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The Zero-Free Intervals for Characteristic Polynomials of Matroids

HUGH EDWARDS, ROBERT HIERONS and BILL JACKSON

Department of Mathematical and Computing Sciences,
Goldsmiths College, London SE14 6NW, UK
(e-mail: b.jackson@gold.ac.uk)

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Let M be a loopless matroid with rank r and c components. Let $P(M,t)$ be the characteristic polynomial of M. We shall show that $(-1)^r P(M,t) \geqslant (1-t)^r$ for $t \in (-\infty, 1)$, that the multiplicity of the zeros of $P(M,t)$ at $t=1$ is equal to c, and that $(-1)^{r+c} P(M,t) \geqslant (t-1)^r$ for $t \in (1, \frac{32}{27}]$. Using a result of C. Thomassen we deduce that the maximal zero-free intervals for characteristic polynomials of loopless matroids are precisely $(-\infty, 1)$ and $(1, \frac{32}{27}]$.

1. Introduction

Characteristic polynomials of matroids were first studied by Rota [6]. Heron [3] defined chromatic polynomials of matroids and showed that they are equivalent to characteristic polynomials. Here, we are concerned with arbitrary matroids, so we will refer to *characteristic polynomials*, reserving the term *chromatic polynomials* for graphs. It is known (see Jackson [4], Tutte [9] and Woodall [12]) that the chromatic polynomial of an arbitrary graph $P(G,t)$ has no zeros in the intervals $(-\infty, 0)$, $(0,1)$ and $(1, \frac{32}{27}]$, that the multiplicity of zeros at 0 is equal to the number of components of the graph and that the multiplicity of zeros at 1 is equal to the number of blocks of the graph. Recently Thomassen [8] has shown that the chromatic zeros of graphs are dense in $(\frac{32}{27}, \infty)$. It follows that the maximal zero-free intervals for chromatic polynomials of graphs are precisely $(-\infty, 0)$, $(0,1)$ and $(1, \frac{32}{27}]$. He also showed that the chromatic polynomial $P(G,t)$ of a connected graph G with n vertices satisfies $|P(G,t)| \geqslant |t(t-1)^{n-1}|$ for all $t \leqslant 32/27$.

The facts that the characteristic polynomial of a graphic matroid has no zeros in the intervals $(-\infty, 1)$ and $(1, \frac{32}{27}]$ and that the multiplicity of zeros at 1 is equal to the number of connected components of the matroid follow from the above results on chromatic polynomials of graphs and the well-known equality $t^c P(M,t) = P(G,t)$ between the chromatic polynomial $P(G,t)$ of a graph G with c components, and the characteristic polynomial $P(M,t)$ of its associated graphic matroid M. It is also known that the characteristic polynomial of a cographic matroid has no zeros in the intervals $(-\infty, 1)$ and $(1, \frac{32}{27}]$, and that the multiplicity of zeros at 1 is equal to the number of connected components of the matroid. This follows immediately from the corresponding results for flow polynomials of graphs (see Wakelin [11]). We shall deduce that both these zero-free intervals hold for the characteristic polynomial $P(M,t)$ of a loopless matroid M with c components and rank r by showing that $(-1)^r P(M,t) \geqslant (1-t)^r$ for $t \in (-\infty, 1)$ and $(-1)^{r+c} P(M,t) \geqslant (t-1)^r$ for $t \in (1, \frac{32}{27}]$. We will also show that the multiplicity of the zeros of $P(M,t)$ at $t = 1$ is equal to c.

Given a matroid M, we shall use S_M, $\mathscr{C}_M$, $\mathscr{I}_M$ and r_M to denote the groundset, the set of circuits, the set of independent sets and the rank function, respectively, of M. These will be simplified to S, $\mathscr{C}$, $\mathscr{I}$ and r when it is obvious which matroid we are referring to. We denote the rank of M by $r(M)$, the number of elements of M by $|M|$ and the number of components of M by $c(M)$. Thus $r(M) = r_M(S)$ and $|M| = |S|$. Let $M - e$ and M/e respectively denote the matroids resulting from the deletion and contraction of an element e.

Sections 2–5 contain definitions and lemmas that will be used in the proofs of the main results of the paper in Section 6. We refer the reader to Oxley [5] for basic definitions not given in this paper.

