &- \mathrm{diag}(1,0,0,0,0,0) \\ &- \tau_1 \mathrm{diag}(-1, I, 0,0,0,0) \\ &- \tau_2 \mathrm{diag}(-1,0,Q_k^{-1},0,0,0) \\ &- \tau_3 \mathrm{diag}(-1,0,0,R_{k+1}^{-1},0,0) \\ &- \tau_4 \mathrm{diag}(-1,0,0,0,I,0) \\ &- \tau_5 \mathrm{diag}(0, -E_k^T E_k, 0,0,0,I)]\eta \le 0. \end{aligned} \tag{29}$$

Equation (29) is written in the following compact form:

$$\begin{aligned} \eta^T [&\Pi^T(\hat{x}_k, u_k) P_{k+1}^{-1} \Pi(\hat{x}_k, u_k) \\ &- \mathrm{diag}(1 - \tau_1 - \tau_2 - \tau_3 - \tau_4, \\ &\tau_1 I - \tau_5 E_k^T E_k, \tau_2 Q_k^{-1}, \tau_3 R_{k+1}^{-1}, \\ &\tau_4 I, \tau_5 I)]\eta \le 0. \end{aligned} \tag{30}$$

By denoting

$$\begin{aligned} \Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) =&\mathrm{diag}(1 - \tau_1 - \tau_2 - \tau_3 - \tau_4, \tau_1 I \\ &- \tau_5 E_k^T E_k, \tau_2 Q_k^{-1}, \tau_3 R_{k+1}^{-1}, \tau_4 I, \tau_5 I). \end{aligned} \tag{31}$$

Equation (30) is written as

$$\eta^T [\Pi^T(\hat{x}_k, u_k) P_{k+1}^{-1} \Pi(\hat{x}_k, u_k) - \Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)]\eta \le 0. \tag{32}$$

We apply Finsler's lemma (Lemma 3) to (24) and (32). Then there exists a τ_6 such that the following inequality holds:

$$\begin{aligned} \Pi^T(\hat{x}_k, u_k) P_{k+1}^{-1} \Pi(\hat{x}_k, u_k) - \Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \\ - \tau_6 \Pi_1^T(\hat{x}_k) \Pi_1(\hat{x}_k) \le 0. \end{aligned} \tag{33}$$

By using Schur complements (Lemma 2), (33) is equivalent to

$$\begin{bmatrix} -P_{k+1} & \Pi(\hat{x}_k, u_k) \\ \Pi^T(\hat{x}_k, u_k) & -\Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) - \tau_6 \Pi_1^T(\hat{x}_k) \Pi_1(\hat{x}_k) \end{bmatrix} \le 0. \tag{34}$$

Thus, if there exist the filter parameters G_k and L_k, $\tau_1 \ge 0$, $\tau_2 \ge 0$, $\tau_3 \ge 0$, $\tau_4 \ge 0$, $\tau_5 \ge 0$, τ_6 such that (34) holds, then one-step ahead state x_{k+1} resides in its state estimation ellipsoid $(x_{k+1} - \hat{x}_{k+1})^T P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \le 1$.

According to the above results, we can summarized as the following theorem.

Theorem 1: For the system (1)–(2) subject to the constraint (6), if the state x_k belongs to its state estimation ellipsoid $(x_k - \hat{x}_k)^T P_k^{-1} (x_k - \hat{x}_k) \le 1$, where $\hat{x}_k$ and $P_k > 0$ are known, then one-step ahead state x_{k+1} resides in its state estimation ellipsoid $(x_{k+1} - \hat{x}_{k+1})^T P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \le 1$, if there exist $P_{k+1}, G_k,\ L_k,\ \tau_1 \ge 0,\ \tau_2 \ge 0,\ \tau_3 \ge 0,\ \tau_4 \ge 0,\ \tau_5 \ge 0,\ \tau_6$ such that

$$\begin{bmatrix} -P_{k+1} & \Pi(\hat{x}_k, u_k) \\ \Pi^T(\hat{x}_k, u_k) & -\Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) - \tau_6 \Pi_1^T(\hat{x}_k) \Pi_1(\hat{x}_k) \end{bmatrix} \le 0. \tag{35}$$

Moreover the center of the state estimate ellipsoid is determined by

$$\hat{x}_{k+1} = G_k \hat{x}_k + F_k u_k + L_k y_{k+1} \tag{36}$$

where $\Pi(\hat{x}_k, u_k)$, $\Pi_1(\hat{x}_k)$ and $\Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ are defined in (22), (25) and (31), respectively.

