Signal Processing 145 (2018) 91–98

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

An event-triggered approach to robust recursive filtering for stochastic discrete time-varying spatial-temporal systems

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ARTICLE INFO

Article history:
Received 16 September 2017
Revised 8 November 2017
Accepted 16 November 2017
Available online 24 November 2017

Keywords:
Event-based mechanism
Robust recursive filtering
Spatial-temporal systems
Time-varying systems
Parameter uncertainties

ABSTRACT

In this paper, the robust recursive filtering problem is investigated for a class of uncertain stochastic discrete time-varying spatial-temporal systems with an event-based communication mechanism. The system under consideration includes a set of sensors located at specified points where each sensor can only receive the measurement output from the system at corresponding position. In order to reduce the communication cost, an event-triggered mechanism is utilized to decide when a certain measurement output should be transmitted from sensors to the centralized filter. By re-organizing the state variables, the uncertain spatial-temporal system is first transformed into an ordinary uncertain difference dynamic system. The aim of this paper is to design a filter such that, for all possible parameter uncertainties, the filtering error covariance is guaranteed to have an upper bound which is then minimized at each time-instant under the event-triggered mechanism. By using the matrix derivation approach, the desired filter gain is obtained in terms of the solution to certain Riccati-like difference equations. Finally, a numerical example is employed to demonstrate the effectiveness of the filtering scheme proposed.

1. Introduction

The past few decades have witnessed constant research interests on various aspects of spatial-temporal systems (STSs) that are capable of modeling many practical situations such as chemical engineering, molecular biology and population dynamic systems, see e.g., [6,11,27,28,38]. Filtering has long been one of the fundamental problems in signal processing, communications and control application [1,8,13,17,30,37,43]. For STSs, filtering problem appears to be especially important since, in most engineering practice, it is of great importance to know/estimate the state information in the prescribed region based only on possibly partial measurement outputs. Therefore, the filtering problem of STSs has recently received increasing research attention and some results have been reported in the literature. For example, in [2], the robust H_∞ filter has been designed for linear stochastic partial differential systems and the corresponding problem has been studied in [3] for nonlinear stochastic partial differential systems where the random external disturbance and measurement noise in the spatial-temporal domain have been taken into account. Recently, in [40], the reliable H_∞ filtering problem has been investigated for a class of stochastic spatial-temporal systems and a reliable H_∞ filter has been designed in presence of sensor saturations and failures. It is noted that almost all existing results have been concerned with *time-invariant* systems. However, in the real world, the systems are likely to be *time-varying* since the system parameters often fluctuate due to the complex environments such as changes of the operating point, aging of components and constant changes of system dynamics in their structure. Therefore, it is of practical significance to develop a recursive filtering approach for time-varying spatial-temporal systems.

In practical engineering, parameter uncertainties are inevitable mainly because of the modeling errors and external disturbances. If the robustness issue against such uncertainties is not taken into appropriate consideration, it would be difficult to guarantee the desired filtering performance by employing conventional filtering approaches. Therefore, in the past decade, many researchers have attempted to develop robust techniques to deal with the effects resulting from parameter uncertainties, and there has been a rich body of results available in existing literature. For example, in [15,20], the polytopic-type uncertainties (where uncertain parameters reside in a polytope) have been well investigated and the corresponding robust controllers/filters have been designed. In [35,41], norm-bounded uncertainties (where uncertain parameters

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https://doi.org/10.1016/j.sigpro.2017.11.011

are bounded by certain norms) have been dealt with and a number of robust analysis/design approaches have been proposed. It should be pointed out that, in the framework of time-varying discrete spatial-temporal systems, the parameter uncertainties have not been properly investigated yet, and this constitutes the first motivation of this paper is to develop robust filtering approach for uncertain discrete time-varying spatial-temporal systems.

It is worth mentioning that, in almost all the existing results on the spatial-temporal systems, the filter/controller design problem has been studied based on the *time-triggered* communication mechanism for the purpose of easy implementation. However, this transmission schedule may lead to many unnecessary information exchanges, thereby wasting considerable communication resource. In order to avoid such a disadvantage of the traditional time-triggered schemes, the event-triggered strategy has recently been proposed, by which the transmission is executed only if certain triggering condition is met. Event-triggered strategy is capable of reducing redundant signal transmissions in an effective way and hence saving the communication cost [9,36]. In the past few years, the event-based control/filtering problems have stirred a great deal of research interest, and many excellent results have been reported in the literature, see e.g. [5,7,10,12,14,18,22,24,26,31,32,42,48]. For example, in [5], the event-triggered leader-follower tracking control problem has been studied for multi-agent systems with general linear dynamics, where the event triggering rules can guarantee bounded tracking errors. The event-triggered H_∞ filtering problem has been addressed for networked Markovian jump system in [39], in which the time-delay modelling method has been employed to describe the event-triggered scheme.

Encouraged by the discussions made so far, it seems a natural idea to consider the event-triggered filtering problem for uncertain time-varying spatial-temporal systems, which is of both theoretical importance and practical significance. Nevertheless, this appears to be a challenging task with two major issues identified as follows: 1) for the spatial-temporal systems, does there exist a valid robust event-triggered filter with a guaranteed upper bound on the filtering error variance? and 2) if such a filter exists, how can we design the (locally) optimal one such that certain upper bound on the filtering error covariance is minimized? It is, therefore, the second motivation of this paper to offer satisfactory answers to these two questions.

In this paper, the event-triggered robust recursive filtering problem is addressed for a class of uncertain stochastic discrete time-varying spatial-temporal systems. The system under consideration, which evolves over both time and space in a given discrete rectangular region, is subject to time-varying parameter uncertainties. Some sensors are located in specified points whose measurement outputs are used for the state estimation. For the purpose of saving communication cost, the event-based communication mechanism is adopted to determine when sensors should transmit certain information to the filter. By using the vector re-organization approach, the spatial-temporal system is represented by an equivalent ordinary difference dynamic system, based on which an upper bound for the filtering error covariance is first obtained in form of Riccati-like difference equations. Then, the desired robust recursive filter is designed such that the obtained upper bound is minimized at each time-instant. Finally, an illustrative example is presented to show the effectiveness of the filtering scheme proposed.

The main contributions of this paper can be briefly stated as follows: 1) the addressed recursive filtering problem for the spatial-temporal systems is new, where the impacts from both the space and the time to the filtering performance are extensively explored; 2) the parameter uncertainties are taken into account from the perspective of the engineering practice; and 3) the event-triggered communication mechanism is introduced to decide if the measurement output should be transmitted to the filter.

Notation In this paper, we denote $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively, as the n dimensional Euclidean space and the set of all $n\times m$ real matrices. The notations $X\geq Y$ and $X>Y$, where X and Y are real symmetric matrices, represent that $X-Y$ is positive semi-definite and positive definite. diag$\{\cdots\}$ stands for a block-diagonal matrix. $\mathbb{N}^+$ indicates the positive integer set. M^T means the transpose of the matrix M. I_n denotes the identity matrix of $n\times n$ dimension. $\mathbb{E}\{x\}$ means the expectation of the stochastic variable x. $\|x\|$ denotes the Euclidean norm of a vector x. In symmetric block matrices, "$*$" is used to denote a term induced by symmetry.

