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Set-membership filtering for polytopic uncertain discrete-time systems

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In this paper, a set-membership filtering problem is considered for systems with polytopic uncertainty. A recursive algorithm for calculating an ellipsoid which always contains the state is developed. In the prediction step, a predicted state ellipsoid is determined; in the update step, a state estimation ellipsoid is computed by combining the predicted state ellipsoid and the set of states compatible with the measurement equation. A smallest possible estimate set is calculated recursively by solving the semi-definite programming problems. Hence, the proposed set-membership filter relies on a two-step prediction–correction structure, which is similar to the Kalman filter. Simulation results are provided to demonstrate the effectiveness of the proposed method.

Keywords: set-membership filtering; predicted state ellipsoid; state estimation ellipsoid; polytopic uncertainty; semi-definite programming

1. Introduction

The filtering problem plays an important role in target tracking, image processing, signal processing, and control engineering (Anderson and Moore 1979). It is now well known that the Kalman filter requires the process noise and measurement noise to be white Gaussian processes (Yang *et al.* 2002). Hence, the Kalman filter may lead to poor performance with non-Gaussian noises. Recently, the H_∞ filtering method has been proposed, which provides a bounded gain of energy for the worst-case estimation error without the need for knowledge of noise statistics (Yang and Hung 2002). In this filtering, process and measurement noises are assumed to be arbitrary rather than Gaussian processes. However, there is no provision in H_∞ filtering to ensure that the variance of the state estimation error lies within acceptable bounds (Yang and Hung 2002). In this respect, it is natural to consider process and measurement noises as unknown but bounded, which belong to given sets in appropriate vector spaces (Schweppe 1968, Morrell and Stirling 1991). All possible state estimates can be characterized by *the set of state estimates* consistent with both the observations received and the constraints on the unknown process and measurement noises, and the true state is contained in this set of state estimates. Thus the actual estimate is a set in state space rather than a single vector. This estimation problem has been referred to as a set-membership (set-value) filtering problem (Bertsekas and Rhodes 1971, Morrell and Stirling 1991, Combettes 1993, Kurzhanski and Valyi 1996, Maskarov and Norton 1996, Shamma and Tu 1999).

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The set-membership filtering problem was first considered by Witsenhausen (1968). The set of all possible values of the states compatible with the observation of outputs is completely characterized by their support functions. An ellipsoidal approximation algorithm with certain computational advantages was provided by Schweppe (1968). In this algorithm, the observations are used to calculate recursively a bounding ellipsoid to the set of possible states, under the assumption that the sets containing the initial condition and the input and observation noises are or can be approximated by ellipsoids. The solution to a set-membership filtering problem with the instantaneous constraints was determined using the results derived for an energy constraint in Bertsekas and Rhodes (1971). The resulting estimator is similar to that proposed by Schweppe (1968), but it has an important advantage in that the gain matrix does not depend on the particular output observations and is, therefore, precomputable. Recently, one has attempted to deal with the set-membership filtering problems for uncertain systems. The different uncertain systems have been considered. For example, a combinational ellipsoidal constraint of the uncertain system matrix and uncertain process noise was introduced for set-membership filtering in Polyak *et al.* (2002). A recursive scheme for constructing an ellipsoidal state estimation set of all states consistent with the measured output, the noises, and unstructured uncertainty was described by a sum quadratic constraint in Savkin and Petersen (1995, 1998) and Ra *et al.* (2004). A convex optimization approach has been applied to the case of norm-bounded uncertainty in the system matrices to provide a set of state estimates in El Ghaoui and Calafiore (1999a,b, 2001).

In this paper, the systems under consideration have polytopic uncertainty. Polytopic uncertainty is probably the most general way of capturing the structured uncertainty that may affect the system parameters. It includes the well-known interval parametric uncertainty (Bhattacharyya *et al.* 1995). We adopt S-procedure technique to determine a *predicted state ellipsoid* for the polytopic uncertain systems subject to unknown-but-bounded process noise. We employ S-procedure and projection techniques to compute a *state estimation ellipsoid* by combining the predicted state ellipsoid and the set of states compatible with the measurement equation subject to unknown-but-bounded measurement noise. A recursive algorithm for calculating an ellipsoid which always contains the state is developed. In each step, the ellipsoid is minimized in some sense by solving the semi-definite programming (SDP) problems. This ellipsoid is a smallest possible estimate set which can be calculated recursively in real time.

The remainder of this paper is organized as follows. The robust set-membership filter design problem is formulated in Section 2 for polytopic uncertain discrete-time systems. A novel algorithm for computing the predicted state ellipsoid and state estimation ellipsoid is developed in Section 3. Section 4 provides an illustrative example to demonstrate the effectiveness of our algorithm. Conclusions are drawn in Section 5.

NOTATION. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). The superscript T stands for matrix transposition. For a matrix U, $U_{\perp}$ denotes any orthogonal complement of U, i.e. a matrix of maximal rank such that $UU_{\perp} = 0$.

2. Problem formulation

Consider the following discrete-time polytopic uncertain system:

$$x_{k+1} = A_k(\alpha)x_k + L_k(\alpha)u_k + B_k(\alpha)w_k, \tag{1}$$

$$y_k = C_k(\alpha)x_k + D_k(\alpha)v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state, $y_k \in \mathbb{R}^m$ is the measurement output, $w_k \in \mathbb{R}^r$ is the process noise, and $v_k \in \mathbb{R}^p$ is the measurement noise.