Remark 3: Theorem 1 provides a clear physical interpretation: the set-membership filter with state constraints is obtained from the unconstrained set-membership filter projecting onto the constrained surface, since the constrained solution is obtained from the unconstrained solution by left-multiplying the transpose of the orthogonal complement of the constrained equation and right-multiplying the orthogonal complement of the constrained equation.

Theorem 1 outlines the principle of determining the current state estimation ellipsoid containing x_{k+1}. However, it does not provide an optimal (minimal) state estimation ellipsoid. Next, we apply the convex optimization approach [16], [29] to determine an optimal ellipsoid. P_{k+1} is obtained by solving the following optimization problem:

$$\min_{P_{k+1},\, G_k,\, L_k,\, \tau_1 \geq 0,\, \tau_2 \geq 0,\, \tau_3 \geq 0,\, \tau_4 \geq 0,\, \tau_5 \geq 0,\, \tau_6} \quad \mathrm{trace}(P_{k+1})$$
$$\text{subject to (35).} \tag{37}$$

Remark 4: Another good measure of the ellipsoid is to choose $\mathrm{logdet}(P_{k+1})$ as the objective function. However, if we change the objective function $\mathrm{trace}(P_{k+1})$ to $\mathrm{logdet}(P_{k+1})$, then the optimization problem (37) is not convex. The existing semi-definite programming (SDP) cannot be used to solve the non-convex optimization problem. In order to transfer this non-convex optimization problem into a convex optimization one, a decoupled technique has been proposed in [8], which provides a unique optimal ellipsoid.

Remark 5: The set-membership filter design in this technical note is different from that of [3]. [3] did a great work on set-membership filtering which can provide a one-to-one correspondence with the Kalman filtering. However, we cannot find the one-to-one correspondence with the Kalman filtering subject to nonlinear state constraints [7], [9], [26]. The reason is that the different methods are employed to obtain the different results. For example, [7] used an iterative method to provide the solution to the Kalman filtering with nonlinear state constraints; [9] proposed a method that utilizes the projection method twice to obtain the solution; [26] provided an analytical solution to the Kalman filtering with nonlinear state constraints. In this technical note, an LMI approach is proposed to provide a solution to the set-membership filtering problem with nonlinear state constraints. The solution is obtained by approximating the nonlinear equality constraint by a set of the linear equality constraints with scaling matrices Ξ_1 and Ξ_2. For some particular nonlinear state constraints, we can find such an approximation. For example, we have a simple and one-dimension nonlinear state constraint: $x_k^2 = d_k$, where $h(x_k) = x_k^2$. We thus get from (17) that $2\hat{x}_k E_k z + \Xi_1 \Delta_3 + \Xi_2 \Delta_2 = d_k - \hat{x}_k^2$. Since $d_k = x_k^2$ and $x_x = \hat{x}_k + E_k z$, we obtain that $2\hat{x}_k E_k z + \Xi_1 \Delta_3 + \Xi_2 \Delta_2 = \hat{x}_k^2 + 2\hat{x}_k E_k z + E_k^T E_k z^2 - \hat{x}_k^2$, which is simplified as $\Xi_1 \Delta_3 + \Xi_2 \Delta_2 = E_k^T E_k z^2$. Therefore, we can choose bigger Ξ_1 and Ξ_2 to cover the nonlinear equality constraint. In this example, we select $\Xi_1 = E_k^T E_k$ and $\Xi_2 = E_k^T E_k$ to guarantee that the nonlinear state constraint $x_k^2 = d_k$ resides within a set of the linear equality constraints for all z with $\|z\| \leq 1$, Δ_2 with $\|\Delta_2\| \leq 1$ and Δ_3 with $\|\Delta_3\| \leq 1$. However, for a general nonlinear equality constraint, it cannot be guaranteed within a set of the linear equality constraints with scaling matrices Ξ_1 and Ξ_2.

Remark 6: From (33), we can see that P_{k+1} is a free variable. When P_{k+1} tends to $+\infty$, the inequality (33) is always satisfied. Since (33) and (35) are equivalent, it is guaranteed that (35) is always feasible. Therefore the optimization problem (37) has feasible solution.

