application of the one-step extrapolation procedure of [3], it is found that the existence of signal $z(t)$ is not valid in the space $R(T)$.

II. NONEXISTENCE OF $z(t)$

In [3, Sec. VI], under the assumption that there exists a signal $z(t) \in R(T)$ such that

$$TBz = Tx \tag{1}$$

the author obtained the following extrapolation equation:

$$x = Bz. \tag{2}$$

Unfortunately, the signal $z(t) \in R(T)$ assumed above does not exist if the observed signal x is bandlimited. Let $p_\Lambda(t)$ be the characteristic function of subset Λ. If a signal $z(t) \in R(T)$ does exist in (1), let $z_0(t) = p_\Lambda(t)z(t)$, and rewrite (2) as

$$x(t) = \int_{-\infty}^{\infty} h(t-\tau)z_0(\tau)\, d\tau \tag{3}$$

where $h(t) = \int_\Omega \exp(2\pi i\omega t)\, d\omega$. After Fourier transformation (FT), (3) becomes

$$X(\omega) = p_\Omega(\omega)Z_0(\omega) \tag{4}$$

where $p_\Omega(\omega)$ denotes the characteristic function of subset Ω, and $Z_0(\omega)$ denotes the FT of $z_0(t)$. Obviously, $X(\omega) = 0$ when $\omega \notin \Omega$, and $Z_0(\omega) = X(\omega)$ on Ω. Since $z_0(t)$ is a time-limited signal, its FT $Z_0(\omega)$ should be an analytical function. However, $X(\omega)$ is a function with support in Ω. In most cases, it is not an analytical function, except that it can be analytically expanded to the whole space. This yields a contradiction.

A simple counterexample is given below. Let $\Lambda = [-1, 1]$, let $\Omega = [-1, 1]$, and define

$$X(\omega) = \begin{cases} 1 - |\omega|, & \omega \in \Omega \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

At the origin, $X(0)$ has no derivative, but $X(\omega)$ should be differentiable according to (4). This counterexample indicates that we cannot find a signal $z(t) \in R(T)$, satisfying (2) even for a simple signal $x(t) = \sin c^2(t)$.

The reason for above result is that the convergence of [3, eq. (19)] depends on the condition $x_0 - x \in B_\Omega$. Actually, we do not know whether [3, eq. (27)] is valid for a signal $z(t) \in R(T)$.

A remedy to remain (2) is to use a specifically defined space instead of L_2. Assume that $\Lambda = [-T, T]$, $\Omega = [-\sigma, \sigma]$, according to [4] for every $x(t) \in R(B)$ and that it can be expanded as

$$x(t) = \sum_k a_k \varphi_k(t) \tag{6}$$

where $\varphi_k(t)$ is the prolate spheroidal wave function for the pair (Λ, Ω). Letting λ_k be the eigenvalue of $\varphi_k(t)$ and defining

$$z(t) = \sum_k \frac{a_k}{\lambda_k}\, \varphi_k(t) \tag{7}$$

we then have the proposed equation

$$x(t) = \int_{-T}^{T} h(t-\tau)z(\tau)\, d\tau \tag{8}$$

where $z(t) \notin R(T)$. On the other hand, for a truncated signal $x(t) = \sum_{n=0}^{N} a_k\varphi_k(t)$, (2) is valid.

III. CONCLUSION

Based on the existence of signal $z(t) \in R(T)$, a one-step extrapolation procedure was developed in [3]. In L_2 space, the existence is not valid. This can be remedied in a specifically defined space by using the prolate spheroidal wave function.

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Robust H_2/H_∞ Filtering for Linear Systems with Error Variance Constraints

Zidong Wang and Biao Huang

***Abstract*—In this correspondence, we consider the robust H_2/H_∞ filtering problem for linear perturbed systems with steady-state error variance constraints. The purpose of this multiobjective problem is to design a linear filter that does not depend on the parameter perturbations such that the following three performance requirements are simultaneously satisfied.**

1) **The filtering process is asymptotically stable.**
2) **The steady-state variance of the estimation error of each state is not more than the individual prespecified value.**
3) **The transfer function from exogenous noise inputs to error state outputs meets the prespecified H_∞ norm upper bound constraint.**

We show that in both continuous and discrete-time cases, the addressed filtering problem can effectively be solved in terms of the solutions of a couple of algebraic Riccati-like equations/inequalities. We present both the existence conditions and the explicit expression of desired robust filters. An illustrative numerical example is provided to demonstrate the flexibility of the proposed design approach.