It is assumed that the initial state x_0 belongs to a given ellipsoid:

$$x_0 \in \mathcal{E}(P_0, \hat{x}_0) = \{x_0 : (x_0 - \hat{x}_0)^{\mathrm{T}} P_0^{-1}(x_0 - \hat{x}_0) \leq 1, P_0 > 0\}, \tag{3}$$

w_k and v_k are assumed unknown-but-bounded noise signals at each time step k, which are assumed to belong to the following given ellipsoids:

$$w_k \in \mathcal{E}(Q_k, 0) = \{w_k : w_k^{\mathrm{T}} Q_k^{-1} w_k \leq 1, Q_k > 0\}, \tag{4}$$

$$v_k \in \mathcal{E}(R_k, 0) = \{v_k : v_k^{\mathrm{T}} R_k^{-1} v_k \leq 1, R_k > 0\}, \tag{5}$$

where $\hat{x}_0$ is known and P_0, Q_k, and R_k are known matrices with compatible dimensions.

The matrices $A_k(\alpha)$, $B_k(\alpha)$, $C_k(\alpha)$, $D_k(\alpha)$, and $L_k(\alpha)$ are unknown time-varying parameters with appropriate dimensions. We assume that $(A_k(\alpha), B_k(\alpha), C_k(\alpha), D_k(\alpha), L_k(\alpha)) \in \Omega$, where Ω is a convex polyhedral set described by N vertices

$$\begin{aligned}\Omega = \{&(A_k(\alpha), B_k(\alpha), C_k(\alpha), D_k(\alpha), L_k(\alpha)) \\ &= \sum_{i=1}^{N} \alpha_i \big(A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)}, L_k^{(i)}\big), \quad \sum_{i=1}^{N} \alpha_i = 1, \quad \alpha_i \geq 0\big\},\end{aligned} \tag{6}$$

where $(A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)}, L_k^{(i)})$ are known for all $i = 1, 2, \ldots, N$.

Remark 1. As compared with the norm-bounded uncertainty in El Ghaoui and Calafiore (2001), the polytopic considered in this paper is more flexible. Polytopic uncertainty is probably the most general way of capturing the structured uncertainty that may affect the system parameters. It includes the well-known interval parametric uncertainty (Bhattacharyya *et al.* 1995).

Our objective is to determine a confidence ellipsoid $\mathcal{E}(P_k, \hat{x}_k)$ for the state x_k, given the measurement information y_k at the time instant k for all unknown matrices $(A_k(\alpha), B_k(\alpha), C_k(\alpha), D_k(\alpha), L_k(\alpha)) \in \Omega$, the process noise $w_k \in \mathcal{E}(Q_k, 0)$, the measurement noise $v_k \in \mathcal{E}(R_k, 0)$, and the initial state $x_0 \in \mathcal{E}(P_0, \hat{x}_0)$, i.e. we look for P_k and $\hat{x}_k$ such that

$$(x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \hat{x}_k) \leq 1 \tag{7}$$

subject to $w_k \in \mathcal{E}(Q_k, 0)$, $v_k \in \mathcal{E}(R_k, 0)$, $x_0 \in \mathcal{E}(P_0, \hat{x}_0)$ and (1) and (2) with (6).

The above filtering problem is referred to as the robust set-membership filtering problem.

3. Robust set-membership filter design

In this section, a robust set-membership filter will be designed for discrete-time polytopic uncertain systems subject to any unknown-but-bounded process noise and measurement noise. In order to derive the filter, we need the following three useful lemmas:

Lemma 3.1. (S-procedure) (Boyd *et al.* 1994) Let $F_0(\eta), F_1(\eta), \ldots, F_p(\eta)$ be quadratic functions of $\eta \in \mathbb{R}^n$

$$F_i(\eta) = \eta^{\mathrm{T}} T_i \eta + 2u_i^{\mathrm{T}} \eta + v_i, \quad i = 0, \ldots, p, \tag{8}$$

with $T_i = T_i^{\mathrm{T}}$. Then, the implication

$$F_1(\eta) \le 0, \ldots, F_p(\eta) \le 0 \Rightarrow F_0(\eta) \le 0 \tag{9}$$

holds if there exist $\tau_1, \ldots, \tau_p \ge 0$ such that

$$F_0(\eta) - \sum_{i=1}^{p} \tau_i F_i(\eta) \le 0, \quad \forall \eta. \tag{10}$$

When $p = 1$, condition (10) is also necessary for (9), provided that there exist some η_0 such that $F_i(\eta_0) < 0$. Notice also that, by homogenization, condition (10) is equivalent to

$$\exists \tau_1, \ldots, \tau_p \ge 0 \quad \text{such that} \begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \le 0. \tag{11}$$

Lemma 3.2. (Schur Complements) (Boyd *et al.* 1994) Given constant matrices $L_1,\ L_2,\ L_3$ where $L_1 = L_1^{\mathrm{T}}$ and $0 < L_2 = L_2^{\mathrm{T}}$, then

$$L_1 - L_3^{\mathrm{T}} L_2^{-1} L_3 \le 0,$$

if and only if

$$\begin{bmatrix} L_1 & L_3^{\mathrm{T}} \\ L_3 & L_2 \end{bmatrix} \le 0,$$

or equivalently

$$\begin{bmatrix} L_2 & L_3 \\ L_3^{\mathrm{T}} & L_1 \end{bmatrix} \le 0.$$

Lemma 3.3. (Finsler's lemma) (Skelton *et al.* 1998) Let $x \in \mathbb{R}^n$, $P = P^{\mathrm{T}} \in \mathbb{R}^{n \times n}$, and $M \in \mathbb{R}^{m \times n}$ such that $\operatorname{rank}(M) = r < n$. The following statements are equivalent:

(1) $x^{\mathrm{T}} P x \le 0,\ \ \forall Mx = 0,\ \ x \ne 0$.
(2) $(M^{\perp})^{\mathrm{T}} P M^{\perp} \le 0$.
(3) $\exists \mu \in \mathbb{R} : P - \mu M^{\mathrm{T}} M \le 0$.
(4) $\exists N \in \mathbb{R}^{m \times n} : P + N^{\mathrm{T}} M + M^{\mathrm{T}} N \le 0$.

