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Brief paper

Set-membership filtering for systems with sensor saturation[☆]

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ABSTRACT

This paper addresses the set-membership filtering problem for a class of discrete time-varying systems with sensor saturation in the presence of unknown-but-bounded process and measurement noises. A sufficient condition for the existence of set-membership filter is derived. A convex optimisation method is proposed to determine a state estimation ellipsoid that is a set of states compatible with sensor saturation and unknown-but-bounded process and measurement noises. A recursive algorithm is developed for computing the ellipsoid that guarantees to contain the true state by solving a time-varying linear matrix inequality. Simulation results are provided to demonstrate the effectiveness of the proposed method.

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1. Introduction

Most filtering approaches require system noises including process noise and measurement noise in a stochastic framework and then provide a *probabilistic state estimation* (Anderson & Moore, 1979; Yang & Hung, 2002; Yang, Wang, & Hung, 2002; Yang, Wang, Ho, & Liu, 2006). The probabilistic nature of the estimates leads to the use of mean and variance to describe the state spreads (distributions). These spreads cannot guarantee that the state is included in some region, because they are not hard bounds. However, in many real-world applications, such as, target tracking and attack, system guidance and navigation, they need 100% confidence to be estimated. This has motivated to develop an *ellipsoidal state estimation* (Kurzhanski & Valyi, 1996). The idea of the ellipsoidal state estimation is to provide a set of state estimates in state space which always contains the true state of the system by assuming *hard bounds* instead of stochastic descriptions on the system noises (Bertsekas & Rhodes, 1971; Schweppe, 1968). The actual estimate is a set in state space rather than a single vector. These methods are therefore known as set-membership or set-valued state estimation (filtering) (Bertsekas & Rhodes, 1971; Morrell & Stirling, 1991; Schweppe, 1968; Witsenhausen, 1968). We prefer to adopt the name set-membership filtering in this paper as it is easy to distinguish between a set estimation and a point estimate in the stochastic framework.

Set-membership filtering problem was first considered by Witsenhausen (1968). The set of all possible values of the states compatible with the measurement of outputs is completely characterised by their support functions. An ellipsoidal approximation algorithm was provided by Schweppe (1968). In this algorithm, the measurements are used to calculate recursively a bounding ellipsoid to the set of possible states, under the assumption that the set containing the initial condition and the process and measurement noises is, or can be approximated by ellipsoids. The solution to a set-membership filtering problem with the instantaneous constraints was determined by using the results derived for an energy constraint in Bertsekas and Rhodes (1971). The resulting estimator is similar to that proposed by Schweppe (1968), but it has an important advantage that the gain matrix does not depend on the particular output measurements and is therefore precomputable. Recently, many researchers have attempted to deal with the set-membership filtering problems with various methods. For example, a convex optimisation approach was applied to deal with the robust set-membership filtering for the systems with norm-bounded uncertainty in the system matrices (Ghaoui & Calafiore, 2001). An ellipsoidal state bounding method was developed to provide an optimal outer approximation of the sum and intersection

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of ellipsoids for set-membership filtering in Durieu, Walter, and Polyak (2001); Maskarov and Norton (1996); Polyak, Nazin, Durieu, and Walter (2002). A recursive scheme for constructing an ellipsoidal state estimation set of all states consistent with the measured output and the given noise and unstructured uncertainty described by a sum quadratic constraint was presented in Petersen and Savkin (1999); Ra, Jin, and Park (2004); Savkin and Petersen (1995, 1998). In the existing literature concerning set-membership filtering techniques, it is implicitly assumed that the measurements are *working under the linear condition* such that the effect of possible amplitude saturation is ignored (Ghaoui & Calafiore, 2001; Polyak et al., 2002; Ra et al., 2004; Savkin & Petersen, 1995, 1998). In practical applications, however, sensor saturation arises frequently in measurements due to physical limitations on sensors (Sadhar & Rajagopalan, 2004). This motivates us to investigate the set-membership filtering problem with sensor saturation.