***Index Terms*—Algebraic Riccati equation, H_∞ filtering, Kalman filtering, quadratic matrix inequality, robust filtering.**

I. INTRODUCTION

In recent years, the study of the so-called cost-guaranteed filters has gained growing interest; see, e.g., [2], [5], [10], and [11]. A common feature of these results is that they have focused on designing a filter that first provides an upper bound on the variance of the estimation error for all admissible parameter perturbations and then minimizes this bound. It is remarkable that in this case, the associated upper bound is

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not specified *a priori*, and the resulting optimal robust filters are often unique in certain cases.

In practical engineering, however, it is often the case that for a large class of filtering problems, the performance objectives are *naturally* described as the upper bounds on the error variances of estimation; see, e.g., [7] and [12]. Unfortunately, it is usually difficult to utilize traditional methods to deal with this class of *constrained variance* filtering problems. A novel filtering method, namely, error covariance assignment (ECA) theory (see, e.g., [12]) was recently developed to provide a closed-form solution for *directly* assigning the specified steady-state estimation error covariance. Subsequently, [8] and [9] extended the ECA theory to the parameter uncertain systems by assigning a prescribed upper bound to the steady-state error variance, but the perturbations were assumed to be time invariant and measurable, and the adopted filter structure depended on the availability of perturbations. This is very restrictive in practical applications.

To overcome the drawback indicated above, this correspondence aims at designing a *perturbation-independent* filter, where the perturbations are not required to be time-invariant and available, such that we have the following.

1) The filtering process is asymptotically stable.
2) The steady-state variance of the estimation error of each state is not more than the individual prespecified value.
3) The transfer function from exogenous noise inputs to error state outputs meets the prespecified H_∞ norm upper bound constraint.

The results obtained improve those of [8] and [9].

II. Problem Formulation: Continuous-Time Case

Consider the following class of linear uncertain continuous-time systems:

$$\begin{aligned}\dot{x}(t) &= (A + \Delta A)x(t) + D_1 w(t)\\ y(t) &= (C + \Delta C)x(t) + D_2 w(t)\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and A, C, D_1, D_2 are known constant matrices. $w(t)$ is a zero mean Gaussian white noise process with covariance $I > 0$. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$ and is uncorrelated with $w(t)$. ΔA and ΔC are real-valued perturbation matrices satisfying

$$\begin{bmatrix}\Delta A\\ \Delta C\end{bmatrix} = \begin{bmatrix}M_1\\ M_2\end{bmatrix} F(t)N \tag{2}$$

where $F(t) \in \mathbb{R}^{i\times j}$ is a real time-varying uncertain matrix meeting $F(t)F^T(t) \le I$, and $M_1, M_2,$ and N are known constant matrices of appropriate dimensions.

Assumption 1: The system matrix A is Hurwitz stable, and the matrix D_2 or M_2 is of full row rank.

In the continuous-time case, the linear full-order filter under consideration is given by

$$\dot{\hat{x}}(t) = G\hat{x}(t) + Ky(t) \tag{3}$$

where $\hat{x}(t)$ denotes the state estimation, and G and K are filter parameters to be determined.

The estimation error covariance in the steady state is denoted by $P := \lim_{t\to\infty} P(t) := \lim_{t\to\infty} E[e(t)e^T(t)]$, where $e(t) = x(t) - \hat{x}(t)$, if the limit exists. By defining

$$\begin{aligned}x_f(t) &:= \begin{bmatrix}x(t)\\ e(t)\end{bmatrix}, \quad A_f := \begin{bmatrix}A & 0\\ A - G - KC & G\end{bmatrix}\\ D_f &:= \begin{bmatrix}D_1\\ D_1 - KD_2\end{bmatrix}\end{aligned} \tag{4}$$

$$\begin{aligned}M_f &:= \begin{bmatrix}M_1\\ M_1 - KM_2\end{bmatrix}, \quad N_f := [N \quad 0]\\ \Delta A_f &= M_f F(t) N_f\end{aligned} \tag{5}$$

and considering (1) and (3), we obtain the following augmented system:

$$\dot{x}_f(t) = (A_f + \Delta A_f)x_f(t) + D_f w(t). \tag{6}$$

When the system (6) is robustly asymptotically stable, the steady-state covariance defined by