Now, we apply Lemmas 3.1–3.3 and the convex combination approach proposed by de Oliviera *et al.* (1999), Peaucelle *et al.* (2000), and Xie *et al.* (2004) to compute the predicted state and state estimation ellipsoids.

Theorem 3.4. For the polytopic uncertain systems (1) and (2), the parameters of which reside in polytope Ω (6) with given vertices $A_k^{(i)}, B_k^{(i)}, C_k^{(i)}, D_k^{(i)}$, and $L_k^{(i)}$ $(i = 1, 2, \ldots, N)$, N is the number of the vertices, if x_k belongs to a current ellipsoid of confidence $\mathcal{E}(P_k, \hat{x}_k) = \left\{x_k : (x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1} (x_k - \hat{x}_k) \le 1, P_k > 0\right\}$, then a one-step-ahead prediction ellipsoid of confidence $\mathcal{E}(P_{k+1|k}, \hat{x}_{k+1|k}) = \{x_{k+1} : (x_{k+1} - \hat{x}_{k+1|k})^{\mathrm{T}} P_{k+1|k}^{-1} (x_{k+1} - \hat{x}_{k+1|k}) \le 1, P_{k+1|k} > 0\}$ can be obtained by solving the optimization problem:

$$\min_{P_{k+1|k} > 0,\, \hat{x}_{k+1|k},\, \tau_1 \ge 0,\, \tau_2 \ge 0} \quad \operatorname{trace}(P_{k+1|k}) \tag{12}$$

subject to

$$\begin{bmatrix} -P_{k+1|k} & \Phi_1^{(i)}(\hat{x}_{k+1|k}) \\ \Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}} & -\mathrm{diag}(1-\tau_1-\tau_2, \tau_1 I, \tau_2 Q_k^{-1}) \end{bmatrix} \leq 0, \tag{13}$$

for all $i \in \{1, 2, \ldots, N\}$, where

$$\Phi_1^{(i)}(\hat{x}_{k+1|k}) = \left[A_k^{(i)}\hat{x}_k - \hat{x}_{k+1|k} + L_k^{(i)}u_k A_k^{(i)}E_k B_k^{(i)}\right], \tag{14}$$

$A_k^{(i)}$, $B_k^{(i)}$, and $L_k^{(i)}$ are the matrices in (6) at the ith vertex of the polytope.

Proof. At a given time instant k, P_k and $\hat{x}_k$ are known. Hence, from that $(x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \hat{x}_k) \leq 1$, we have

$$x_k = \hat{x}_k + E_k z, \tag{15}$$

where E_k comes from $P_k = E_k E_k^{\mathrm{T}}$ by means of Cholesky factorization, and $\|z\| \leq 1$. Then the prediction error $x_{k+1} - \hat{x}_{k+1|k}$ for the vertex i is written as

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1|k} &= A_k^{(i)}x_k + L_k^{(i)}u_k + B_k^{(i)}w_k - \hat{x}_{k+1|k} \\ &= A_k^{(i)}\hat{x}_k - \hat{x}_{k+1|k} + L_k^{(i)}u_k + A_k^{(i)}E_k z + B_k^{(i)}w_k. \end{aligned} \tag{16}$$

Define

$$\eta = \begin{bmatrix} 1 \\ z \\ w_k \end{bmatrix}, \tag{17}$$

we can rewrite (16) as follows:

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1|k} &= \left[A_k^{(i)}\hat{x}_k - \hat{x}_{k+1|k} + L_k^{(i)}u_k A_k^{(i)}E_k B_k^{(i)}\right]\eta \\ &= \Phi_1^{(i)}(\hat{x}_{k+1|k})\eta, \end{aligned} \tag{18}$$

where

$$\Phi_1^{(i)}(\hat{x}_{k+1|k}) = \left[A_k^{(i)}\hat{x}_k - \hat{x}_{k+1|k} + L_k^{(i)}u_k A_k^{(i)}E_k B_k^{(i)}\right]. \tag{19}$$

Hence, $(x_{k+1} - \hat{x}_{k+1|k})^{\mathrm{T}} P_{k+1|k}^{-1}(x_{k+1} - \hat{x}_{k+1|k}) \leq 1$ can be written as

$$\eta^{\mathrm{T}}\,\mathrm{diag}(-1,0,0)\eta + \eta^{\mathrm{T}}\Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}}P_{k+1|k}^{-1}\Phi_1^{(i)}(\hat{x}_{k+1|k})\eta \leq 0. \tag{20}$$