Analysis and design for systems with sensor saturation have been addressed by several authors (Cao, Lin, & Chen, 2003; Hu & Lin, 2001; Koplon, Hantus, & Sontag, 1994; Kreisselmeier, 1996; Lin & Hu, 2001; Xiao, Cao, & Lin, 2004; Zuo, Wang, & Huang, 2005). For example, the stabilization for linear systems subject to sensor saturation has been considered in Kreisselmeier (1996); Lin and Hu (2001). An output feedback control with sensor saturation has been proposed for linear continuous-time systems in Cao et al. (2003). An extension to linear discrete-time systems has been presented in Zuo et al. (2005). The observability of linear systems with saturated outputs has been discussed in Koplon et al. (1994). A robust filter has been designed for discrete-time systems with sensor saturation and applied to the digital transmultiplexer systems in Xiao et al. (2004). In this paper, we consider the set-membership filtering problem for discrete-time systems subject to sensor saturation. Our goal is to provide a region of state estimate where the true state resides despite the presence of both measurement saturation, and unknown-but-bounded process and measurement noises. For this purpose, a sufficient condition for the existence of set-membership filter is derived for discrete-time systems with sensor saturation. A convex optimisation method is proposed to determine a *state estimation ellipsoid* that is a set of states compatible with sensor saturation and unknown-but-bounded process and measurement noises. A recursive algorithm is developed for computing the ellipsoid that guarantees to contain the true state by solving a time-varying linear matrix inequality.

The rest of this paper is organised as follows. The set-membership filter design problem is formulated in Section 2 for discrete time-varying systems subject to sensor saturation. A convex optimisation method for computing the state estimation ellipsoid is developed in Section 3. Section 4 provides an illustrative example to demonstrate the effectiveness of our algorithm. Conclusions and future directions are described in Section 5.

Notation. The notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). The superscript T stands for matrix transpose. The notation trace(P) denotes the trace of P.

2. Problem formulation

Consider a discrete time-varying system in the presence of sensor saturation

$$x_{k+1} = A_k x_k + F_k u_k + B_k w_k, \tag{1}$$

$$y_k = \sigma(C_k x_k) + D_k v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state; $u_k \in \mathbb{R}^l$ is the known deterministic input; $y_k \in \mathbb{R}^m$ is the measurement output; A_k, B_k, C_k, D_k and F_k are known time-varying matrices with appropriate dimensions; The saturation function $\sigma(\cdot)\colon \mathbb{R}^m \mapsto \mathbb{R}^m$ is defined as

$$\sigma(r) = \begin{bmatrix} \sigma_1(r_1) \\ \sigma_2(r_2) \\ \cdots \\ \sigma_m(r_m) \end{bmatrix} \tag{3}$$

with $\sigma_i(r_i) = \text{sign}(r_i)\min\{r_{i,\max}, |r_i|\}$, where $r_{i,\max}$ denotes the ith element of the vector $r_{\max}$, the saturation level; $w_k \in \mathbb{R}^r$ is the process noise and $v_k \in \mathbb{R}^p$ is the measurement noise, which is assumed to be confined to specified ellipsoidal sets:

$$\mathcal{W}_k := \{w_k : w_k^{\mathrm{T}} Q_k^{-1} w_k \leq 1\}, \tag{4}$$

$$\mathcal{V}_k := \{v_k : v_k^{\mathrm{T}} R_k^{-1} v_k \leq 1\}, \tag{5}$$

where $Q_k = Q_k^{\mathrm{T}} > 0$ and $R_k = R_k^{\mathrm{T}} > 0$ are known matrices with compatible dimensions; the initial state x_0 belongs to a given ellipsoid:

$$\mathcal{X}_0 := \{x_0 : (x_0 - \hat{x}_0)^{\mathrm{T}} P_0^{-1} (x_0 - \hat{x}_0) \leq 1\}, \tag{6}$$

where $\hat{x}_0$ is an estimate of x_0 which is assumed to be given, and $P_0 = P_0^{\mathrm{T}} > 0$ is a known matrix.

Remark 1. Sensor saturation exists in many practical applications due to the use of cheaper sensors with inadequate range in instrument device, amplifier saturation of electronic circuit, rate limitation in mechanical systems. The examples are flight control systems (Cao et al., 2003) and restoration of images (Sadhar & Rajagopalan, 2004). Here we only consider the *Occasional* measurement saturations. Deep saturations are beyond our study in this paper.

In this paper, a filter based on the current measurement is considered for the system (1)–(2) subject to the saturation (3), which is of the form:

$$\hat{x}_{k+1} = G_k \hat{x}_k + F_k u_k + L_k y_{k+1}, \tag{7}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimate of x_k. G_k and L_k are the filter parameters to be determined.