$$\begin{aligned}X &:= \lim_{t\to\infty} X(t) := \lim_{t\to\infty} E\left[x_f(t)x_f^T(t)\right]\\ &:= \begin{bmatrix}X_{xx} & X_{xe}\\ X_{xe}^T & P\end{bmatrix}\end{aligned} \tag{7}$$

exists and satisfies the following Lyapunov matrix equation:

$$(A_f + \Delta A_f)X + X(A_f + \Delta A_f)^T + D_f D_f^T = 0. \tag{8}$$

Our objective is to seek the filter parameters G and K such that for all admissible parameter perturbations ΔA and ΔC, the following three requirements are simultaneously satisfied.

1) The augmented system (6) is asymptotically stable.
2) The steady-state error covariance P meets $[P]_{ii} \le \sigma_i^2$, $i = 1, 2, \ldots, n$, where $[P]_{ii}$ stands for the ith diagonal element of P. σ_i^2 $(i = 1, 2, \ldots, n)$ denotes the steady-state estimation error variance constraint on the ith state, which is not less than the minimal value obtained from the (robust) minimum variance filtering theory.
3) The H_∞ norm of the transfer function $H(s) = C_f[sI - (A_f + \Delta A_f)]^{-1}D_f$ from disturbances $w(t)$ to error state outputs $Le(t)$ (or $C_f x_f(t)$) satisfies the constraint $\|H(s)\|_\infty \le \gamma$, where L is the known error state output matrix, $C_f := [0 \ L]$, $\|H(s)\|_\infty = \sup_{\omega\in R} \sigma_{\max}[H(j\omega)]$, $\sigma_{\max}[\cdot]$ denotes the largest singular value of $[\cdot]$, and γ is a given positive constant.

III. Main Results: Continuous-Time Case

Prior to providing the main results, we first make the following definitions for notational simplicity:

$$\begin{aligned}\hat{A} &:= A + \left(\varepsilon M_1 M_1^T + D_1 D_1^T\right) P_1^{-1}\\ \hat{C} &:= C + \left(\varepsilon M_2 M_1^T + D_2 D_1^T\right) P_1^{-1}\end{aligned} \tag{9}$$

$$\begin{aligned}R &= \varepsilon M_2 M_2^T + D_2 D_2^T\\ \tilde{A} &:= \hat{A} - \left(\varepsilon M_1 M_2^T + D_1 D_2^T\right) R^{-1}\hat{C}.\end{aligned} \tag{10}$$

Theorem 1: Let $\delta_1 > 0$ and $\delta_2 > 0$ be sufficiently small constants, and let $U \in \mathbb{R}^{p\times p}$ be an arbitrary orthogonal matrix. If there exist a scalar $\varepsilon > 0$ and a matrix $H \in \mathbb{R}^{n\times p}$ such that the Riccati equations

$$\begin{aligned}&AP_1 + P_1 A^T + \varepsilon M_1 M_1^T + \varepsilon^{-1} P_1 N^T N P_1\\ &\quad + D_1 D_1^T + \delta_1 I = 0\end{aligned} \tag{11}$$

$$\begin{aligned}&\tilde{A}P_2 + P_2\tilde{A}^T - P_2(\hat{C}^T R^{-1}\hat{C} - \gamma^{-2} L^T L)P_2\\ &\quad + \varepsilon M_1 M_1^T + D_1 D_1^T + HH^T - \left(\varepsilon M_1 M_2^T + D_1 D_2^T\right)\\ &\quad \times R^{-1}\left(\varepsilon M_1 M_2^T + D_1 D_2^T\right)^T + \delta_2 I = 0\end{aligned} \tag{12}$$

respectively, have positive definite solutions $P_1 > 0$ and $P_2 > 0$, then with the parameters determined by

$$\begin{aligned}K &= \left(P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T\right) R^{-1} + HUR^{-1/2}\\ G &= \hat{A} - K\hat{C}\end{aligned} \tag{13}$$

the filter (3) will be such that for all admissible perturbations ΔA and ΔC, we have the following.

1) The augmented system (6) is asymptotically stable.
2) The steady-state error covariance P exists and meets $P < P_2$.
3) $\|H(s)\|_\infty \le \gamma$.