2. Problem formulation

In this paper, we consider the following class of uncertain stochastic time-varying discrete spatial-temporal systems:

$$\begin{aligned}x_{m,n}(k+1)=&(\kappa(k)+\Delta\kappa(k))\{x_{m+1,n}(k)+x_{m-1,n}(k)\\&+x_{m,n+1}(k)+x_{m,n-1}(k)-4x_{m,n}(k)\}\\&+(A(k)+\Delta A(k))\{x_{m+1,n}(k)+x_{m,n+1}(k)\\&-2x_{m,n}(k)\}+(B(k)+\Delta B(k))x_{m,n}(k)\\&+G(k)w_{m,n}(k)\end{aligned}\tag{1}$$

where the location $(m,\ n)$ is the crossing point of mth row and nth column in the discrete rectangular region that is given as $\{0,1,2,\dots,M\}\times\{0,1,2,\dots,N\}$. At the location $(m,\ n)$, $x_{m,n}(k)\in\mathbb{R}^{n_x}$ is the state vector and $w_{m,n}(k)\in\mathbb{R}^{n_w}$ is a zero-mean Gaussian white noise sequence. $\kappa(k)$, $A(k)$, $B(k)$ and $G(k)$ are known real time-varying matrices with appropriate dimensions.

Matrices $\Delta\kappa(k)$, $\Delta A(k)$ and $\Delta B(k)$ represent parameter uncertainties of the following form:

$$\begin{aligned}&\begin{bmatrix}\Delta\kappa(k) & \Delta A(k) & \Delta B(k)\end{bmatrix}\\&=\Gamma(k)E(k)\begin{bmatrix}Q_1(k) & Q_2(k) & Q_3(k)\end{bmatrix}\end{aligned}\tag{2}$$

where $\Gamma(k)$, $Q_1(k)$, $Q_2(k)$ and $Q_3(k)$ are known real time-varying matrices with appropriate dimensions, and $E(k)$ is an unknown matrix satisfying $E(k)E(k)^T\leq I$.

Remark 1. It should be mentioned that, the underlying systems (1) can be regarded as the discrete-space version of continuous-spatial temporal systems. Specifically, in systems (1), the first and second terms are originally from the discretization of the second and first orders continuous partial derivatives, respectively.

The ith sensor measurement output model is described as follows:

$$\begin{aligned}y_{m_i,n_i}(k)=&C_{m_i,n_i}(k)x_{m_i,n_i}(k)+D_{m_i,n_i}(k)\upsilon_{m_i,n_i}(k),\\&i=1,2,\cdots,l\end{aligned}\tag{3}$$

where $y_{m_i,n_i}(k)\in\mathbb{R}^{n_y}$ is the measurement output received by the sensor at the location $(m_i,\ n_i)$, and $\upsilon_{m_i,n_i}(k)\in\mathbb{R}^{n_\upsilon}$ stands for the sequence of measurement noises. $C_{m_i,n_i}(k)$ and $D_{m_i,n_i}(k)$ are known real time-varying matrices with appropriate dimensions.

Assumption 1. $w_{m,n}(k)$ and $\upsilon_{m_i,n_i}(k)$ are mutually uncorrelated zero-mean Gaussian white-noise sequences with covariances $S_{m,\ n}(k)>0$ and $R_{m_i,n_i}(k)>0$, respectively.

Assumption 2. The initial states $x_{m,\ n}(0)$ are uncorrelated with both $w_{m,n}(k)$ and $\upsilon_{m_i,n_i}(k)$, which have the means $x^0_{m,n}$ and covariances $U^0_{m,n}$.

Assumption 3. On the boundary, the values of states satisfy the Dirichlet boundary condition, i.e., $x_{m,n}(k)=0$ for $m=0,\ M$ or $n=0,\ N$.

To describe the event-triggering mechanism, we introduce the transmission instants for the ith sensor ($i=1,2,\cdots,l$), which can be represented as $0 \le s_0^i \le s_1^i \le s_2^i \cdots$, and $s_{t_i+1}^i$ satisfies

$$s_{t_i+1}^i \triangleq \min\left\{k \in \mathbb{N} | k > s_{t_i}^i, f_{m_i,n_i}(e_{m_i,n_i}(k), \delta_{m_i,n_i}) > 0\right\}$$

where $e_{m_i,n_i}(k)$ and event generator functions $f_{m_i,n_i} : \mathbb{R}^{n_y} \times \mathbb{R} \to \mathbb{R}$ are defined in the following form:

$$e_{m_i,n_i}(k) \triangleq y_{m_i,n_i}(s_{t_i}^i) - y_{m_i,n_i}(k),$$

$$f_{m_i,n_i}(e_{m_i,n_i}(k), \delta_{m_i,n_i}) \triangleq e_{m_i,n_i}^T(k) e_{m_i,n_i}(k) - \delta_{m_i,n_i}$$

with constants $\delta_{m_i,n_i} > 0$.

Before proceeding further, let's define an index $\mathfrak{H} = \{h = n(M+1) + m + 1 | m = 0, n = 0, 1, \cdots, N$ or $m = M, n = 0, 1, \cdots, N$ or $m = 0, 1, \cdots, M, n = 0$ or $m = 0, 1, \cdots, M, n = N\}$. Denoting $h = n(M+1) + m + 1$ and $L = (M+1)(N+1)$ and setting $x_h(k) = x_{m,n}(k)$, we can reorganize all state variables as follows:

$$\begin{aligned} x(k) &\triangleq \left[x_{0,0}^T(k) \ldots x_{m,0}^T(k) \ldots x_{M,0}^T(k) \ldots x_{m,n}^T(k) \right. \\ &\quad \left. \ldots x_{0,N}^T(k) \ldots x_{m,N}^T(k) \ldots x_{M,N}^T(k)\right]^T, \\ &= \left[x_1^T(k) \quad x_2^T(k) \quad x_3^T(k) \quad \ldots \quad x_{L-1}^T(k) \quad x_L^T(k)\right]^T. \end{aligned}$$

In the similar way, we can define $w_h(k)$, $U_h(0)$ and $S_h(k)$ from $w_{m,n}(k)$, $U_{m,n}^0$ and $S_{m,\,n}(k)$, respectively.