Our objective is to determine an ellipsoid

$$\mathcal{X}_{k+1} := \{x_{k+1} : (x_{k+1} - \hat{x}_{k+1})^{\mathrm{T}} P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \leq 1\}, \tag{8}$$

for the state x_{k+1}, given the measurement information y_{k+1} at the time instant $k+1$ for the process noise $w_k \in \mathcal{W}_k$ and the measurement noise $v_k \in \mathcal{V}_k$ subject to the saturation (3). In other words, we look for P_{k+1} and $\hat{x}_{k+1}$ such that

$$(x_{k+1} - \hat{x}_{k+1})^{\mathrm{T}} P_{k+1}^{-1} (x_{k+1} - \hat{x}_{k+1}) \leq 1, \tag{9}$$

subject to $w_k \in \mathcal{W}_k$, $v_k \in \mathcal{V}_k$ and (3).

This problem will be referred to as a set-membership filter design problem.

3. Set-membership filter design with sensor saturation

In this section, we will design a set-membership filter for the system (1)–(2) subject to sensor saturation. We start with some basic notions that we are interested in. We then recall two useful lemmas for our following development. After that we derive the existence conditions for the set-membership filter that the true state is guaranteed to reside in a set of state estimates. We finally provide the convex optimisation recursive algorithm for designing the set-membership filter.

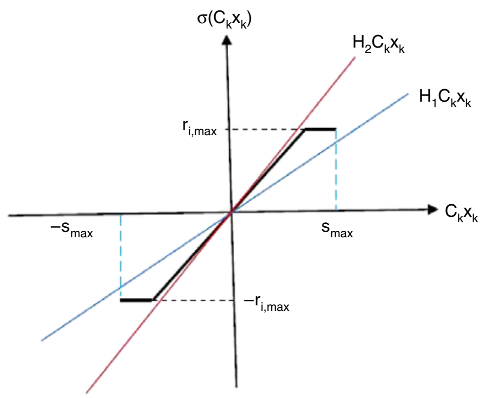

Fig. 1. The saturation type nonlinearity.

Definition 1 (*Cao et al., 2003; Khalil, 1996*). A nonlinearity $\psi: \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if

$$(\psi(v) - K_1 v)^{\mathrm{T}}(\psi(v) - K_2 v) \leq 0, \quad \forall v \in \mathbb{R}^m \tag{10}$$

for some real matrices $K_1, K_2 \in \mathbb{R}^{m\times m}$, where $K = K_2 - K_1$ is a positive-definite symmetric matrix. In this case, we say that ψ belongs to the sector $[K_1, K_2]$.

If we assume that there exist the diagonal matrices H_1 and H_2 such that $0 \leq H_1 < I \leq H_2$, then the saturation function $\sigma(C_k x_k)$ in (2) can be written as follows

$$\sigma(C_k x_k) = H_1 C_k x_k + \psi(C_k x_k), \tag{11}$$

where $\psi(C_k x_k)$ is a nonlinear vector-valued function which satisfies a sector condition with $K_1 = 0$ and $K_2 = H$, in which $H = H_2 - H_1$, i.e., $\psi(C_k x_k)$ satisfies the following inequality:

$$\psi^{\mathrm{T}}(C_k x_k)(\psi(C_k x_k) - HC_k x_k) \leq 0. \tag{12}$$

Remark 2. In theory, we should choose H_1 as zero to guarantee that the inequality (12) holds in whole. However, in practical applications, $C_k x_k$ is limited, i.e., $C_k x_k$ is not greater than $s_{\max}$ in Fig. 1. So we can use a smaller sector (a bigger H_1) to cover the saturation type nonlinearities and guarantee that the inequality (12) is satisfied. If H_1 is chosen too small, the saturation function will belong to a bigger sector (see Fig. 1), and the estimated region will become bigger too.

In the following set-membership filter design for sensor saturation, we will take into account the saturation constraints (11) and (12). Before providing the solution to the set-membership filtering problem with sensor saturation, we recall the following two useful lemmas:

Lemma 1 (*S-Procedure Boyd, Ghaoui, Feron, & Balakrishnan, 1994; Skelton, Iwasaki, & Grigoriadis, 1998*). *Let $Y_0(\eta), Y_1(\eta), \ldots, Y_p(\eta)$ be quadratic functions of $\eta \in \mathbb{R}^n$*

$$Y_i(\eta) = \eta^{\mathrm{T}} T_i \eta, \quad i = 0, 1, \ldots, p, \tag{13}$$

with $T_i = T_i^{\mathrm{T}}$. Then, the implication

$$Y_1(\eta) \leq 0, \ldots, Y_p(\eta) \leq 0 \Longrightarrow Y_0(\eta) \leq 0, \tag{14}$$

holds if there exist $\tau_1, \ldots, \tau_p > 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i \leq 0. \tag{15}$$