Proof:

1) For a scalar $\varepsilon > 0$ and a matrix $P_f > 0$, it is easy to prove that $(\Delta A_f)P_f + P_f(\Delta A_f)^T \le \varepsilon M_f M_f^T + \varepsilon^{-1} P_f N_f^T N_f P_f$. Next, by setting $P_f :=$ Block-diag(P_1, P_2), we have

$$(A_f + \Delta A_f)P_f + P_f(A_f + \Delta A_f)^T + \gamma^{-2} P_f C_f^T C_f P_f + D_f D_f^T \le \Psi := \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^T & \Psi_{22} \end{bmatrix} \tag{14}$$

where

$$\Psi_{11} = AP_1 + P_1 A^T + \varepsilon M_1 M_1^T + \varepsilon^{-1} P_1 N^T N P_1 + D_1 D_1^T \tag{15}$$

$$\Psi_{12} = P_1(A - G - KC)^T + \varepsilon M_1 (M_1 - KM_2)^T + D_1(D_1 - KD_2)^T \tag{16}$$

$$\Psi_{22} = GP_2 + P_2 G^T + \varepsilon (M_1 - KM_2)(M_1 - KM_2)^T + \gamma^{-2} P_2 L^T L P_2 + (D_1 - KD_2)(D_1 - KD_2)^T. \tag{17}$$

Equation (11) means that $\Psi_{11} = -\delta_1 I < 0$, and the expression of G in (13) and (16) implies $\Psi_{12} = 0$. Now, substitute the expression $G = \hat{A} - K\hat{C}$ into (17), and it is not difficult to verify that

$$\begin{aligned}\Psi_{22} = {} & \hat{A}P_2 + P_2\hat{A}^T + \varepsilon M_1 M_1^T + \gamma^{-2} P_2 L^T L P_2 + D_1 D_1^T \\ & + \left[KR^{1/2} - \left(P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T\right) R^{-1/2}\right] \\ & \times \left[KR^{1/2} - \left(P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T\right) R^{-1/2}\right]^T \\ & - \left(P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T\right) \\ & \times R^{-1}\left(P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T\right)^T.\end{aligned} \tag{18}$$

Furthermore, taking into account the expression of K in (13) and noticing the facts that $UU^T = I$ and $KR^{1/2} - (P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T)R^{-1/2} = HU$, we easily obtain from (18) that $\Psi_{22} = \tilde{A}P_2 + P_2\tilde{A}^T - P_2(\hat{C}^T R^{-1}\hat{C} - \gamma^{-2} L^T L)P_2 + \varepsilon M_1 M_1^T + D_1 D_1^T + HH^T - (\varepsilon M_1 M_2^T + D_1 D_2^T)R^{-1}(\varepsilon M_1 M_2^T + D_1 D_2^T)^T$.

Finally, it results from (12) that $\Psi_{22} = -\delta_2 I < 0$. Now, we have the conclusion that $\Psi < 0$, and thus, $(A_f + \Delta A_f)P_f + P_f(A_f + \Delta A_f)^T \le -(\gamma^{-2} P_f C_f^T C_f P_f + D_f D_f^T) + \Psi < 0$, which shows from Lyapunov stability theory that the augmented system (6) is asymptotically stable.

2) Since $A_f + \Delta A_f$ remains asymptotically stable, the steady-state covariance X exists and meets (8). Define $\Xi := \Psi - [(A_f + \Delta A_f)P_f + P_f(A_f + \Delta A_f)^T + \gamma^{-2} P_f C_f^T C_f P_f + D_f D_f^T]$. Clearly, $\Xi \ge 0$, and subsequently

$$(A_f + \Delta A_f)P_f + P_f(A_f + \Delta A_f)^T + \gamma^{-2} P_f C_f^T C_f P_f + D_f D_f^T - \Psi + \Xi = 0. \tag{19}$$

Subtract (8) from (19) to obtain $(A_f + \Delta A_f)(P_f - X) + (P_f - X)(A_f + \Delta A_f)^T + \gamma^{-2} P_f C_f^T C_f P_f - \Psi + \Xi = 0$, or equivalently, $P_f - X = \int_0^\infty \exp[(A_f + \Delta A_f)t](\gamma^{-2} P_f C_f^T C_f P_f - \Psi + \Xi)\exp[(A_f + \Delta A_f)^T t]\, dt > 0$, which means that $X < P_f$. Conclusion 2) follows from $P = [X]_{22}, P_2 = [P_f]_{22}$ directly, where $[\cdot]_{22}$ is the 22-sub-block of $[\cdot]$.