With the notations defined previously, the spatial-temporal systems (1) can be rewritten by the following augmented system

$$x(k+1) = (\bar{A}(k) + \Delta\bar{A}(k))x(k) + \bar{G}(k)w(k) \tag{4}$$

where

$$\begin{aligned} &T_1 \triangleq \left[\; T_{11}^T \quad T_{12}^T \quad \ldots \quad T_{1L}^T \;\right]^T, \quad \bar{G}(k) \triangleq I_L \otimes G(k), \\ &T_2 \triangleq \left[\; T_{21}^T \quad T_{22}^T \quad \ldots \quad T_{2L}^T \;\right]^T, \quad \bar{\Gamma}(k) \triangleq I_L \otimes \Gamma(k), \\ &\bar{A}(k) \triangleq (I_L \otimes \kappa(k))T_1 + (I_L \otimes A(k))T_2 + I_L \otimes B(k), \\ &\Delta\bar{A}(k) \triangleq (I_L \otimes \Gamma(k))(I_L \otimes E(k))[(I_L \otimes Q_1(k))T_1 \\ &\qquad +(I_L \otimes Q_2(k))T_2 + I_L \otimes Q_3(k)], \\ &\bar{Q}(k) \triangleq (I_L \otimes Q_1(k))T_1 + (I_L \otimes Q_2(k))T_2 + I_L \otimes Q_3(k), \\ &w(k) \triangleq \left[\; w_1(k)^T \quad w_2(k)^T \quad \ldots \quad w_L(k)^T \;\right]^T, \\ &\bar{U}(0) \triangleq \mathrm{diag}\{U_1(0), U_2(0), \ldots, U_L(0)\}, \\ &\bar{S}(k) \triangleq \mathrm{diag}\{S_1(k), S_2(k), \ldots, S_L(k)\}, \\ &\bar{E}(k) \triangleq I_L \otimes E(k). \end{aligned}$$

$$T_{1h} = \begin{cases} \underbrace{[\overbrace{0\ldots0}^{h-M-2} \quad I_n \quad \overbrace{0\ldots0}^{M-1} \quad I_n \quad -4I_n \quad I_n \quad \overbrace{0\ldots0}^{M-1} \quad I_n \quad \overbrace{0\ldots0}^{L-h-M-1}}_{L}], & h \notin \mathfrak{H}, \\ 0, \quad h \in \mathfrak{H}, \end{cases}$$

$$T_{2h} = \begin{cases} \underbrace{[\overbrace{0\ldots0}^{h-1} \quad -2I_n \quad I_n \quad \overbrace{0\ldots0}^{M-1} \quad I_n \quad \overbrace{0\ldots0}^{L-h-M-1}}_{L}], & h \notin \mathfrak{H}, \\ 0, \quad h \in \mathfrak{H}. \end{cases}$$

For the convenience of expression, the measurements from l sensors $y(k)$, the measurement signals based on event-trigger mechanism $y^t(k)$ (for $s_{t_i}^i \le k < s_{t_i+1}^i$ with $i = 1, 2, \cdots, l$), the measurement noises $\nu(k)$, the covariance $\bar{R}(k)$ and the difference $e(k)$ are organized as

$$\begin{aligned} &y(k) \triangleq [y_{m_1,n_1}^T(k) \quad y_{m_2,n_2}^T(k) \quad \ldots \quad y_{m_l,n_l}^T(k)]^T, \\ &y^t(k) \triangleq [y_{m_1,n_1}^T(s_{t_1}^1) \quad y_{m_2,n_2}^T(s_{t_2}^2) \quad \ldots \quad y_{m_l,n_l}^T(s_{t_l}^l)]^T, \\ &\nu(k) \triangleq [\nu_{m_1,n_1}^T(k) \quad \nu_{m_2,n_2}^T(k) \quad \ldots \quad \nu_{m_l,n_l}^T(k)]^T, \\ &\bar{R}(k) \triangleq \mathrm{diag}\{R_{m_1,n_1}(k), R_{m_2,n_2}(k), \ldots, R_{m_l,n_l}(k)\}, \\ &e(k) \triangleq y^t(k) - y(k). \end{aligned}$$

Subsequently, by defining $j_i = n_i(M+1) + m_i + 1$ and the following auxiliary matrices for $i = 1, 2, \ldots, l$

$$C_i(k) \triangleq \underbrace{[\overbrace{0\ldots0}^{j_i-1} \quad C_{m_i,n_i}(k) \quad 0\ldots0}_{L}],$$

the sensor measurement model (3) can be transformed the following augmented form

$$y(k) = \bar{C}(k)x(k) + \bar{D}(k)\nu(k) \tag{5}$$

where

$$\begin{aligned} &\bar{C}(k) \triangleq [C_1^T(k) \quad C_2^T(k) \quad \ldots \quad C_l^T(k)]^T, \\ &\bar{D}(k) \triangleq \mathrm{diag}\{D_{m_1,n_1}(k), D_{m_2,n_2}(k), \ldots, D_{m_l,n_l}(k)\}. \end{aligned}$$

The event-based filter structure is constructed as follows:

$$\hat{x}(k+1) = F(k)\hat{x}(k) + K(k)(y^t(k) - \bar{C}\hat{x}(k)) \tag{6}$$

where $\hat{x}(k) \in \mathbb{R}^{n_x L}$ is the state estimate, and $F(k)$ and $K(k)$ are the parameter matrices to be determined. The estimate of initial state $\hat{x}(0) \in \mathbb{R}^{n_x L}$ is selected as $\hat{x}(0) = \mathbb{E}\{x(0)\}$.

By letting $\tilde{x}(k) \triangleq \left[\; x^T(k) \quad \hat{x}^T(k) \;\right]^T$, it follows from (4)–(6) that

$$\begin{aligned} \tilde{x}(k+1) =& (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\tilde{x}(k) + \tilde{K}(k)e(k) \\ &+ \tilde{G}(k)w(k) + \tilde{D}(k)\nu(k) \end{aligned} \tag{7}$$

where

$$\tilde{A}(k) \triangleq \begin{bmatrix} \bar{A}(k) & 0 \\ K(k)\bar{C}(k) & F(k) - K(k)\bar{C}(k) \end{bmatrix},$$

$$\tilde{\Gamma}(k) \triangleq \begin{bmatrix} \bar{\Gamma}(k) \\ 0 \end{bmatrix}, \quad \tilde{Q}(k) \triangleq \begin{bmatrix} \bar{Q}(k) & 0 \end{bmatrix},$$

$$\tilde{K}(k) \triangleq \begin{bmatrix} 0 \\ K(k) \end{bmatrix}, \quad \tilde{G}(k) \triangleq \begin{bmatrix} \bar{G}(k) \\ 0 \end{bmatrix},$$

$$\tilde{D}(k) \triangleq \begin{bmatrix} 0 \\ K(k)\bar{D}(k) \end{bmatrix}.$$

In this paper, our aim is to parameterize the filter gains $F(k)$ and $K(k)$ in the filter (6) so as to the filtering error covariance $P(k) \triangleq \mathbb{E}\{(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^T\}$ has an upper bounded and such an upper bound is minimized at each instant.

3. Main results

In this section, an upper bound for the filtering error covariance $P(k)$ is first expressed in terms of the Riccati-like difference equations. Based on the aforementioned upper bound condition, the algorithm of the solution for the desired filter parameters $F(k)$ and $K(k)$ are then given.

Before proceeding, we introduce the following lemmas that will be in the proof of our main results.