Now, $\|z\| \leq 1$ and $w_k^{\mathrm{T}}Q_k^{-1}w_k \leq 1$ are also written as

$$\eta^{\mathrm{T}}\,\mathrm{diag}(-1, I, 0)\eta \leq 0, \tag{21}$$

$$\eta^{\mathrm{T}}\,\mathrm{diag}(-1, 0, Q_k^{-1})\eta \leq 0. \tag{22}$$

According to Lemma 3.1, the sufficient condition for the inequalities (20)–(22) to hold is that there exist non-negative scalars τ_1 and τ_2 such that

$$\begin{aligned} &\eta^{\mathrm{T}}\,\mathrm{diag}(-1,0,0)\eta + \eta^{\mathrm{T}}\Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}}P_{k+1|k}^{-1}\Phi_1^{(i)}(\hat{x}_{k+1|k})\eta \\ &\quad - \tau_1\eta^{\mathrm{T}}\,\mathrm{diag}(-1, I, 0)\eta - \tau_2\eta^{\mathrm{T}}\,\mathrm{diag}(-1, 0, Q_k^{-1})\eta \leq 0. \end{aligned} \tag{23}$$

A necessary and sufficient condition for (23) to hold for all η is

$$\operatorname{diag}(-1,0,0) + \Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}} P_{k+1|k}^{-1} \Phi_1^{(i)}(\hat{x}_{k+1|k}) \\ - \tau_1 \operatorname{diag}(-1, I, 0) - \tau_2 \operatorname{diag}(-1, 0, Q_k^{-1}) \leq 0, \tag{24}$$

i.e.

$$-\operatorname{diag}(1-\tau_1-\tau_2, \tau_1 I, \tau_2 Q_k^{-1}) + \Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}} P_{k+1|k}^{-1} \Phi_1^{(i)}(\hat{x}_{k+1|k}) \leq 0. \tag{25}$$

Using Schur complements in Lemma 3.2, (25) is equivalent to

$$\begin{bmatrix} -P_{k+1|k} & \Phi_1^{(i)}(\hat{x}_{k+1|k}) \\ \Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}} & -\operatorname{diag}(1-\tau_1-\tau_2, \tau_1 I, \tau_2 Q_k^{-1}) \end{bmatrix} \leq 0. \tag{26}$$

For all $i \in \{1, 2, \ldots, N\}$, we obtain the following result by the convex combination approach (de Oliviera *et al.* 1999, Peaucelle *et al.* 2000, Xie *et al.* 2004):

$$\begin{bmatrix} P_{k+1|k} & \Phi_1(\hat{x}_{k+1|k}) \\ \Phi_1(\hat{x}_{k+1|k})^{\mathrm{T}} & \operatorname{diag}(1-\tau_1-\tau_2, \tau_1 I, \tau_2 Q_k^{-1}) \end{bmatrix} \eta \\ = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} P_{k+1|k} & \Phi_1^{(i)}(\hat{x}_{k+1|k}) \\ \Phi_1^{(i)}(\hat{x}_{k+1|k})^{\mathrm{T}} & \operatorname{diag}(1-\tau_1-\tau_2, \tau_1 I, \tau_2 Q_k^{-1}) \end{bmatrix} \leq 0, \tag{27}$$

by noting that $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$, where

$$\Phi_1(\alpha)(\hat{x}_{k+1|k}) = \left[A_k(\alpha)\hat{x}_k - \hat{x}_{k+1|k} + L_k(\alpha)u_k A_k(\alpha)E_k B_k(\alpha)\right]. \tag{28}$$

□

Theorem 3.5. For the polytopic uncertain system (1) and (2), the parameters of which reside in polytope Ω (6) with given vertices $A_k^{(i)}$, $B_k^{(i)}$, $C_k^{(i)}$, $D_k^{(i)}$, and $L_k^{(i)}$ ($i = 1, 2, \ldots, N$), N is the number of the vertices, if x_{k+1} belongs to a one-step-ahead prediction ellipsoid of confidence $\mathcal{E}\left(P_{k+1|k}, \hat{x}_{k+1|k}\right) = \left\{x_{k+1} : (x_{k+1} - \hat{x}_{k+1|k})^{\mathrm{T}} P_{k+1|k}^{-1}(x_{k+1} - \hat{x}_{k+1|k}) \leq 1, P_{k+1|k} > 0\right\}$, then an estimation ellipsoid of confidence $\mathcal{E}(P_{k+1}, \hat{x}_{k+1}) = \left\{x_{k+1} : (x_{k+1} - \hat{x}_{k+1})^{\mathrm{T}} P_{k+1}^{-1}(x_{k+1} - \hat{x}_{k+1}) \leq 1, P_{k+1} > 0\right\}$ can be obtained by solving the optimisation problem:

$$\min_{P_{k+1}>0, \hat{x}_{k+1}, \tau_3 \geq 0, \tau_4 \geq 0} \operatorname{trace}(P_{k+1}) \tag{29}$$

subject to

$$\begin{bmatrix} -P_{k+1} & \Phi_2(\hat{x}_{k+1})]\Phi_{3\perp}^{(i)} \\ \Phi_{3\perp}^{(i)\mathrm{T}}\Phi_2(\hat{x}_{k+1})^{\mathrm{T}} & -\Phi_{3\perp}^{(i)T}[\operatorname{diag}(1-\tau_3-\tau_4, \tau_3 I, \tau_4 R_{k+1}^{-1})\Phi_{3\perp}^{(i)} \end{bmatrix} \leq 0, \tag{30}$$

for all $i \in \{1, 2, \ldots, N\}$, where

$$\Phi_2(\hat{x}_{k+1}) = \left[\hat{x}_{k+1|k} - \hat{x}_{k+1}\, E_{k+1|k} 0\right], \tag{31}$$

$\Phi_{3\perp}^{(i)}$ is an orthogonal complement of $\Phi_3^{(i)}$, and

$$\Phi_3^{(i)} = \left[C_{k+1}^{(i)}\hat{x}_{k+1|k} - y_{k+1} C_{k+1}^{(i)} E_{k+1|k} D_{k+1}^{(i)}\right], \tag{32}$$

$C_{k+1}^{(i)}$ is the matrix in (6) at the ith vertex of the polytope.