3) Since $-\Psi + \Xi > 0$, the proof of $\|H(s)\|_\infty \le \gamma$ can be completed by a standard manipulation of (19) [9]. The proof of Theorem 1 is then completed.

□

In view of Theorem 1, if the positive definite solutions P_1 and P_2 to (11) and (12) exist, and $P_2 > 0$ meets $[P_2]_{ii} \le \sigma_i^2$, $i = 1, 2, \ldots, n$, we will have the following conclusions.

1) The augmented system (6) is asymptotically stable.
2) $\|H(s)\|_\infty \le \gamma$.
3) $[P]_{ii} < [P_2]_{ii} \le \sigma_i^2$, $i = 1, 2, \ldots, n$.

Hence, with the filter (3), whose parameters K and G are determined by (13), the variance-constrained robust H_2/H_∞ filtering gain design task will be accomplished, and we can see that the key step in designing the expected filters is to deal with the solvability of the Riccati equations (11) and (12). □

Lemma 1 [5]: If A is stable and $\|N(sI - A)^{-1} M_1\|_\infty < 1$, then there exists a constant $\sigma > 0$ such that for all $\varepsilon \in (0, \sigma)$, the matrix Riccati equation (11) has a positive definite solution P_1.

In addition, since (12) is a parameter-dependent continuous-time Riccati matrix equation, the related numerical algorithms can be found in many papers, such as [5] and [10].

Moreover, we can use the modified quadratic matrix inequalities (QMI's) to restate Theorem 1 in a clearer sense and obtain the following results immediately.

Theorem 2: Let $U \in R^{p \times p}$ be an arbitrary orthogonal matrix. If there exists a positive scalar $\varepsilon > 0$ such that the QMI's

$$AP_1 + P_1 A^T + \varepsilon M_1 M_1^T + \varepsilon^{-1} P_1 N^T N P_1 + D_1 D_1^T < 0 \tag{20}$$

$$\begin{aligned}\Lambda := {} & \tilde{A}P_2 + P_2\tilde{A}^T - P_2(\hat{C}^T R^{-1}\hat{C} - \gamma^{-2} L^T L)P_2 \\ & + \varepsilon M_1 M_1^T + D_1 D_1^T - \left(\varepsilon M_1 M_2^T + D_1 D_2^T\right) \\ & \times R^{-1}\left(\varepsilon M_1 M_2^T + D_1 D_2^T\right)^T < 0\end{aligned} \tag{21}$$

respectively, have positive definite solutions $P_1 > 0$ and $P_2 > 0$, then the filter (3) with parameters

$$\begin{aligned} K &= \left(P_2\hat{C}^T + \varepsilon M_1 M_2^T + D_1 D_2^T\right) R^{-1} + EUR^{-1/2} \\ G &= \hat{A} - K\hat{C}\end{aligned} \tag{22}$$

where $E \in \mathbb{R}^{n \times p}$ $(p \le n)$ is an arbitrary matrix meeting $\Lambda + EE^T < 0$, and Λ is defined in (21) will be such that for all admissible perturbations ΔA and ΔC, we have the following.

1) The augmented system (6) is asymptotically stable.
2) The steady-state error covariance P exists and meets $P < P_2$.
3) $\|H(s)\|_\infty \le \gamma$.

Theorem 3: If there exist positive definite solutions, $P_1 > 0$ and $P_2 > 0$, respectively, to Riccati matrix equations (11) and (12) or QMI's (20), (21), and $[P_2]_{ii} \le \sigma_i^2$ $(i = 1, 2, \ldots, n)$, then the filter with parameters determined by (11) or (12) will satisfy the desired robust filtering performance constraints.

Remark 1: In practical applications, it is very desirable to directly solve Riccati matrix equations (11) and (12) or QMI's (20) and (21), subject to the constraint $[P_2]_{ii} \le \sigma_i^2$ $(i = 1, 2, \ldots, n)$ and then obtain the expected filter parameters readily from (13) or (22). When we deal with the QMI's (20) and (21), the local numerical searching algorithms suggested in [1] are very effective for a relatively low-order model. A related discussion of the solving algorithms for QMI's can also be found in [6].