Lemma 1. *[44] Let the matrices A, H, E and F ($FF^T \le I$) with compatible dimensions be given, X be a symmetric positive definite matrix, and $\beta > 0$ be an arbitrary positive constant such that $\beta^{-1}I - EXE^T > 0$. Then, the inequality holds as follows:*

$$\begin{aligned} &(A + HFE)X(A + HFE)^T \\ &\le A(X^{-1} - \beta E^T E)^{-1} A^T + \beta^{-1} H H^T. \end{aligned}$$

Lemma 2. *[22] For $0 \le k \le N$, suppose that $X = X^T \ge 0$, $Y = Y^T \ge 0$ and $h_k(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. If*

$$h_k(X) \le h_k(Y), \quad \forall X \le Y$$

then the solutions $W(k+1)$ *and* $M(k+1)$ *to the following difference equations*

$$W(k+1) = h_k(W(k)), \quad M(k+1) \le h_k(M(k)),$$
$$M(0) = W(0)$$

satisfy

$$M(k+1) \le W(k+1).$$

Lemma 3. *[34] For* $0 \le k \le N$*, suppose that* $X = X^T \ge 0$, $Y = Y^T \ge 0$, $h_k(\cdot) = h_k^T(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ *and* $g_k(\cdot) = g_k^T(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$*. If*

$$h_k(X) \le h_k(Y), \quad \forall X \le Y,$$
$$h_k(Y) \le g_k(Y),$$

then the solutions $W(k+1)$ *and* $M(k+1)$ *to the following difference equations*

$$W(k+1) = h_k(W(k)), \quad M(k+1) = g_k(M(k)),$$
$$M(0) = W(0) > 0$$

satisfy $W(k+1) \le M(k+1)$.

Lemma 4. *Let A, B, C and D be given matrices of appropriate dimensions with A, D and* $D^{-1} + CA^{-1}B$ *being invertible, then*

$$(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}.$$

Now, we introduce the binary variables $\gamma_j(k)$ for $j = 1, 2, \cdots, l$ as follows: $\gamma_j(k) = 0$ when the event generator conditions are satisfied at the time k for sensor j, while $\gamma_j(k) = 1$ when there is no event-triggered. Furthermore, we denote

$$\tilde{\gamma}(k) \triangleq \mathrm{diag}\{\gamma_1(k)I_{n_y}, \gamma_2(k)I_{n_y}, \ldots, \gamma_l(k)I_{n_y}\},$$
$$\bar{\gamma}(k) \triangleq I_{n_y l} - \tilde{\gamma}(k). \tag{8}$$

In the following theorem, an upper bound is obtained for the filtering error covariance $P(k)$.

Theorem 1. *Let positive scalars* α, $\lambda \triangleq \sum_{i=1}^{l} \delta_{m_i,n_i}$, $\beta(k)$ *and the filter gains* $F(k)$, $K(k)$ *be given. If there exists a set of real-valued matrices* $\Xi(k) = \Xi^T(k) \ge 0$ $(0 \le k \le N-1)$ *satisfying the following Riccati-like difference equation (9) under the constraint (10) and the initial condition (11):*

$$\begin{aligned}
&\Xi(k+1)\\
\triangleq&(1+\alpha)\tilde{A}(k)(\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}\tilde{A}^T(k)\\
&+ (1+\alpha)\beta^{-1}(k)\tilde{\Gamma}(k)\tilde{\Gamma}^T(k) + \lambda(1+\alpha^{-1})\tilde{K}(k)\tilde{K}^T(k)\\
&+ \tilde{D}(k)(\bar{\gamma}(k)\bar{R}(k)\bar{\gamma}(k) - \tilde{\gamma}(k)\bar{R}(k)\tilde{\gamma}(k))\tilde{D}^T(k)\\
&+ \tilde{G}(k)\bar{S}(k)\tilde{G}^T(k)\\
\triangleq&g_k(\Xi(k)),
\end{aligned} \tag{9}$$

$$\beta^{-1}(k)I - \tilde{Q}(k)\Xi(k)\tilde{Q}^T(k) > 0, \tag{10}$$

$$\Xi(0) \triangleq \mathbb{E}\{\tilde{x}(0)\tilde{x}^T(0)\}, \tag{11}$$

then we obtain

$$P(k) \le \Theta(k) \triangleq [I \quad -I]\Xi(k)[I \quad -I]^T. \tag{12}$$

Proof. See the Appendix A. □

Having obtained an upper bound of filtering error covariance, we are now in a position to deal with the design problem of the filter gains $F(k)$ and $K(k)$. For the convenience of design, the matrices $\Xi(k)$ are rewritten as the following form

$$\Xi(k) = \begin{bmatrix} \Xi_{11}(k) & \Xi_{12}(k) \\ \Xi_{12}^T(k) & \Xi_{22}(k) \end{bmatrix}.$$

Now, according to the results obtained above, we are going to minimize the upper bound $\Theta(k+1)$ in Theorem 1 and obtain the following theorem by designing the desired filter gains $F(k)$ and $K(k)$.

Theorem 2. *Let positive scalars* α, $\lambda \triangleq \sum_{i=1}^{l} \delta_{m_i,n_i}$ *and* $\beta(k)$ *be given. For* $0 \le k \le N-1$*, Assume that the matrices* $\Xi(k)$ *are the solution to (9) subject to (10) and (11), and the real-valued matrices* $\Pi_1(k)$, $\Pi_2(k)$, $\Pi_3(k)$ *and* $\Pi_4(k)$ *satisfy the following conditions*

$$\begin{aligned}
\Pi_1(k) \triangleq&\,\Xi_{12}(k) - \Xi_{22}(k) - (\Xi_{11}(k) - \Xi_{12}^T(k))\bar{Q}^T(k)\\
&\times(\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\bar{Q}(k)\Xi_{12}(k),\\
\Pi_2(k) \triangleq&\,\Xi_{12}^T(k)\bar{Q}^T(k)(\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
&\times\bar{Q}(k)\Xi_{12}(k) - \Xi_{22}(k),\\
\Pi_3(k) \triangleq&\,\Xi_{11}(k) - \Xi_{12}^T(k) - \Xi_{12}(k) + \Xi_{22}(k)\\
&+ (\Xi_{11}(k) - \Xi_{12}^T(k))\bar{Q}^T(k)\\
&\times(\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
&\times\bar{Q}(k)(\Xi_{11}(k) - \Xi_{12}(k)),\\
\Pi_4(k) \triangleq&\,\Xi_{22}(k) - \Xi_{12}^T(k) - \Xi_{12}^T(k)\bar{Q}^T(k)\\
&\times(\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
&\times\bar{Q}(k)(\Xi_{11}(k) - \Xi_{12}(k)).
\end{aligned} \tag{13}$$

Then, the filter gains $F(k)$ *and* $K(k)$ *that minimize the upper bound* $\Theta(k+1)$ *at each iteration are given as follows:*

$$F(k) = \bar{A}(k) + (\bar{A}(k) - K(k)\bar{C}(k))\Pi_1(k)\Pi_2^{-1}(k), \tag{14}$$

$$\begin{aligned}
K(k) =&(1+\alpha)\bar{A}(k)(\Pi_3(k) + \Pi_1(k)\Pi_2^{-1}(k)\Pi_4(k))\\
&\times\bar{C}^T(k)\Omega^{-1}(k)
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\Omega(k) \triangleq&(1+\alpha)\bar{C}(k)(\Pi_3(k) + \Pi_1(k)\Pi_2^{-1}(k)\Pi_4(k))\bar{C}^T(k)\\
&+ \bar{D}(k)(\bar{\gamma}(k)\bar{R}(k)\bar{\gamma}(k) - \tilde{\gamma}(k)\bar{R}(k)\tilde{\gamma}(k))\bar{D}^T(k)\\
&+ \lambda(1+\alpha^{-1}).
\end{aligned}$$

Proof. See the Appendix B. □

Until now, we have derived an upper bound for the filtering error covariance, and the filter gains $F(k)$ and $K(k)$ given in (14) and (15) minimize such an upper bound on the filtering error covariance at each iteration. The addressed filter design process can be summarised in the following Table 1.