Equation (37) provides the computation of the state estimation ellipsoid of the minimal size in the sense of trace. Now we summarize a recursive algorithm for the set-membership filtering as follows:

1) The Set-Membership Filtering Recursive Algorithm:

Step 1: Step 1) Given the initial values $(\hat{x}_0,\ P_0)$ and the recursive times N. Set $k = 0, E = E_0, \hat{x} = \hat{x}_0$, where $P_0 = E_0 E_0^T$;

Step 2: Step 2) Given E, $\hat{x}$, and current measurement y, compute the shape of the state estimation ellipsoid P_{k+1}, and filter parameters G_k and L_k by solving the optimization problem (37) via the semi-definite programming software [12];

Step 3: Step 3) Compute the state estimate $\hat{x}_{k+1}$ by using (36);

Step 4: Step 4) Find the matrix E_{k+1} such that $P_{k+1} = E_{k+1} E_{k+1}^T$;

Step 5: Step 5) Set $E = E_{k+1}$, $\hat{x} = \hat{x}_{k+1}$. If $k = N$, then Stop, otherwise, $k = k + 1$ and go to Step 2.

IV. An Illustrative Example

This section presents a simple example to illustrate the theory of this technical note. Consider a discrete-time dynamic system

$$x_{k+1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k + u_k + w_k \tag{38}$$

$$y_k = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 2 \\ 1 \end{bmatrix} v_k \tag{39}$$

where $x_k = [\, x_{k1} \quad x_{k2} \quad x_{k3} \quad x_{k4} \,]^T$, $u_k = [\, u_{k1} \quad u_{k2} \quad u_{k3} \quad u_{k4} \,]^T$, $w_k = [\, w_{k1} \quad w_{k2} \quad w_{k3} \quad w_{k4} \,]^T$, $y_k = [\, y_{k1} \quad y_{k2} \,]^T$. There are an extra nonlinear state constraint on the above system described as follows:

$$(r - x_{k1})^2 + x_{k2}^2 = r^2. \tag{40}$$

This nonlinear constraint is quadratic and is assumed to only take into account the first two states for simplicity as the trajectory of two states can be plotted in the plane whereas four states are in the hyperplane.

From (40) and according to (12) and (13), we have the Jacobian matrix H_k as

$$H_k = [\, -2(r - \hat{x}_{k1}) \quad 2\hat{x}_{k2} \quad 0 \quad 0 \,] \tag{41}$$

and $h(\hat{x}_k)$ as

$$h(\hat{x}_k) = (r - \hat{x}_{k1})^2 + \hat{x}_{k2}^2. \tag{42}$$

In this example, r is chosen as 30, $\Xi_1 = [\, 5 \quad 3 \quad 2 \quad 1 \,]$, and $\Xi_2 = 100$. w_k and v_k are assumed as $0.3\sin(2k)$ and $0.2\sin(30k)$, respectively. The initial state is set as $x_0 = [0 \quad 0 \quad 0 \quad 0]^T$, which belongs to the ellipsoid $\mathcal{E}(P_0, \hat{x}_0) = \{x_0 : (x_0 - \hat{x}_0)^T P_0^{-1} (x_0 - \hat{x}_0) \leq 1$, where $\hat{x}_0 = [0 \quad 0 \quad 0 \quad 0]^T$, and $P_0 = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ for all k, $Q_k = 1$ and $R_k = 1$.

The simulation results are obtained by solving the semi-definite programming problem (37) under Matlab 6.5 with YALMIP 3.0 and SeDuMi 1.1 [12]. Fig. 1 shows the trajectory of x_{k1} vs x_{k2} and the trajectory of the estimate of x_{k1} vs the estimate of x_{k2}. It can be seen that the state variables x_{k1} vs x_{k2} satisfy the nonlinear equality constraint (40), which is a half-circle. Figs. 2 and 3 display the true values of the state, its estimates, its upper bounds and lower bounds by using the constrained set-membership filter. The results confirm that the true signals x_{k1} and x_{k2} always reside between their upper bounds and lower bounds. Therefore, our method provides an ellipsoidal