IV. MAIN RESULTS: DISCRETE-TIME CASE

In this section, we will briefly state the main results for discrete-time systems and only give the necessary sketches of the proofs. Consider the following linear uncertain discrete-time stochastic system:

$$x(k+1) = (A+\Delta A)x(k) + D_1 w(k)$$
$$y(k) = (C+\Delta C)x(k) + D_2 w(k) \tag{23}$$

where x, y, A, C, D_1, D_2, ΔA, and ΔC have the similar meanings to the continuous-time case. $w(k)$ is a zero mean Gaussian white noise sequence with covariance $I > 0$. The initial state $x(0)$ has the mean $\bar{x}(0)$ and covariance $P(0)$ and is uncorrelated with $w(k)$.

Assumption 2: The matrix A is Schur stable and nonsingular, and the matrix D_2 or M_2 is of full row rank.

In the discrete-time case, the adopted linear full-order filter is of the following structure:

$$\hat{x}(k+1) = G\hat{x}(k) + Ky(k) \tag{24}$$

where $\hat{x}(k)$ stands for the state estimation, and G and K are filter parameters to be scheduled.

The steady-state estimation error covariance is defined by $P := \lim_{k\to\infty} P(k) := \lim_{k\to\infty} E[e(k)e^T(k)]$, $e(t) = x(t) - \hat{x}(t)$ if the limit exists. Furthermore, define $x_f(k) := [\, x^T(k) \quad e^T(k)\,]^T$ and A_f, D_f, M_f, N_f, and ΔA_f as in (4) and (5), and we obtain the augmented system

$$x_f(k+1) = (A_f + \Delta A_f)x_f(k) + D_f w(k). \tag{25}$$

When the augmented system (25) is robustly asymptotically stable, the steady-state covariance given by

$$X := \lim_{k\to\infty} X(k) := \lim_{k\to\infty} E\left[x_f(k)x_f^T(k)\right] := \begin{bmatrix} X_{xx} & X_{xe} \\ X_{xe}^T & P \end{bmatrix} \tag{26}$$

exists and meets the discrete-time Lyapunov equation $X = (A_f + \Delta A_f)X(A_f+\Delta A_f)^T + D_f D_f^T$.

The purpose of this section is to design the filter parameters G and K such that for all admissible perturbations ΔA and ΔC, we have the following.

1) The augmented system (25) is asymptotically stable.
2) $[P]_{ii} \le \sigma_i^2,\ i = 1, 2, \ldots, n$.
3) The H_∞ norm of the transfer function $H(z) := C_f[zI - (A_f + \Delta A_f)]^{-1} D_f$ from disturbances $w(k)$ to error state outputs $Le(k)$ (or $C_f x_f(k)$) satisfies the constraint $\|H(z)\|_\infty \le \gamma$, where L is the known error state output matrix, $C_f := [0 \;\; L]$, and $\|H(z)\|_\infty = \sup_{\theta\in[0,2\pi]} \sigma_{\max}[H(e^{j\theta})]$.

Lemma 2 (see, e.g., [11]): Let a positive scalar $\varepsilon > 0$ and a positive definite matrix $Q_f > 0$ be such that $N_f Q_f N_f^T < \varepsilon I$; then, $(A_f + \Delta A_f)Q_f(A_f+\Delta A_f)^T \le A_f(Q_f^{-1} - \varepsilon^{-1}N_f^T N_f)^{-1}A_f^T + \varepsilon M_f M_f^T$ holds.

For technical convenience, we define the following additional notation:

$$\Phi := \left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1} A^T$$
$$\hat{A} := A + \left(\varepsilon M_1 M_1^T + D_1 D_1^T\right)\Phi^{-1} \tag{27}$$
$$\hat{C} := C + \left(\varepsilon M_2 M_1^T + D_2 D_1^T\right)\Phi^{-1}$$
$$\Gamma := \Phi^{-1}\left(P_1^{-1} - \varepsilon^{-1}N^T N\right)^{-1}(\Phi^{-1})^T \tag{28}$$
$$\tilde{P}_2 := P_2 + P_2 L^T(\gamma^2 I - LP_2L^T)^{-1}LP_2 \tag{29}$$
$$\Theta := \hat{A}\tilde{P}_2\hat{C}^T + \left(\varepsilon M_1 M_1^T + D_1 D_1^T\right)\Gamma\left(\varepsilon M_2 M_1^T + D_2 D_1^T\right)^T \varepsilon M_1 M_2^T + D_1 D_2^T \tag{30}$$
$$R := \hat{C}\tilde{P}_2\hat{C}^T + \left(\varepsilon M_2 M_1^T + D_2 D_1^T\right)\Gamma\left(\varepsilon M_2 M_1^T + D_2 D_1^T\right)^T + \varepsilon M_2 M_2^T + D_2 D_2^T. \tag{31}$$