Table 1
The filter design algorithm.

Step 1	Give positive scalars α, δ_{m_i,n_i} $(i = 1, 2, \ldots, l)$ and $\beta(0)$ satisfying (10), and set the initial conditions as $\hat{x}(0) = \mathbb{E}\{x(0)\}$ and $\Xi(0) = \mathbb{E}\{\tilde{x}(0)\tilde{x}^T(0)\}$;
Step 2	For the sampling instant k, calculate the filter parameter matrices $F(k)$ and $K(k)$ from (14) and (15), respectively;
Step 3	Compute the state estimate $\hat{x}(k+1)$ from the filter structure (6) and $\Xi(k+1)$ obtained from the Riccati-like difference equation (9), and set $k = k+1$ and a positive scalar $\beta(k+1)$ satisfying (10);
Step 4	If $k < N$, then go to Step 2, else go to Step 5;
Step 5	Stop.

Remark 2. In Theorems 1 and 2, an upper bound is constructed and the robust recursive filter is designed to minimize such an upper bound for systems (1) with an event-based communication mechanism. Intuitively, the increase of the threshold δ_{m_i,n_i} (and therefore λ) would lead to less transmission frequency for more energy-saving, but this is likely to result in a larger upper bound

of the filtering error covariance. Clearly, there are trade-offs between the energy-saving in the event triggered scheme and the upper bound expressed in terms of the solution to Riccati-like difference equations.

Remark 3. In Theorems 1 and 2, we have solved the event-triggered filtering problem for a class of uncertain time-varying spatial-temporal systems by solving certain constrained Riccati-like recursive equations, in which all the information about the systems (i.e. system parameters from both space and time, bounds on the parameter uncertainties, thresholds on the event-triggers) have been reflected. For the addressed spatial-temporal systems, we have obtained the existence conditions for the desired robust event-triggered filter with a guaranteed upper bound on the filtering error variance, and then we have designed the (locally) optimal filter gain to minimize such upper bound at each iteration. Comparing to the existing literature, the distinctive novelties lie mainly in the consideration of spatial-temporal systems as well as the event-triggering scheme in the robust filter design.

4. An illustrative example

In this section, we provide a numerical simulation example to validate the filtering algorithm proposed in this paper.

Given the region [0, 3] × [0, 3], the parameters of the underlying systems (1) with the sensor measurement model (3) are set as follows:

$$\kappa(k) = \begin{bmatrix} 0.02+0.01\cos(k) & 0.002 \\ 0 & 0.01 \end{bmatrix}, \quad G(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$A(k) = \begin{bmatrix} 0.02 & 0.01 \\ 0.02 & 0.01+0.01\sin(k) \end{bmatrix}, \quad \Gamma(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$B(k) = \begin{bmatrix} 0.02+0.01\sin(k) & 0 \\ 0.01 & 0.03 \end{bmatrix},$$

$$C_{m_1,n_1}(k) = C_{1,1}(k) = [0.9+0.1\cos(k) \quad 0.8],$$
$$C_{m_2,n_2}(k) = C_{2,1}(k) = [0.7 \quad 1+\sin(k)],$$
$$C_{m_3,n_3}(k) = C_{1,2}(k) = [0.6+0.1\sin(k) \quad 0.9],$$
$$C_{m_4,n_4}(k) = C_{2,2}(k) = [1.2 \quad 0.9+0.1\cos(k)],$$
$$D_{m_1,n_1}(k) = D_{1,1}(k) = 1, \quad D_{m_2,n_2}(k) = D_{2,1}(k) = 1.5,$$
$$D_{m_3,n_3}(k) = D_{1,2}(k) = 2, \quad D_{m_4,n_4}(k) = D_{2,2}(k) = 1,$$
$$Q_1(k) = [0.01\cos(k) \quad 0.02],$$
$$Q_2(k) = [0.01 \quad 0.05\sin(k)],$$
$$Q_3(k) = [0.02\cos(k) \quad 0.01].$$

The event generator function parameters δ_{m_1,n_1}, δ_{m_2,n_2}, δ_{m_3,n_3} and δ_{m_4,n_4} are taken as 0.1, 0.11, 0.12 and 0.08, respectively. α is selected as 1. With the above parameters, the Riccati-like difference equation (9) is solved by using the Matlab software and, according to Theorem 2, the desired filter gains $F(k)$ and $K(k)$ can be obtained iteratively.

In the simulation, the covariances for $w_{m,n}(k)$ and $v_{m_i,n_i}(k)$ are selected as $S_{m,n}(k) = 0.1$ and $R_{m_i,n_i}(k) = 0.1$, the boundary condition is given as $x_{0,0}(k) = x_{0,3}(k) = x_{3,0}(k) = x_{3,3}(k) = 0$ and the initial covariances at the inner points are chosen as $U^0_{1,1} = U^0_{1,2} = U^0_{2,1} = U^0_{2,2} = \text{diag}\{0.1, 0.1\}$.

The simulation results are shown in Figs. 1–5. Figs. 1-2 depict the state trajectories and their estimates, which are written as $x_{m,n}(k) = \begin{bmatrix} x^1_{m,n}(k) & x^2_{m,n}(k) \end{bmatrix}^T$ and $\hat{x}_{m,n}(k) = \begin{bmatrix} \hat{x}^1_{m,n}(k) & \hat{x}^2_{m,n}(k) \end{bmatrix}^T$. Figs. 3-4 show an upper bound of the error covariance matrix for the state at the different positions. The triggering instants of each sensor node can be seen in Fig. 5. The simulation results have illustrated the feasibility of the proposed filter in this paper.

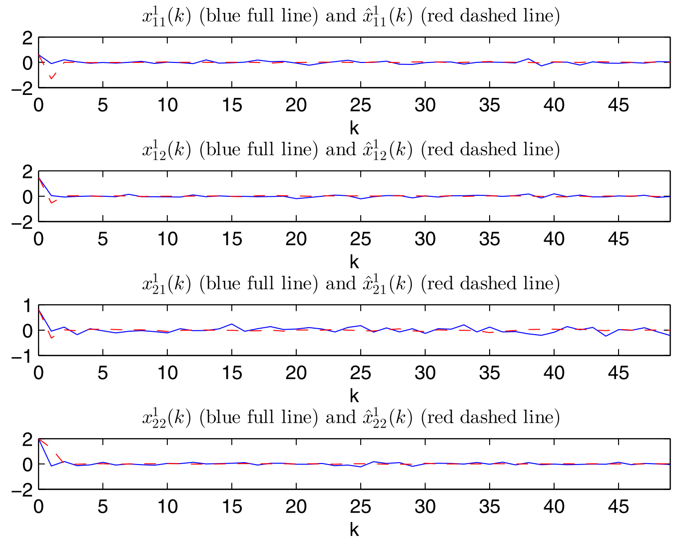

Fig. 1. State $x^1_{m,n}(k)$ and its estimation $\hat{x}^1_{m,n}(k)$.