2. Some elementary matroid results

Lemma 2.1. ([2], Proposition 4.1) *Let M be a matroid with $e \in S$, $A \subseteq S - e$. Then*

(a) $r_{M-e}(A) = r_M(A)$
(b) $r(M-e) = r(M) - 1$ *if e is a coloop of M*
(c) $r(M-e) = r(M)$ *if e is not a coloop of M.*

By repeated application of Lemma 2.1(a) we obtain the following lemma.

Lemma 2.2. *Let M be a matroid with $A, B \subseteq S, A \cap B = \emptyset$. Then $r_M(A) = r_{M-B}(A)$.*

Lemma 2.3. ([2], Proposition 4.1) *Let M be a matroid with $e \in S$, $A \subseteq S - e$. Suppose e is not a loop in M. Then*

(a) $r_{M/e}(A) = r_M(A \cup \{e\}) - 1$
(b) $r(M/e) = r(M) - 1$.

Lemma 2.4. ([5], Proposition 3.1.26) *Let M be a matroid with $e, f \in S$. Then $(M-e)/f = (M/f) - e$.*

Lemma 2.5. ([9], Theorem 6.5) *Let $e \in S$ be an element of a connected matroid M. Then either $M - e$ or M/e is connected.*

Lemma 2.6. ([9], Theorem 6.6) *Let e be an element of a 3-connected matroid M. Then, provided $|M| \geqslant 4$, both $M - e$ and M/e are connected.*

3. Characteristic polynomials

The *characteristic polynomial* $P(M,t)$ of a matroid M is a polynomial in t defined by

$$P(M,t) = \sum_{A \subseteq S} (-1)^{|A|} t^{r_M(S) - r_M(A)}.$$

Lemma 3.1. ([3], Lemma 1.4) *If the matroid M has a loop, then $P(M,t) \equiv 0$.*

Lemma 3.2. ([3], Lemma 1.2) *If M is a matroid and $e \in S$ is not a loop or a coloop, then*

$$P(M,t) = P(M-e,t) - P(M/e,t).$$

Lemma 3.3. ([3], Lemma 1.5) *If e is a coloop in a loopless matroid M, then $P(M,t) = (t-1)P(M-e,t)$.*

Lemma 3.4. ([3], Lemma 3.10) *Let M be a matroid and $M_1, M_2, \ldots, M_n$ be the components of M. Then $P(M,t) = \prod_{i=1}^{n} P(M_i,t)$.*

Lemma 3.5. *If e is a parallel element in a matroid M, then $P(M,t) = P(M-e,t)$.*

Proof. Follows from Lemmas 3.1 and 3.2. □

4. Parallel connections, series connections and 2-sums

Let M_1 and M_2 be matroids. We shall denote the *direct sum* of M_1 and M_2 by $M_1 \oplus M_2$. We shall denote the *parallel connection*, *series connection*, and *2-sum* of M_1 and M_2 about basepoint p by $M_1 \parallel^p M_2$, $M_1 *^p M_2$, and $M_1 \oplus_2^p M_2$, respectively.

Given matroids M_i, $1 \leqslant i \leqslant n$, such that $S_{M_i} \cap S_{M_j} = \{p\}$ for $1 \leqslant i < j \leqslant n$, where p is not a loop or a coloop in M_i and $|M_i| \geqslant 3$ for $1 \leqslant i \leqslant n$, we define the *2-sum* of $M_1, \ldots, M_n$ to be the matroid

$$M_1 \oplus_2^p M_2 \oplus_2^p \cdots \oplus_2^p M_n = (M_1 \parallel^p M_2 \parallel^p \cdots \parallel^p M_n) - p.$$

Lemma 4.1. ([5], Corollary 7.1.23) *The class of graphic matroids is closed under parallel connection, series connection and 2-sum.*

Lemma 4.2. *Let $M = M_1 \parallel^p M_2 \parallel^p \cdots \parallel^p M_n$.*

(a) $r(M) = \sum_{i=1}^{n} r(M_i) - n + 1$.
(b) $c(M) = \sum_{i=1}^{n} c(M_i) - n + 1$.
(c) $M/p = M_1/p \oplus M_2/p \oplus \cdots \oplus M_n/p$.
(d) $P(M,t) = (t-1)^{-n+1} \prod_{i=1}^{n} P(M_i,t)$.