Proof. At a given time instant $k+1$, $P_{k+1|k}$ and $\hat{x}_{k+1|k}$ are known. Hence, from that $(x_{k+1}-\hat{x}_{k+1|k})^{\mathrm{T}}P_{k+1|k}^{-1}(x_{k+1}-\hat{x}_{k+1|k})\leq 1$, we have

$$x_{k+1}=\hat{x}_{k+1|k}+E_{k+1|k}z, \tag{33}$$

where $E_{k+1|k}$ comes from $P_{k+1|k}=E_{k+1|k}E_{k+1|k}^{\mathrm{T}}$ by means of Cholesky factorization, and $\|z\|\leq 1$. Then the estimation error $x_{k+1}-\hat{x}_{k+1}$ is written as

$$x_{k+1}-\hat{x}_{k+1}=\hat{x}_{k+1|k}-\hat{x}_{k+1}+E_{k+1|k}z. \tag{34}$$

Define

$$\eta=\begin{bmatrix}1\\ z\\ v_{k+1}\end{bmatrix}, \tag{35}$$

we can write $(x_{k+1}-\hat{x}_{k+1})^{\mathrm{T}}P_{k+1}^{-1}(x_{k+1}-\hat{x}_{k+1})\leq 1$, $\|z\|\leq 1$, and $v_{k+1}^{\mathrm{T}}R_{k+1}^{-1}v_{k+1}\leq 1$ in the form of η, respectively,

$$\eta^{\mathrm{T}}\operatorname{diag}(-1,0,0)\eta+\eta^{\mathrm{T}}\Phi_2(\hat{x}_{k+1})^{\mathrm{T}}P_{k+1}^{-1}\Phi_2(\hat{x}_{k+1})\eta\leq 0, \tag{36}$$

$$\eta^{\mathrm{T}}\operatorname{diag}(-1,I,0)\eta\leq 0, \tag{37}$$

$$\eta^{\mathrm{T}}\operatorname{diag}(-1,0,R_{k+1}^{-1})\eta\leq 0, \tag{38}$$

where

$$\Phi_2(\hat{x}_{k+1})=\left[\hat{x}_{k+1|k}-\hat{x}_{k+1}E_{k+1|k}0\right]. \tag{39}$$

According to Lemma 3.1, the sufficient condition for the inequalities (36)–(38) to hold is that there exist non-negative scalars τ_3 and τ_4 such that

$$\begin{aligned}&\eta^{\mathrm{T}}\operatorname{diag}(-1,0,0)\eta+\eta^{\mathrm{T}}\Phi_2(\hat{x}_{k+1})^{\mathrm{T}}P_{k+1}^{-1}\Phi_2(\hat{x}_{k+1})\eta\\ &\quad-\tau_3\eta^{\mathrm{T}}\operatorname{diag}(-1,I,0)\eta-\tau_4\eta^{\mathrm{T}}\operatorname{diag}(-1,0,R_{k+1}^{-1})\eta\leq 0.\end{aligned} \tag{40}$$

On the other hand, the measurement equation at time instant $k+1$ in (2) for the vertex i can be written as

$$C_{k+1}^{(i)}\hat{x}_{k+1|k}-y_{k+1}+C_{k+1}^{(i)}E_{k+1|k}z+D_{k+1}v_{k+1}=0, \tag{41}$$

Hence, (41) can be expressed in the form of η:

$$\Phi_3^{(i)}\eta=0, \tag{42}$$

where

$$\Phi_3^{(i)}=\left[C_{k+1}^{(i)}\hat{x}_{k+1|k}-y_{k+1}C_{k+1}^{(i)}E_{k+1|k}D_{k+1}\right]. \tag{43}$$

Using Lemma 3.3, a necessary and sufficient condition for (40) and (42) to hold for all η is that there exists M such that

$$-\Phi_{3\perp}^{(i)\mathrm{T}}\left[\operatorname{diag}(1-\tau_3-\tau_4,\tau_3 I,\tau_4 R_{k+1}^{-1})+\Phi_2(\hat{x}_{k+1})^{\mathrm{T}}P_{k+1}^{-1}\Phi_2(\hat{x}_{k+1})\right]\Phi_{3\perp}^{(i)}\leq 0. \tag{44}$$

Using Schur complements in Lemma 3.2, (44) is equivalent to

$$\begin{bmatrix} -P_{k+1} & \Phi_2(\hat{x}_{k+1})\Phi_{3\perp}^{(i)} \\ \Phi_{3\perp}^{(i)\mathrm{T}}\Phi_2(\hat{x}_{k+1})^{\mathrm{T}} & -\Phi_{3\perp}^{(i)T}\,\mathrm{diag}(1-\tau_3-\tau_4, \tau_3 I, \tau_4 R_{k+1}^{-1})\Phi_{3\perp}^{(i)} \end{bmatrix} \leq 0. \tag{45}$$

The following proof is similar to Theorem 3.4 and is omitted. □

Theorems 3.4 and 3.5 provide the computation of the ellipsoids of the minimal sizes for the prediction step and measurement update step. Now, we use these two-step updates to form a recursive algorithm for the set-membership filtering as follows.