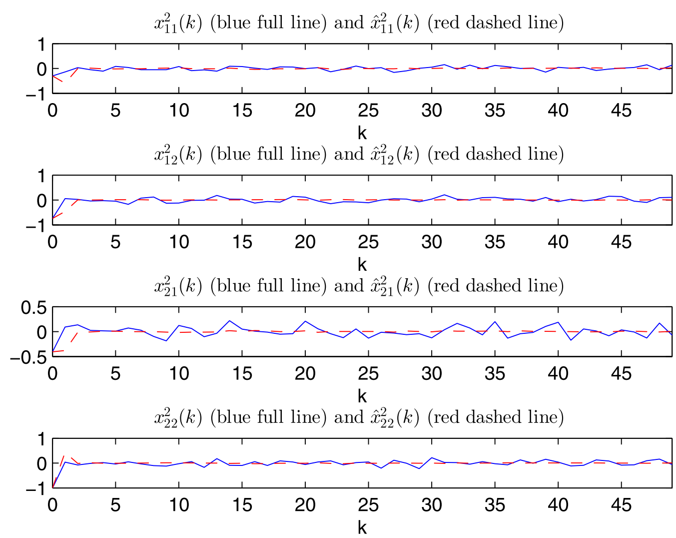

Fig. 2. State $x^2_{m,n}(k)$ and its estimation $\hat{x}^2_{m,n}(k)$.

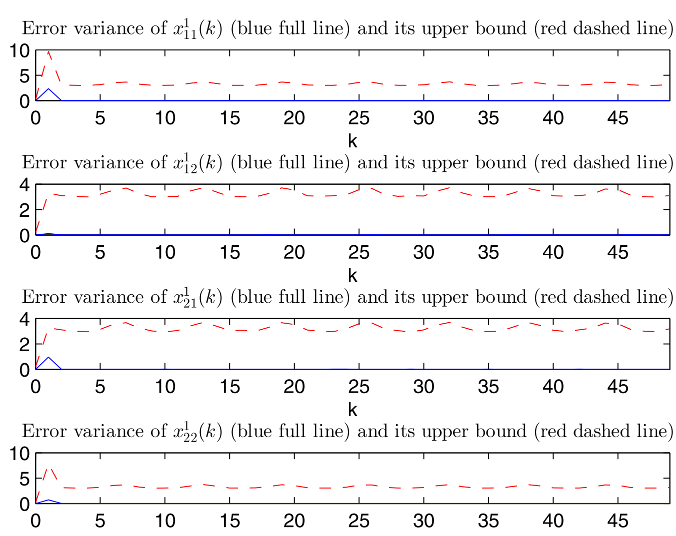

Fig. 3. Error variance of $x^1_{m,n}(k)$ and its upper bound.

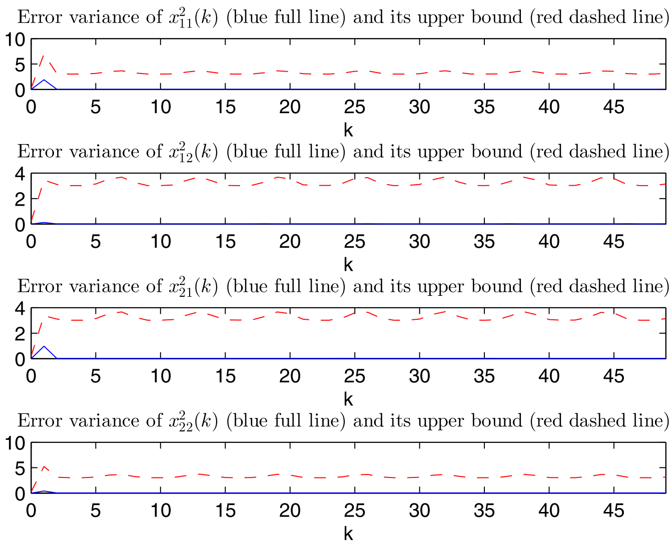

Fig. 4. Error variance of $x_{m,n}^2(k)$ and its upper bound.

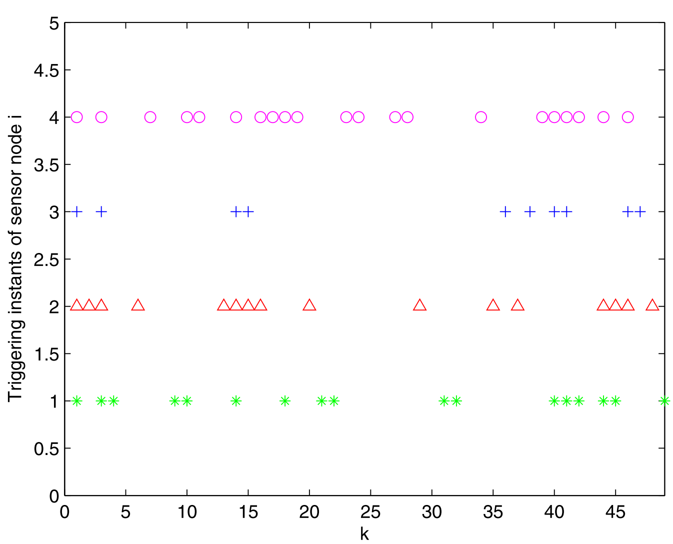

Fig. 5. Triggering instants of each sensor node i.

5. Conclusion

In this paper, the robust recursive filtering problem has been studied for a class of uncertain stochastic time-varying discrete spatial-temporal systems. The event-triggered mechanism has been taken into consideration to save communication energy cost. Based on the triggering information measured by sensor nodes, the desired robust recursive filtering scheme has been designed according to the solution of Riccati-like difference equations. Such a filter has guaranteed the bounded filtering error covariance and an upper bound is minimized at each iteration. Finally, the filtering performance of the proposed filtering scheme has been shown via a numerical example. Future research topics would include the extension of the main results to more complicated systems, see [4,16,19,21,23,25,29,33,45–47].

Acknowledgements

This work was supported in part by the Royal Society of the U.K., the National Natural Science Foundation of China under Grants 61329301, 61473076 and 61703093, the Program for Professor of Special Appointment (Eastern Scholar) at Shanghai Institutions of Higher Learning of China, and the Alexander von Humboldt Foundation of Germany.