Proof. Follows from [1, Theorem 6.16(i) and (v), Propositions 5.5 and 5.8] by using induction on n. □

Lemma 4.3. ([1], Proposition 5.8) *Let M be a connected matroid and $M/p = M_1 \oplus M_2$, where both M_1 and M_2 are nonempty. Then $M = (M - S_{M_1}) \parallel^p (M - S_{M_2})$.*

Lemma 4.4. ([1], Theorem 6.16(i), (v) and Propositions 4.6, 4.7) *Let $M = M_1 *^p M_2$.*

(a) $r(M) = r(M_1) + r(M_2)$.
(b) $c(M) = c(M_1) + c(M_2) - 1$.
(c) $M - p = (M_1 - p) \oplus (M_2 - p)$.
(d)

$$\begin{aligned} P(M,t) &= (t-2)(t-1)^{-1}P(M_1,t)P(M_2,t) + P(M_1,t)P(M_2/p,t) \\ &+ P(M_1/p,t)P(M_2,t). \end{aligned}$$

Lemma 4.5. ([1], Proposition 4.10) *Let M be a connected matroid and $M-p = M_1 \oplus M_2$, where both M_1 and M_2 are nonempty. Then*

$$M = (M/S_{M_1}) *^p (M/S_{M_2}).$$

Lemma 4.6. *Let $M = M_1 \oplus_2^p M_2$. Then*

(a) $r(M) = r(M_1) + r(M_2) - 1$
(b) $c(M) = c(M_1) + c(M_2) - 1$
(c)

$$\begin{aligned} P(M,t) &= P(M_1/p,t)P(M_2 - p,t) \\ &+ P(M_2,t)[(t-1)^{-1}P(M_1,t) - P(M_1/p,t)]. \end{aligned}$$

Proof. We obtain (a) and (b) by applying Lemma 2.1(c) to Lemma 4.2(a) and (b). To prove (c) we let $M' = M_1 \parallel^p M_2$. By Lemma 3.2,

$$P(M,t) = P(M',t) + P(M'/p,t). \tag{4.1}$$

By Lemma 4.2(d),

$$P(M',t) = P(M_1,t)P(M_2,t)(t-1)^{-1}. \tag{4.2}$$

By Lemma 4.2(c), $M'/p = M_1/p \oplus M_2/p$, so by Lemma 3.4,

$$P(M'/p,t) = P(M_1/p,t)P(M_2/p,t). \tag{4.3}$$

Hence, by equations (4.1), (4.2) and (4.3),

$$P(M,t) = P(M_1,t)P(M_2,t)(t-1)^{-1} + P(M_1/p,t)P(M_2/p,t). \tag{4.4}$$

By Lemma 3.2, $P(M_2/p,t) = P(M_2 - p,t) - P(M_2,t)$, so from (4.4) we obtain

$$\begin{aligned} (t-1)P(M,t) &= P(M_1,t)P(M_2,t) \\ &\quad +(t-1)P(M_1/p,t)(P(M_2-p,t) - P(M_2,t)) \\ &= (t-1)P(M_1/p,t)P(M_2-p,t) \\ &\quad +P(M_2,t)(P(M_1,t) - (t-1)P(M_1/p,t)). \end{aligned}$$

□

Lemma 4.7. *Let M be a matroid that can be expressed as a 2-sum about a basepoint p. Then there exist matroids $M_1, M_2, \ldots, M_k$ such that $M = M_1 \oplus_2^p M_2 \oplus_2^p \cdots \oplus_2^p M_k$ and M_i/p is connected for $1 \leqslant i \leqslant k$.*

Proof. Suppose the number of components of the 2-sum, k, is as large as possible. Suppose the lemma is false and there exists a component M_i, $1 \leqslant i \leqslant k$ such that M_i/p is disconnected. Without loss of generality assume this component is M_k. Because M_k/p is disconnected, there exist matroids N_k and N_{k+1} such that $M_k/p = N_k \oplus N_{k+1}$. By Lemma 4.3, M_k is the parallel connection of some matroids N'_k and N'_{k+1} about basepoint p. But this means we have $M = M_1 \oplus_2^p M_2 \oplus_2^p \cdots \oplus_2^p M_{k-1} \oplus_2^p N_k \oplus_2^p N_{k+1}$, contradicting the maximality of k and completing the proof of Lemma 4.7. □