Proposed set-membership filtering algorithm:

Step 1: Given the initial values ($\hat{x}_0$, P_0), and $k = 0$.
Step 2: Calculate the predicted state ellipsoid $\hat{x}_{k+1|k}$ and $P_{k+1|k}$ by solving the optimization problem (12).
Step 3: Calculate the updated state ellipsoid $\hat{x}_{k+1}$ and P_{k+1} by solving the optimization problem (29).
Step 4: If $k = K_N$, then Stop, otherwise $k = k + 1$ and go to Step 2

Remark 1. We can see from Theorem 3.4 that the inequality (13) is linear to the variables $P_{k+1|k}, \hat{x}_{k+1|k}, \tau_1, \tau_2$ and from Theorem 3.5 that the inequality (30) is linear to the variables $P_{k+1}, \hat{x}_{k+1}, \tau_3, \tau_4$. (13) and (30) therefore are linear matrix inequalities. Hence, the optimization problems (12) and (29) can be solved by the existing SDP via interior point approach.

Remark 2. The proposed algorithm is similar to the Kalman filtering algorithm which consists of two-step recursions of prediction and update. The difference is that Riccati equations are solved in Kalman filtering, and the SDP optimization problems should be solved in the proposed set-membership filtering. Therefore, more intensive computation is needed in set-membership filtering than in Kalman filtering.

Remark 3. The geometrical interpretation of Theorems 3.4 and 3.5 can be described as follows: the prediction step in Theorem 3.4 is to compute the predicted state ellipsoid. The update step in Theorem 3.4 is to compute the intersection of two sets, the predicted state ellipsoid and the set of states compatible with the measurement equation. The traces of $P_{k+1|k}$ and P_{k+1} are optimized at each time step in an effort to find the smallest ellipsoid for the prediction and update steps, respectively. Other measures of the ellipsoid can also be introduced, for example, determinant (Durieu *et al.* 2001, El Ghaoui and Calafiore 2001).

4. An illustrative example

Consider a tracking system

$$x_{k+1} = \left(\begin{bmatrix} 0.9+\alpha & T \\ 0 & 0.9 \end{bmatrix}\right) x_k + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} w_k, \quad y_k = \begin{bmatrix} 0.5 & 1 \end{bmatrix} x_k + v_k, \tag{46}$$

where T is the sample period, the state $x_k = [\, s(k) \quad \dot{s}(k) \,]^{\mathrm{T}}$ are the position and velocity of the target at time kT, respectively, and y_k is the measured output. Due to modelling errors, α is unknown but it belongs to the known interval $[\alpha_{\min}, \alpha_{\max}]$.

In the simulation, the sample period T is chosen as 0.001 s, and w_k and v_k as $\sin(k)$ and $\sin(2k)$, respectively. The initial state is set as $x_0 = [50 \quad -20]^{\mathrm{T}}$, which belongs to the ellipsoid $\mathcal{E}(P_0, \hat{x}_0) = \{x_0 : (x_0 - \hat{x}_0)^{\mathrm{T}} P_0^{-1}(x_0 - \hat{x}_0) \le 1\}$, where

$$\hat{x}_0 = \begin{bmatrix} 50 & -20 \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad P_0 = \begin{bmatrix} 200 & 20 \\ 20 & 10 \end{bmatrix}.$$

First, we consider the known system, i.e. $\alpha = 0$ in (46). The simulation results are obtained by solving the SDP problems (12) and (13) in Theorem 3.4 and the SDP problems (29) and (30) in Theorem 3.5 under Matlab 6.5 with YALMIP 3.0 and SeDuMi 1.1 (Löfberg 2004). Figure 1 shows the phase-plane estimation using the proposed set-membership filter. It can be seen that the estimated ellipsoids always contain the true states. Figures 2 and 3 further confirm that the true position and true velocity reside between the upper bound and lower bound. At any time step, we therefore always know that the target belongs the estimated region. The target can be fully tracked. Moreover, we can see from Figures 1–3 that the estimated ellipsoid will become small and the upper bound will approach to the lower bound after a number of recursions. Hence, the proposed algorithm has a good convergence.

Second, we consider the polytopic uncertain system, i.e. $\alpha = [-0.02, \quad 0.02]$ in (46). The simulation results are obtained by solving the SDP problems (12) and (13) in Theorem 3.4 and the SDP problems (29) and (30) in Theorem 3.5 under Matlab 6.5 with YALMIP 3.0 and SeDuMi 1.1 (Löfberg 2004). Figure 4 shows the phase-plane estimation using the proposed set-membership filter. Due to the uncertain parameters α, the true states distribute in the phase plane with respect to the different α. However, they are always contained in the estimated ellipsoid. It is also shown from Figures 5 and 6 that the true