Appendix A

To prove Theorem 1, for the covariance matrix $\tilde{\Xi}(k) \triangleq \mathbb{E}\{\tilde{x}(k)\tilde{x}^T(k)\}$, we have the following equation that governs its evolution:

$$
\begin{aligned}
\tilde{\Xi}(k+1) &= \mathbb{E}\{\tilde{x}(k+1)\tilde{x}^T(k+1)\} \\
&= (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\mathbb{E}\{\tilde{x}(k)\tilde{x}^T(k)\} \\
&\quad \times (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))^T + \tilde{K}(k)\mathbb{E}\{e(k)e^T(k)\}\tilde{K}^T(k) \\
&\quad + \tilde{G}(k)\mathbb{E}\{w(k)w^T(k)\}\tilde{G}^T(k) + \tilde{D}(k)\mathbb{E}\{v(k)v^T(k)\}\tilde{D}^T(k) \\
&\quad + \Psi_{1k} + \Psi_{1k}^T + \Psi_{2k} + \Psi_{2k}^T + \Psi_{3k} + \Psi_{3k}^T + \Psi_{4k} + \Psi_{4k}^T \\
&\quad + \Psi_{5k} + \Psi_{5k}^T + \Psi_{6k} + \Psi_{6k}^T
\end{aligned} \tag{16}
$$

where

$$
\begin{aligned}
\Psi_{1k} &\triangleq (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\mathbb{E}\{\tilde{x}(k)e^T(k)\}\tilde{K}^T(k), \\
\Psi_{2k} &\triangleq (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\mathbb{E}\{\tilde{x}(k)w^T(k)\}\tilde{G}^T(k), \\
\Psi_{3k} &\triangleq (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\mathbb{E}\{\tilde{x}(k)v^T(k)\}\tilde{D}^T(k), \\
\Psi_{4k} &\triangleq \tilde{K}(k)\mathbb{E}\{e(k)w^T(k)\}\tilde{G}^T(k), \\
\Psi_{5k} &\triangleq \tilde{K}(k)\mathbb{E}\{e(k)v^T(k)\}\tilde{D}^T(k), \\
\Psi_{6k} &\triangleq \tilde{G}(k)\mathbb{E}\{w(k)v^T(k)\}\tilde{D}^T(k).
\end{aligned} \tag{17}
$$

From the following facts

$$
\begin{aligned}
&\mathbb{E}\{\tilde{x}(k)w^T(k)\} = 0, \quad \mathbb{E}\{\tilde{x}(k)v^T(k)\} = 0, \\
&\mathbb{E}\{e(k)w^T(k)\} = 0, \quad \mathbb{E}\{w(k)v^T(k)\} = 0,
\end{aligned} \tag{18}
$$

it can be easily derived that $\Psi_{2k} = 0$, $\Psi_{3k} = 0$, $\Psi_{4k} = 0$, $\Psi_{6k} = 0$. It should be mentioned that, we need to further calculate the expectations of some cross terms in Ψ_{1k} and Ψ_{5k} which are non-zero.

Based on the elementary inequality

$$
(\alpha^{1/2}M - \alpha^{-1/2}N)(\alpha^{1/2}M - \alpha^{-1/2}N)^T \geq 0
$$

where M and N are matrices with compatible dimensions, it can be easily known from (17) that

$$
\begin{aligned}
\Psi_{1k} + \Psi_{1k}^T \leq & \alpha(\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\mathbb{E}\{\tilde{x}(k)\tilde{x}^T(k)\} \\
& \times (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))^T \\
& + \alpha^{-1}\tilde{K}(k)\mathbb{E}\{e(k)e^T(k)\}\tilde{K}^T(k).
\end{aligned} \tag{19}
$$

On the other hand, it is not difficult to see that the following inequality

$$
e^T(k)e(k) \leq \lambda
$$

is always true. Then, by using the properties of matrix operations, we obtain

$$
e(k)e^T(k) \leq \|e(k)\|^2 I = e^T(k)e(k)I \leq \lambda I,
$$

and hence

$$
\mathbb{E}\{e(k)e^T(k)\} \leq \lambda I. \tag{20}
$$

According to

$$
e(k) = y^t(k) - y(k) = y^t(k) - (\bar{C}(k)x(k) + \bar{D}(k)v(k)),
$$

we have

$$
\begin{aligned}
\mathbb{E}\{e(k)v^T(k)\} =& \mathbb{E}\{(y^t(k) - (\bar{C}(k)x(k) + \bar{D}(k)v(k)))v^T(k)\} \\
=& -\bar{D}(k)\bar{\gamma}(k)\mathbb{E}\{v(k)v^T(k)\}.
\end{aligned} \tag{21}
$$

By noting $\tilde{K}(k)\bar{D}(k) = \tilde{D}(k)$ and (21), it is easily known that

$$
\Psi_{5k} + \Psi_{5k}^T = \tilde{D}(k)\bar{\gamma}(k)\mathbb{E}\{v(k)v^T(k)\}\bar{\gamma}(k)\bar{D}^T(k)
$$

$$
\begin{aligned}
&- \tilde{D}(k)\tilde{\gamma}(k)\mathbb{E}\{v(k)v^T(k)\}\tilde{\gamma}(k)\tilde{D}^T(k)\\
&- \tilde{D}(k)\mathbb{E}\{v(k)v^T(k)\}\tilde{D}^T(k). \qquad (22)
\end{aligned}
$$

Furthermore, it follows from (18)–(20), (22) and the facts of $\mathbb{E}\{w(k)w^T(k)\} = \bar{S}(k)$ and $\mathbb{E}\{v(k)v^T(k)\} = \bar{R}(k)$ that

$$
\begin{aligned}
\tilde{\Xi}(k+1) \le\ & (1+\alpha)(\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\mathbb{E}\{\tilde{x}(k)\tilde{x}^T(k)\}\\
& \times (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))^T + \tilde{G}(k)\mathbb{E}\{w(k)w^T(k)\}\tilde{G}^T(k)\\
& + \lambda(1+\alpha^{-1})\tilde{K}(k)\tilde{K}^T(k) + \tilde{D}(k)(\bar{\gamma}(k)\mathbb{E}\{v(k)v^T(k)\}\bar{\gamma}(k)\\
& - \tilde{\gamma}(k)\mathbb{E}\{v(k)v^T(k)\}\tilde{\gamma}(k))\tilde{D}^T(k)\\
=\ & (1+\alpha)(\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\tilde{\Xi}(k)\\
& \times (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))^T\\
& + \tilde{D}(k)(\bar{\gamma}(k)\bar{R}(k)\bar{\gamma}(k) - \tilde{\gamma}(k)\bar{R}(k)\tilde{\gamma}(k))\tilde{D}^T(k)\\
& + \lambda(1+\alpha^{-1})\tilde{K}(k)\tilde{K}^T(k) + \tilde{G}(k)\bar{S}(k)\tilde{G}^T(k) \qquad (23)
\end{aligned}
$$

Now, we denote $\bar{\Xi}(k)$ (with the initial condition $\tilde{\Xi}(0) = \bar{\Xi}(0)$) and the function $h_k(\bar{\Xi}(k))$ as follows:

$$
\begin{aligned}
\bar{\Xi}(k+1) \triangleq\ & h_k(\bar{\Xi}(k))\\
\triangleq\ & (1+\alpha)(\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))\bar{\Xi}(k)\\
& \times (\tilde{A}(k) + \tilde{\Gamma}(k)\bar{E}(k)\tilde{Q}(k))^T\\
& + \tilde{D}(k)(\bar{\gamma}(k)\bar{R}(k)\bar{\gamma}(k) - \tilde{\gamma}(k)\bar{R}(k)\tilde{\gamma}(k))\tilde{D}^T(k)\\
& + \lambda(1+\alpha^{-1})\tilde{K}(k)\tilde{K}^T(k) + \tilde{G}(k)\bar{S}(k)\tilde{G}^T(k),
\end{aligned}
$$

then we derive from Lemma 2 that

$$\tilde{\Xi}(k+1) \le \bar{\Xi}(k+1). \qquad (24)$$

Subsequently, we consider the fact that $\Xi(k)$ satisfies $\Xi(0) = \bar{\Xi}(0)$ and $\Xi(k+1) = g_k(\Xi(k))$. By using (10) and Lemma 1, it can be easily checked that functions $h_k(\cdot)$ and $g_k(\cdot)$ satisfy the conditions in Lemma 3, which implies

$$\bar{\Xi}(k+1) \le \Xi(k+1). \qquad (25)$$

Finally, by considering (24) and (25), we can induce that

$$
\begin{aligned}
P(k) &= \mathbb{E}\{(x(k) - \hat{x}(k))(x(k) - \hat{x}(k))^T\}\\
&= [I \quad -I]\tilde{\Xi}(k)[I \quad -I]^T \le [I \quad -I]\bar{\Xi}(k)[I \quad -I]^T\\
&\le [I \quad -I]\Xi(k)[I \quad -I]^T = \Theta(k).
\end{aligned}
$$

Therefore, the proof of Theorem 1 is now complete.