Lemma 4.8. *Let $M = M_1 \oplus_2^p M_2$ and $q \in S_{M_1} - \{p\}$, where q is not parallel to p in M_1. Then $(M_1 \oplus_2^p M_2)/q = (M_1/q \oplus_2^p M_2)$.*

Proof. This follows from [1, Proposition 5.8], the definition of 2-sum, and Lemma 2.4. □

Lemma 4.9. *Let M_1, M_2 and M_3 be matroids such that $S_{M_i} \cap S_{M_j} = \{p\}$ for $1 \leqslant i < j \leqslant 3$. Then $M_1 \oplus_2^p (M_2 *^p M_3) = (M_1 *^p M_2) \oplus_2^p M_3$.*

Proof. It can be seen that the set of circuits of the matroids on each side of the inequality is given by $\mathscr{C}_{M_1-p} \cup \mathscr{C}_{M_2-p} \cup \mathscr{C}_{M_3-p} \cup \{(C_1-p) \cup (C_2-p) \cup (C_3-p) \,:\, p \in C_i \in \mathscr{C}_{M_i}, 1 \leqslant i \leqslant 3\}$. □

Lemma 4.10. ([7], 2.6) *A connected matroid M is not 3-connected if and only if $M = M_1 \oplus_2^p M_2$ for some matroids M_1 and M_2.*

5. Generalized coloops and 3-circuits.

Let $M_\triangle = A \parallel^p B$, where A and B are each isomorphic to $U_{2,3}$, the graphic matroid of a circuit of length three. Thus $M_\triangle$ is a matroid with five elements, one of which is distinguished as p, such that p is in a pair of 3-circuits, and such that $M_\triangle - p$ is a 4-circuit. Because A and B both have rank 2, it follows from Lemma 4.2(a) that the rank of $M_\triangle$ is 3. As A and B are both connected and graphic, by Lemmas 4.2(b) and 4.1, it follows that $M_\triangle$ is also connected and graphic.

Given a matroid M we define a *parallel subdivision* of M to be a matroid M_{PS} obtained by first choosing a copy $M^i_\triangle$ of $M_\triangle$ with distinguished element p_i, then relabelling one of the elements of M as p_i, and putting $M_{PS} = M \oplus_2^{p_i} M^i_\triangle$. By Lemma 4.6(a), $r(M_{PS}) = r(M) + r(M_\triangle) - 1 = r(M) + 2$. By Lemma 4.6(b), if M is connected so is M_{PS}. It also follows, by Lemma 4.1, that if M is graphic so is M_{PS}. This operation has the effect of replacing $p_i \in M$ by a pair of 2-cocircuits, which themselves form a 4-circuit.

A *generalized p-coloop* is either the rank one matroid with one element p, or $M_\triangle - p$, or any matroid that can be obtained from $M_\triangle - p$ by a sequence of parallel subdivisions. Given a generalized p-coloop (other than the one-element matroid) $M = (M_\triangle - p) \oplus_2^{p_1}$

$M_\triangle^1 \oplus_2^{p_2} M_\triangle^2 \oplus_2^{p_3} \cdots \oplus_2^{p_n} M_\triangle^n$, we define its *p-extension* to be the matroid $M' = M_\triangle \oplus_2^{p_1} M_\triangle^1 \oplus_2^{p_2} M_\triangle^2 \oplus_2^{p_3} \cdots \oplus_2^{p_n} M_\triangle^n$. Thus the p-extension of $M_\triangle - p$ is $M_\triangle$ and, in general, $p \in S_{M'}$.

Because $M_\triangle - p$ has rank 3, is connected and is graphic, by Lemma 4.6(b) and Lemma 4.1 its parallel subdivisions are connected and graphic. Hence generalized p-coloops are the graphic matroids corresponding to the generalized edges defined in [4]. Because every parallel subdivision increases the rank by 2, it follows that all generalized p-coloops have odd rank.