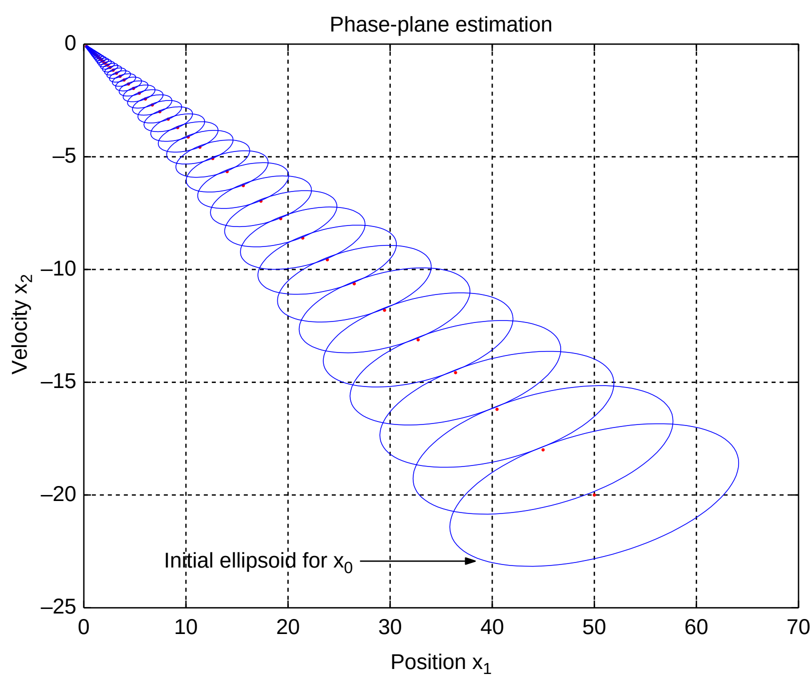

Figure 1. The phase-plane estimation using the proposed set-membership filter for the known system.

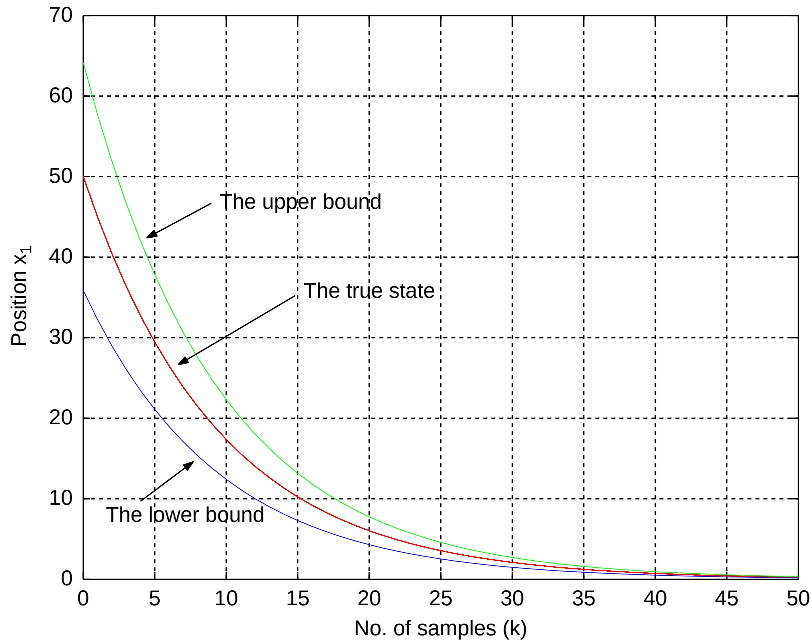

Figure 2. The true state value and its bounds for the known system.

position and true velocity reside between the upper bound and lower bound for all uncertain parameters α. Therefore, our proposed filtering algorithm is useful for target tracking and target attack, when the target systems are hardly exactly modelled.

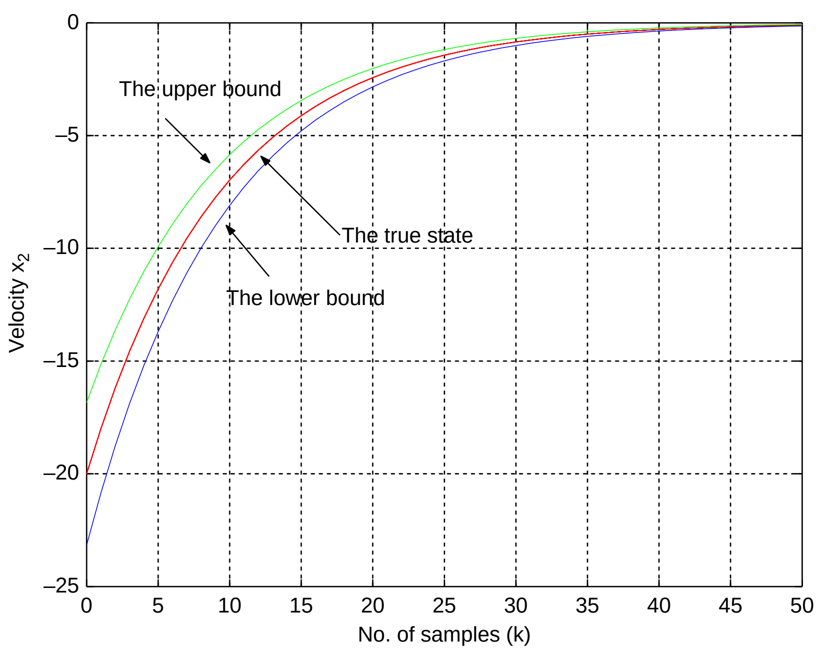

Figure 3. The true state value and its bounds for the known system.