Theorem 4: Let $\delta_1 > 0$ and $\delta_2 > 0$ be sufficiently small positive constants, and let $U \in \mathbb{R}^{p\times p}$ be an arbitrary orthogonal matrix. If there exist a scalar $\varepsilon > 0$ and a matrix $H \in \mathbb{R}^{n\times p}$ such that

$$AP_1A^T - P_1 + AP_1N^T(\varepsilon I - NP_1N^T)^{-1}NP_1A^T + \varepsilon M_1M_1^T + D_1D_1^T + \delta_1 I = 0 \tag{32}$$
$$\hat{A}\tilde{P}_2\hat{A}^T - P_2 - \Theta R^{-1}\Theta^T + \left(\varepsilon M_1M_1^T + D_1D_1^T\right) \times \Gamma\left(\varepsilon M_1M_1^T + D_1D_1^T\right)^T + \varepsilon M_1M_1^T + D_1D_1^T + HH^T + \delta_2 I = 0 \tag{33}$$

together with the inequality constraints

$$NP_1N^T \le \varepsilon I, \quad LP_2L^T \le \gamma^2 I \tag{34}$$

respectively, have positive definite solutions $P_1 > 0$ and $P_2 > 0$, then the filter (24) with the parameters determined by $K = \Theta R^{-1} + HUR^{-1/2}$, $G = \hat{A} - K\hat{C}$ will be such that for all admissible perturbations ΔA and ΔC, we have the following.

1) The augmented system (25) is asymptotically stable.
2) The steady-state error covariance P exists and meets $P < P_2$.
3) $\|H(z)\|_\infty \le \gamma$.

Sketch of the Proof: To start with, we set $P_f := \text{Block-diag}(P_1, P_2)$, $Q_f := \text{Block-diag}(P_1, \tilde{P}_2)$, and by means of Lemma 2 and (27)–(31), it is easy to verify that $Q_f = P_f + P_fC_f^T(\gamma^2 I - C_fP_fC_f^T)^{-1}C_fP_f$ and

$$(A_f+\Delta A_f)\left[P_f + P_fC_f^T\left(\gamma^2 I - C_fP_fC_f^T\right)^{-1}C_fP_f\right] \times (A_f+\Delta A_f)^T - P_f + D_fD_f^T \le A_f\left(Q_f^{-1} - \varepsilon^{-1}N_f^TN_f\right)^{-1}A_f^T + \varepsilon M_fM_f^T - P_f + D_fD_f^T := \Psi. \tag{35}$$

Then, similar derivation as in the proof of Theorem 1 shows $\Psi < 0$, and therefore, the augmented system (25) is Schur stable. The proof techniques of second and third conclusions are along the lines of those used in Theorem 1 and [4, Lemma 5.1], and thus, the detailed proofs are omitted here.

We now briefly discuss the solvability of the Riccati equations (32) and (33).

Lemma 3 [3]: If the uncertain system (23) is quadratically stable, then there must exist a scalar $\varepsilon > 0$ and a matrix $P_1 > 0$ that satisfy $NP_1N^T < \varepsilon I$ and the discrete-time Riccati equation (32).

Next, noting that (33) is actually a generalized parameter-dependent Riccati equation, we can deal with it by using the approach proposed in [6] and [11]. We further restate Theorem 1 in terms of two QMI's and then obtain our main results for the discrete-time case.