In the following, we will prove Theorem 2. From (9), it is known that

$$
\begin{aligned}
\Theta(k+1) =\ & [I \quad -I]\Xi(k+1)[I \quad -I]^T\\
=\ & \Xi_{11}(k) - \Xi_{12}^T(k) - \Xi_{12}(k) + \Xi_{22}(k)\\
=\ & (1+\alpha)\beta^{-1}(k)\bar{\Gamma}(k)\bar{\Gamma}^T(k) + \lambda(1+\alpha^{-1})K(k)K^T(k)\\
& + K(k)\bar{D}(k)(\bar{\gamma}(k)\bar{R}(k)\bar{\gamma}(k) - \tilde{\gamma}(k)\bar{R}(k)\tilde{\gamma}(k))\bar{D}^T(k)K^T(k)\\
& + (1+\alpha)\big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - F(k)\big]\\
& \times (\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}\\
& \times \big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - F(k)\big]^T\\
& + \bar{G}(k)\bar{S}(k)\bar{G}^T(k). \qquad (26)
\end{aligned}
$$

In order to determine optimal filter parameters $F(k)$ and $K(k)$ that minimize $\Theta(k+1)$, we take the partial derivative of (26) with respect to $F(k)$ and $K(k)$, respectively, and it follows that

$$
\begin{aligned}
& 2(1+\alpha)\big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - F(k)\big]\\
& \times (\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}[0 \quad -I]^T = 0, \qquad (27)
\end{aligned}
$$

and

$$
\begin{aligned}
& 2K(k)\bar{D}(k)(\bar{\gamma}(k)\bar{R}(k)\bar{\gamma}(k) - \tilde{\gamma}(k)\bar{R}(k)\tilde{\gamma}(k))\bar{D}^T(k)\\
& + 2(1+\alpha)\big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - F(k)\big]\\
& \times (\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}\big[-\bar{C}(k) \quad \bar{C}(k)\big]^T\\
& + 2\lambda(1+\alpha^{-1})K(k) = 0. \qquad (28)
\end{aligned}
$$

By using Lemma 4, it is easily seen that

$$
\begin{aligned}
& (\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}\\
& = \Xi(k)\tilde{Q}^T(k)(\beta^{-1}(k)I - \tilde{Q}(k)\Xi(k)\tilde{Q}^T(k))^{-1}\\
& \times \tilde{Q}(k)\Xi(k) + \Xi(k). \qquad (29)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
& \big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - F(k)\big]\\
& = \big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - \bar{A}(k) + \bar{A}(k) - F(k)\big]\\
& = (\bar{A}(k) - K(k)\bar{C}(k))[I \quad -I]\\
& + (\bar{A}(k) - F(k))[0 \quad I]. \qquad (30)
\end{aligned}
$$

Then, by noting (27), (29), (30) and the facts that

$$
\begin{aligned}
\Pi_1(k) =\ & \Xi_{12}(k) - \Xi_{22}(k) - (\Xi_{11}(k) - \Xi_{12}^T(k))\bar{Q}^T(k)\\
& \times (\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
& \times \bar{Q}(k)\Xi_{12}(k),\\
=\ & [I \quad -I](\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}[0 \quad -I]^T\\
\Pi_2(k) =\ & \Xi_{12}^T(k)\bar{Q}^T(k)(\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
& \times \bar{Q}(k)\Xi_{12}(k) - \Xi_{22}(k)\\
=\ & [0 \quad I](\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}[0 \quad -I]^T,
\end{aligned}
$$

the optimal filter parameter $F(k)$ is determined as follows:

$$F(k) = \bar{A}(k) + (\bar{A}(k) - K(k)\bar{C}(k))\Pi_1(k)\Pi_2^{-1}(k). \qquad (31)$$

Next, our task is to derive another optimal filter parameter $K(k)$. It follows from (13) that

$$
\begin{aligned}
\Pi_3(k) =\ & \Xi_{11}(k) - \Xi_{12}^T(k) - \Xi_{12}(k) + \Xi_{22}(k)\\
& + (\Xi_{11}(k) - \Xi_{12}^T(k))\bar{Q}^T(k)\\
& \times (\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
& \times \bar{Q}(k)(\Xi_{11}(k) - \Xi_{12}(k))\\
=\ & [I \quad -I](\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}[I \quad -I]^T,\\
\Pi_4(k) =\ & \Xi_{22}(k) - \Xi_{12}^T(k) - \Xi_{12}^T(k)\bar{Q}^T(k)\\
& \times (\beta^{-1}(k)I - \bar{Q}(k)\Xi_{11}(k)\bar{Q}^T(k))^{-1}\\
& \times \bar{Q}(k)(\Xi_{11}(k) - \Xi_{12}(k))\\
=\ & [0 \quad -I](\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}\\
& \times [I \quad -I]^T. \qquad (32)
\end{aligned}
$$

Then, by noting (31) and (32), we have

$$
\begin{aligned}
& \big[\bar{A}(k) - K(k)\bar{C}(k) \quad K(k)\bar{C}(k) - F(k)\big]\\
& \quad \times (\Xi^{-1}(k) - \beta(k)\tilde{Q}^T(k)\tilde{Q}(k))^{-1}[I \quad -I]^T\\
& \quad = (\bar{A}(k) - K(k)\bar{C}(k))\Pi_3(k) - (\bar{A}(k) - F(k))\Pi_4(k)\\
& \quad = -[\bar{A}(k) - (\bar{A}(k) + (\bar{A}(k) - K(k)\bar{C}(k))\Pi_1(k)\Pi_2^{-1}(k))]\\
& \quad \times \Pi_4(k) + (\bar{A}(k) - K(k)\bar{C}(k))\Pi_3(k)\\
& \quad = (\bar{A}(k) - K(k)\bar{C}(k))(\Pi_3(k) + \Pi_1(k)\Pi_2^{-1}(k)\Pi_4(k)). \qquad (33)
\end{aligned}
$$

Finally, substituting (33) to (28), the optimal filter parameter $K(k)$ is given by

$$
\begin{aligned}
K(k) =\ & (1+\alpha)\bar{A}(k)(\Pi_3(k) + \Pi_1(k)\Pi_2^{-1}(k)\Pi_4(k))\\
& \times \bar{C}^T(k)\Omega^{-1}(k), \qquad (34)
\end{aligned}
$$

which ends the proof.

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