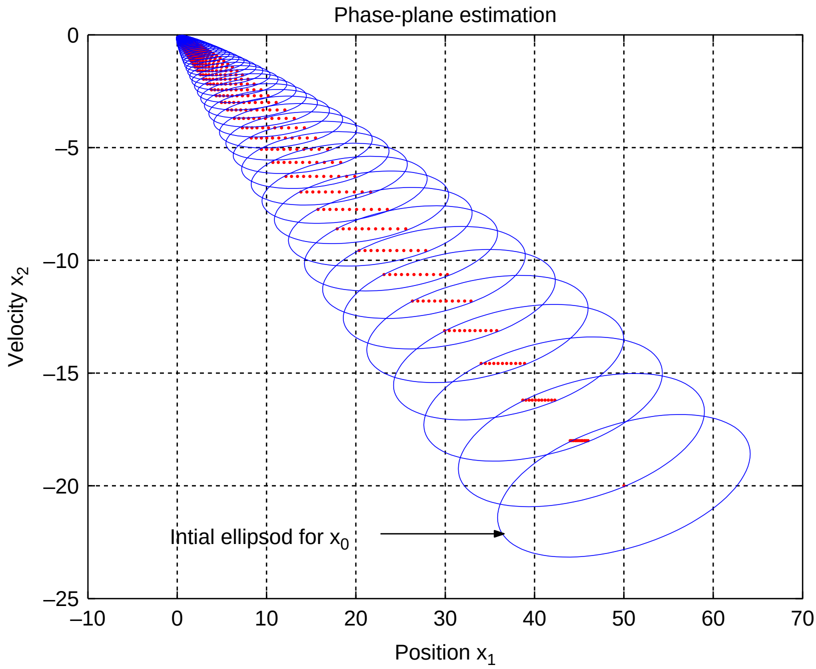

Figure 4. The phase-plane estimation using the proposed set-membership filter for the polytopic uncertain system.

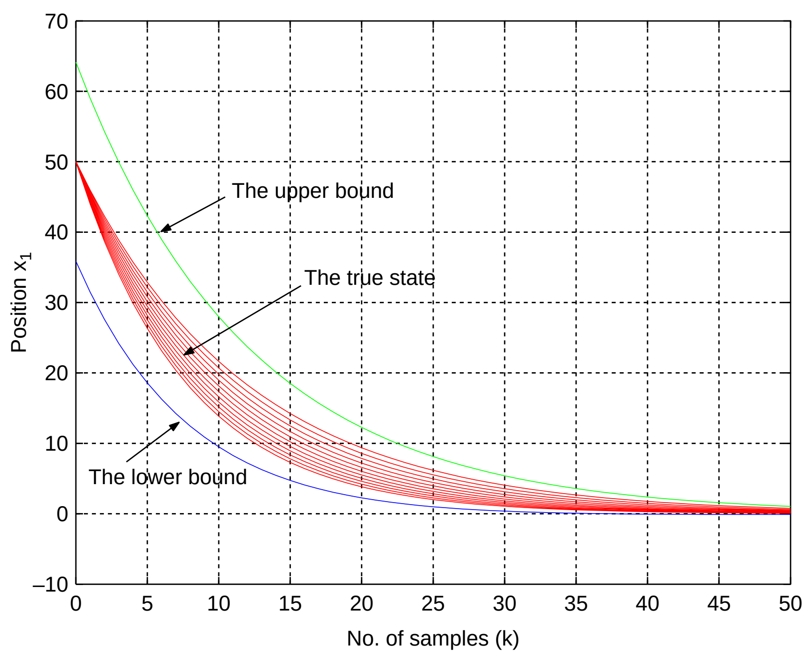

Figure 5. The true state value and its bounds for the polytopic uncertain system.

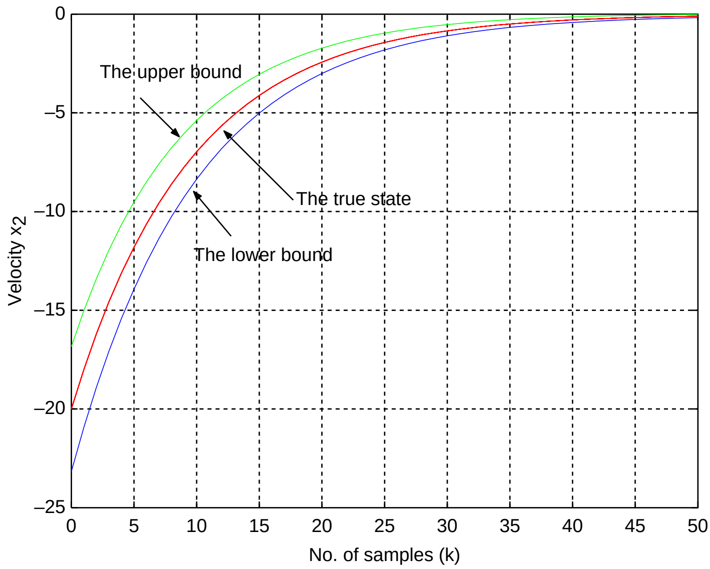

Figure 6. The true state value and its bounds for the polytopic uncertain system.

5. Conclusions

In this paper, a set-membership filtering problem has been considered for discrete-time systems with polytopic uncertainty. A recursive algorithm for computing an ellipsoid which always contains the state has been developed. An illustrative example has demonstrated the feasibility of the proposed filtering methods. The proposed filtering algorithm is similar to the Kalman filtering algorithm based on a two-step prediction–correction structure. The algorithm is computationally attractive for online systems with polytopic uncertainties and unknown but bounded noises.

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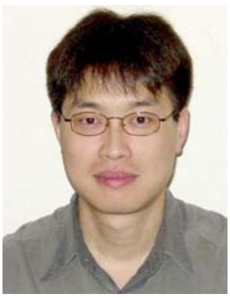

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