Lemma 2 (*Schur Complements Boyd et al., 1994*). *Given constant matrices L_1, L_2, L_3 where $L_1 = L_1^{\mathrm{T}}$ and $L_2 = L_2^{\mathrm{T}} < 0$, then*

$$L_1 - L_3^{\mathrm{T}} L_2^{-1} L_3 \leq 0,$$

if and only if

$$\begin{bmatrix} L_1 & L_3^{\mathrm{T}} \\ L_3 & L_2 \end{bmatrix} \leq 0,$$

or equivalently

$$\begin{bmatrix} L_2 & L_3 \\ L_3^{\mathrm{T}} & L_1 \end{bmatrix} \leq 0.$$

We are now in position to provide the main results in the following theorem. The existence conditions are developed for the set-membership filter that the true state is guaranteed to reside in a set of state estimates.

Theorem 1. *For the system (1)–(2) subject to the sensor saturation (3), suppose that the state x_k belongs to its state estimation ellipsoid $(x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \hat{x}_k) \leq 1$. Then one-step ahead state x_{k+1} resides in its state estimation ellipsoid $(x_{k+1} - \hat{x}_{k+1})^{\mathrm{T}} P_{k+1}^{-1}(x_{k+1} - \hat{x}_{k+1}) \leq 1$, if there exist $P_{k+1} > 0$, G_k, L_k, $\tau_1 > 0$, $\tau_2 > 0$, $\tau_3 > 0$, $\tau_4 > 0$ such that*

$$\begin{bmatrix} -P_{k+1} & \Pi(\hat{x}_k, u_k) \\ \Pi^{\mathrm{T}}(\hat{x}_k, u_k) & -\Theta(\tau_1, \tau_2, \tau_3, \tau_4) \end{bmatrix} \leq 0. \tag{16}$$

Moreover the centre of the state estimate ellipsoid is determined by

$$\hat{x}_{k+1} = G_k \hat{x}_k + F_k u_k + L_k y_{k+1}, \tag{17}$$

where $\Theta(\tau_1, \tau_2, \tau_3, \tau_4)$ and $\Pi(\hat{x}_k, u_k)$ are given in Box I, and Φ_k is given in Box II.

Proof. For simplicity, we denote $\psi(C_k x_k)$ by ψ_k, then one-step ahead estimation error $x_{k+1} - \hat{x}_{k+1}$ is written as:

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1} &= A_k x_k + F_k u_k + B_k w_k - G_k \hat{x}_k - F_k u_k - L_k y_{k+1} \\ &= A_k x_k + B_k w_k - G_k \hat{x}_k - L_k[\sigma(C_{k+1} x_{k+1}) + D_{k+1} v_{k+1}] \\ &= A_k x_k + B_k w_k - G_k \hat{x}_k - L_k(H_1 C_{k+1} x_{k+1} + \psi_{k+1} + D_{k+1} v_{k+1}) \\ &= (I - L_k H_1 C_{k+1}) A_k x_k - G_k \hat{x}_k - L_k H_1 C_{k+1} F_k u_k \\ &\quad + (I - L_k H_1 C_{k+1}) B_k w_k - L_k \psi_{k+1} - L_k D_{k+1} v_{k+1}. \end{aligned} \tag{20}$$

On the other hand, if $(x_k - \hat{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \hat{x}_k) \leq 1$, then there exists a z with $\|z\| \leq 1$ such that (Ghaoui & Calafiore, 2001)

$$x_k = \hat{x}_k + E_k z, \tag{21}$$

where E_k is a factorisation of $P_k = E_k E_k^{\mathrm{T}}$.