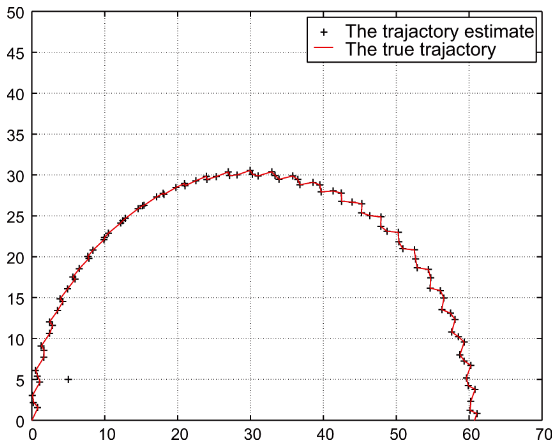

Fig. 1. Plots of state 1 versus state 2 and the estimate of state 1 versus the estimate of state 2.

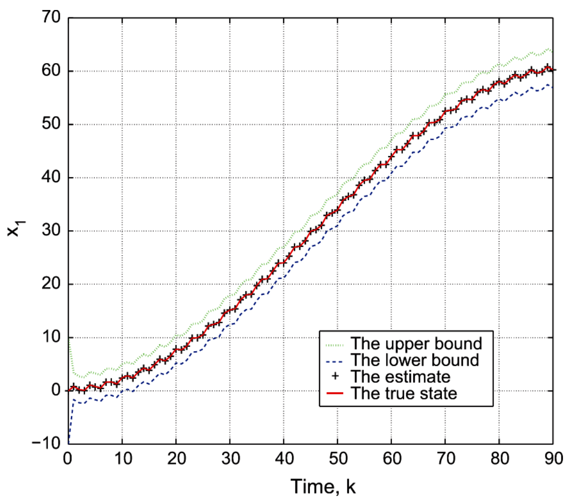

Fig. 2. Upper bound, the lower bound, the estimate and the true value of State 1.

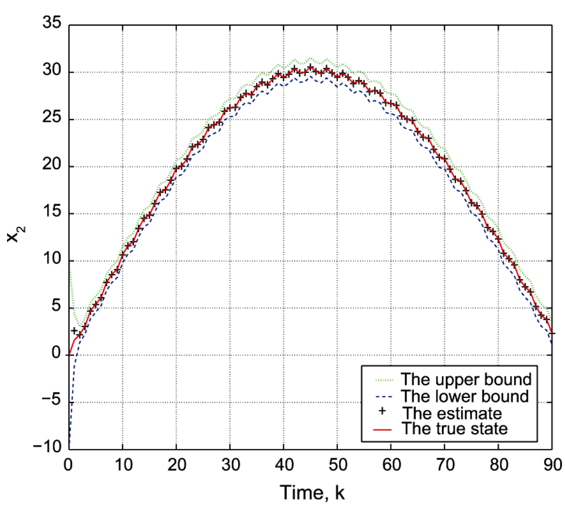

Fig. 3. Upper bound, the lower bound, the estimate and the true value of State 2.

state estimate, which is a set of state estimates that contains the true state regardless of the process noise $w_k \in \mathcal{W}_k$ and the measurement noise $v_k \in \mathcal{V}_k$.

V. Conclusion

This technical note has considered the problem of set-membership filtering for discrete-time systems with nonlinear equality constraint between their state variables. The set-membership filtering with nonlinear equality constraint has been regarded as a set-membership filtering with linear equality constraint with two uncertain quantities by linearizing. The solution to the unconstrained set-membership filtering problem has first been derived by S-procedure method. The set-membership filter with state constraint has then been developed from projecting the unconstrained set-membership filter onto the constrained surface by using Finsler's Lemma. A recursive time-varying LMI optimization algorithm has been developed for computing the state estimate ellipsoid that guarantees to contain the true state. An illustrative example has demonstrated the feasibility of the proposed set-membership filtering with nonlinear equality constraint. Our design method is quite different from the Kalman filter design ones. They provide *an estimation point* accurately to estimate the true state, whereas our method is to provide *an estimation ellipsoid* that contains the true value. Our future research topics will focus on the steady-state analysis of the set-membership filtering with nonlinear equality constraint, the convergence of the algorithms and the conservatism of the possible estimation sets. In this technical note, we only consider one-step delay state constraint $h(x_k) = d_k$ at time t_k. Another challenging work is the state constraint $h(x_{k+1}) = d_{k+1}$ at time t_{k+1} which is also our future research topic.

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