Theorem 5: Let $U \in R^{p\times p}$ be an arbitrary orthogonal matrix. If there exists a positive scalar $\varepsilon > 0$ such that the following two QMI's

$$AP_1A^T - P_1 + AP_1N^T(\varepsilon I - NP_1N^T)^{-1}NP_1A^T + \varepsilon M_1M_1^T + D_1D_1^T < 0 \tag{36}$$

$$\Lambda := \hat{A}\tilde{P}_2\hat{A}^T - P_2 - \Theta R^{-1}\Theta^T + \left(\varepsilon M_1 M_1^T + D_1 D_1^T\right) \times \Gamma\left(\varepsilon M_1 M_1^T + D_1 D_1^T\right)^T + \varepsilon M_1 M_1^T + D_1 D_1^T < 0 \quad (37)$$

together with the inequality constraints (34), respectively, have positive definite solutions $P_1 > 0$ and $P_2 > 0$, then the filter (24) with the parameters determined by $K = \Theta R^{-1} + EUR^{-1/2}$, $G = \hat{A} - K\hat{C}$, where $E \in \mathbb{R}^{n\times p}$ $(p \le n)$ is an arbitrary matrix meeting $\Lambda + EE^T < 0$ and Λ is defined in (37), will be such that we have the following.

1) The augmented system (25) is asymptotically stable.
2) $P < P_2$.
3) $\|H(z)\|_\infty \le \gamma$.

Theorem 6: Subject to the constraints (34), if there exist positive definite solutions $P_1 > 0$ and $P_2 > 0$, respectively, to Riccati matrix equations (32) and (33) or QMI's (36), (37), and $[P_2]_{ii} \le \sigma_i^2$ $(i = 1, 2, \ldots, n)$, then the filter with parameters determined by Theorem 4 or Theorem 5 will, respectively, meet the desired robust H_2/H_∞ filtering performance requirements.

V. Numerical Example

Consider a linear discrete-time uncertain stochastic system (23) with the following parameters:

$$A = \begin{bmatrix} 0.8 & 0.05 \\ -0.08 & -0.5 \end{bmatrix}, \quad C = [1 \quad 0]$$
$$D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_2 = [0.1 \quad 0.08]$$
$$M_1 = \begin{bmatrix} 0.08 \\ 0.06 \end{bmatrix}, \quad M_2 = 0.1$$
$$N = [0.5 \quad 0.5], \quad L = [0.01 \quad 0.01].$$

We wish to design a filter (24) such that the augmented system (25) is asymptotically stable, the steady-state error covariance P meets $[P]_{11} \le \sigma_1^2 = 0.5$, $[P]_{22} \le \sigma_2^2 = 1.2$, and $\|H(z)\|_\infty \le \gamma = 0.9$.

Choose $\varepsilon = 0.5$, then a positive definite solution to QMI (36), and therefore, Φ, $\hat{A}$, $\hat{C}$, Γ can be obtained as follows:

$$P_1 = \begin{bmatrix} 0.0410 & -0.0006 \\ -0.0006 & 0.0174 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0.0335 & -0.0032 \\ 0.0007 & -0.0088 \end{bmatrix}$$
$$\hat{A} = \begin{bmatrix} 1.2029 & -0.3712 \\ 0.0194 & -1.8813 \end{bmatrix}, \quad \hat{C} = [1.4468 \quad -1.4175]$$
$$\Gamma = \begin{bmatrix} 40.2445 & 26.0997 \\ 26.0997 & 232.2372 \end{bmatrix}.$$

Then, solve the QMI (37) to give

$$P_2 = \begin{bmatrix} 0.4271 & -0.1484 \\ -0.1484 & 1.1617 \end{bmatrix}$$

and it is easily seen that $[P_2]_{ii} < \sigma_i^2$ $(i = 1, 2)$, and (34) is satisfied. Next, choose parameter E, which meets $\Lambda + EE^T < 0$ [Λ is defined in (37)] as $E = [0.0800 \;\; 0.1000]^T$. Then, for the two cases of $U_1 = 1$ and $U_2 = -1$, we obtain the corresponding desired filter parameters, respectively, as

$$K_1 = \begin{bmatrix} 0.4812 \\ 0.9643 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.5068 & 0.3108 \\ -1.3758 & -0.5144 \end{bmatrix}$$
$$K_2 = \begin{bmatrix} 0.4002 \\ 0.8631 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.6240 & 0.1960 \\ -1.2293 & -0.6579 \end{bmatrix}.$$

It is not difficult to test that the precsribed performance objectives are all realized.

VI. Conclusion

In this correspondence, attention has been focused on designing linear perturbation-independent filters that achieve the multiple prescribed objectives of filtering process:

- robust stability,;
- H_∞ norm;
- steady-state estimation error variance constraints.

The further study will be the development of efficient algorithms with guaranteed convergence.

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