By using (21), (20) can be rewritten as

$$\begin{aligned} x_{k+1} - \hat{x}_{k+1} = {} & (I - L_k H_1 C_{k+1}) A_k \hat{x}_k - G_k \hat{x}_k - L_k H_1 C_{k+1} F_k u_k \\ & + (I - L_k H_1 C_{k+1}) A_k E_k z + (I - L_k H_1 C_{k+1}) B_k w_k \\ & - L_k \psi_{k+1} - L_k D_{k+1} v_{k+1}. \end{aligned} \tag{22}$$

Denoting

$$\eta = \begin{bmatrix} 1 \\ z \\ w_k \\ v_{k+1} \\ \psi_{k+1} \end{bmatrix} \tag{23}$$

and $\Pi(\hat{x}_k, u_k)$, given in Box III, we obtain a compact form for (21) as follows:

$$x_{k+1} - \hat{x}_{k+1} = \Pi(\hat{x}_k, u_k)\eta, \tag{24}$$

$$\Theta(\tau_1, \tau_2, \tau_3, \tau_4) = \tau_1 \Phi_k + \mathrm{diag}(1 - \tau_2 - \tau_3 - \tau_4, \tau_2 I, \tau_3 Q_k^{-1}, \tau_4 R_{k+1}^{-1}, 0), \quad (18)$$

$$\Pi(\hat{x}_k, u_k) = \begin{bmatrix}(I - L_k H_1 C_{k+1})A_k\hat{x}_k - G_k\hat{x}_k - L_k H_1 C_{k+1} F_k u_k & (I - L_k H_1 C_{k+1})A_k E_k & (I - L_k H_1 C_{k+1})B_k & -L_k D_{k+1} & -L_k\end{bmatrix} \quad (19)$$

Box I.

$$\Phi_k = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & -(HC_{k+1}A_k\hat{x}_k + HC_{k+1}F_k u_k)^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & -(HC_{k+1}A_k E_k)^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & -(HC_{k+1}B_k)^{\mathrm{T}} \\ 0 & 0 & 0 & 0 & 0 \\ -HC_{k+1}A_k\hat{x}_k - HC_{k+1}F_k u_k & -HC_{k+1}A_k E_k & -HC_{k+1}B_k & 0 & 2 \end{bmatrix}$$

Box II.

$$\Pi(\hat{x}_k, u_k) = \begin{bmatrix}(I - L_k H_1 C_{k+1})A_k\hat{x}_k - G_k\hat{x}_k - L_k H_1 C_{k+1} F_k u_k & (I - L_k H_1 C_{k+1})A_k E_k & (I - L_k H_1 C_{k+1})B_k & -L_k D_{k+1} & -L_k\end{bmatrix}$$

Box III.

Hence, $(x_{k+1} - \hat{x}_{k+1})^{\mathrm{T}} P_{k+1}^{-1}(x_{k+1} - \hat{x}_{k+1}) \le 1$ can be written as

$$\eta^{\mathrm{T}}[\Pi^{\mathrm{T}}(\hat{x}_k, u_k)P_{k+1}^{-1}\Pi(\hat{x}_k, u_k) - \mathrm{diag}(1, 0, 0, 0, 0)]\eta \le 0. \quad (25)$$

From (12), we have

$$\psi_{k+1}^{\mathrm{T}}(\psi_{k+1} - HC_{k+1}x_{k+1}) \le 0. \quad (26)$$

Substituting (1) and (21) into (26) yields

$$\psi_{k+1}^{\mathrm{T}}(\psi_{k+1} - HC_{k+1}A_k\hat{x}_k - HC_{k+1}F_k u_k - HC_{k+1}A_k E_k z - HC_{k+1}B_k w_k) \le 0 \quad (27)$$

which can be written in η as

$$\eta^{\mathrm{T}}\Phi_k\eta \le 0, \quad (28)$$

where Φ_k is as given in Box IV.

From (4), (5) and (21), z, w_k and v_{k+1} have the following constraints

$$\begin{cases} \|z\| \le 1, \\ w_k^{\mathrm{T}} Q_k^{-1} w_k \le 1, \\ v_{k+1}^{\mathrm{T}} R_{k+1}^{-1} v_{k+1} \le 1. \end{cases} \quad (29)$$

which can be expressed in η as:

$$\begin{cases} \eta^{\mathrm{T}}\mathrm{diag}(-1, I, 0, 0, 0)\eta \le 0, \\ \eta^{\mathrm{T}}\mathrm{diag}(-1, 0, Q_k^{-1}, 0, 0), \eta \le 0, \\ \eta^{\mathrm{T}}\mathrm{diag}(-1, 0, 0, R_{k+1}^{-1}, 0)\eta \le 0. \end{cases} \quad (30)$$

Therefore we need to find a condition to make (25) hold subject to the saturation constraints (28), and constraints (30). By using the S-procedure (Lemma 1), the sufficient condition such that the inequalities (28) and (30) imply (25) to hold is that there exist positive scalars τ_1, τ_2, τ_3, and τ_4 such that

$$\Pi^{\mathrm{T}}(\hat{x}_k, u_k)P_{k+1}^{-1}\Pi(\hat{x}_k, u_k) - \mathrm{diag}(1, 0, 0, 0, 0) - \tau_1\Phi_k - \tau_2\mathrm{diag}(-1, I, 0, 0, 0) - \tau_3\mathrm{diag}(-1, 0, Q_k^{-1}, 0, 0) - \tau_4\mathrm{diag}(-1, 0, 0, R_{k+1}^{-1}, 0) \le 0. \quad (31)$$

(31) is written in the following compact form:

$$\Pi^{\mathrm{T}}(\hat{x}_k, u_k)P_{k+1}^{-1}\Pi(\hat{x}_k, u_k) - \tau_1\Phi_k - \mathrm{diag}(1 - \tau_2 - \tau_3 - \tau_4, \tau_2 I, \tau_3 Q_k^{-1}, \tau_4 R_{k+1}^{-1}, 0) \le 0. \quad (32)$$

By denoting

$$\Theta(\tau_1, \tau_2, \tau_3, \tau_4) = \tau_1\Phi_k + \mathrm{diag}(1 - \tau_2 - \tau_3 - \tau_4, \tau_2 I, \tau_3 Q_k^{-1}, \tau_4 R_{k+1}^{-1}, 0), \quad (33)$$

(32) is written as

$$\Pi^{\mathrm{T}}(\hat{x}_k, u_k)P_{k+1}^{-1}\Pi(\hat{x}_k, u_k) - \Theta(\tau_1, \tau_2, \tau_3, \tau_4) \le 0. \quad (34)$$

By using the Schur complements (Lemma 2), (34) is equivalent to:

$$\begin{bmatrix} -P_{k+1} & \Pi(\hat{x}_k, u_k) \\ \Pi^{\mathrm{T}}(\hat{x}_k, u_k) & -\Theta(\tau_1, \tau_2, \tau_3, \tau_4) \end{bmatrix} \le 0. \quad (35)$$

Thus, if there exist the filter parameters G_k and L_k, and scalars $\tau_1 > 0$, $\tau_2 > 0$, $\tau_3 > 0$, $\tau_4 > 0$ such that (16) holds, then one-step ahead state x_{k+1} resides in its state estimation ellipsoid $(x_{k+1} - \hat{x}_{k+1})^{\mathrm{T}}P_{k+1}^{-1}(x_{k+1} - \hat{x}_{k+1}) \le 1$. □

Theorem 1 outlines the principle of determining the current state estimation ellipsoid containing x_{k+1}. However, it does not provide an optimal (minimal) state estimation ellipsoid. Next, we apply the convex optimisation approach (Nesterov & Nemirovski, 1994; Vandenberghe & Boyd, 1996) to determine an optimal ellipsoid. P_{k+1} is obtained by solving the following optimisation problem:

$$\min_{P_{k+1}>0, G_k, L_k, \tau_1>0, \tau_2>0, \tau_3>0, \tau_4>0} \mathrm{trace}(P_{k+1}) \quad \text{subject to (16)} \quad (36)$$

Remark 3. We can see from Theorem 1 that the inequality (16) is linear to the variables P_{k+1}, G_k, L_k, τ_1, τ_2, τ_3, τ_4. Hence, the optimisation problem (36) subject to (16) can be solved by the existing semi-definite programming (SDP) via interior-point approach (Nesterov & Nemirovski, 1994; Vandenberghe & Boyd, 1996).

Remark 4. The sufficient conditions are provided to estimate a state ellipsoid. In order to reduce the conservativeness, the convex optimisation method has been proposed in (36). The trace of P_{k+1} is optimised at each time step in an effort to find the smallest ellipsoid for the state estimate. Other measures of the ellipsoid can also be introduced, for example, determinant (Durieu et al., 2001; Ghaoui & Calafiore, 2001).

Remark 5. It is worth mentioning that a set-value state estimation problem has been proposed in Petersen and Savkin (1999); Savkin and Petersen (1995, 1998) by designing a robust Kalman filter and then deriving the bound of $(x_{k+1}-\hat{x}_{k+1})^{\mathrm{T}}P_{k+1}^{-1}(x_{k+1}-\hat{x}_{k+1})$ via using *an analytical method*. In this paper, we assume a general structure of the filter as the centre of the state estimate ellipsoid, and then