Lemma 5.1. *Let M be a generalized p-coloop with at least four elements and M' be the p-extension of M. Then M'/p is disconnected with exactly two components.*

Proof. Since $M_\triangle/p$ is disconnected with exactly two components we may suppose that $|M| > 4$. Let $M = (M_\triangle - p) \oplus_2^{p_1} M_\triangle^1 \oplus_2^{p_2} M_\triangle^2 \oplus_2^{p_3} \cdots \oplus_2^{p_n} M_\triangle^n$. Then $M' = M_\triangle \oplus_2^{p_1} M_\triangle^1 \oplus_2^{p_2} M_\triangle^2 \oplus_2^{p_3} \cdots \oplus_2^{p_n} M_\triangle^n$ and $M'/p = (M_\triangle \oplus_2^{p_1} M_\triangle^1 \oplus_2^{p_2} M_\triangle^2 \oplus_2^{p_3} \cdots \oplus_2^{p_n} M_\triangle^n)/p = (M_\triangle/p) \oplus_2^{p_1} M_\triangle^1 \oplus_2^{p_2} M_\triangle^2 \oplus_2^{p_3} \cdots \oplus_2^{p_n} M_\triangle^n$, by Lemma 4.8. Since $M_\triangle/p$ has exactly two components, it follows by repeated application of Lemma 4.6(b) that M'/p is disconnected with exactly two components. □

A *generalized 3-circuit* is either the graphic matroid of a 3-circuit, $U_{2,3}$, or any matroid which can be obtained from $U_{2,3}$ by a sequence of parallel subdivisions.

Because $U_{2,3}$ has rank 2, is connected, and is graphic, it follows from Lemma 4.6(b) and Lemma 4.1 that all generalized 3-circuits have even rank, are connected and graphic. Hence generalized 3-circuits are the graphic matroids corresponding to the generalized triangles defined in [4].

Lemma 5.2. *Let M be a matroid with the following properties.*

(a) *M is connected.*
(b) *For every $e \in S$, $M - e$ is disconnected with exactly two components.*
(c) *Whenever M is the 2-sum of n connected matroids $M_i (1 \leqslant i \leqslant n)$, with basepoint p, such that M_i/p is connected for all i, $1 \leqslant i \leqslant n$, then n is odd.*
(d) *Whenever M is the 2-sum of two matroids M_1' and M_2', with basepoint q and such that $M_1 = M_1' - q$ is a generalized q-coloop, then $M_2 = M_2' - q$ has exactly two components.*

Then M is a generalized 3-circuit.

Proof. Suppose Lemma 5.2 is false and let M be a counterexample with $|M|$ as small as possible. Because M is connected and, by (b), has no parallel elements, if $|M| = 3$ then $M = U_{2,3}$ and hence is a generalized 3-circuit. So it follows that $|M| \geqslant 4$.

Claim 1. *M can be expressed as a 2-sum of matroids Y_1, Y_2 and Y_3 about the same basepoint p.*

Proof. By (a) and (b), M is connected but not 3-connected. Thus, by Lemma 4.10, M is the 2-sum of matroids, M_1' and M_2' say, about basepoint p, $p \notin M$. Claim 1 now follows from Lemma 4.7 and (c). □

Claim 2. *M can be expressed as a 2-sum of matroids Y_1, Y_2 and Y_3 about some basepoint p, where Y_1 and Y_2 are both 3-circuits.*

Proof. By Claim 1, M can be expressed as $M = Y_1 \oplus_2^p Y_2 \oplus_2^p Y_3$. Suppose p, Y_1 and Y_2 have been chosen such that $|Y_1 \oplus_2^p Y_2|$ is as small as possible. It follows from (a) and Lemma 4.6(b) that Y_1, Y_2 and Y_3 are connected. Assume that Y_1 is not 3-connected. Hence, by Lemma 4.10, Y_1 can be expressed as a 2-sum. By Lemma 4.7, $Y_1 = X_1 \oplus_2^q X_2 \oplus_2^q \cdots \oplus_2^q X_k$, and X_i/q is connected, $(1 \leqslant i \leqslant k)$. Using the minimality of $|Y_1 \oplus_2^p Y_2|$, we have $q \neq p$. Without loss of generality we may assume $p \in X_k$. We now have

$$\begin{aligned} M &= (X_1 \oplus_2^q X_2 \oplus_2^q \cdots \oplus_2^q X_{k-1} \oplus_2^q X_k) \oplus_2^p Y_2 \oplus_2^p Y_3 \\ &= X_1 \oplus_2^q X_2 \oplus_2^q \cdots \oplus_2^q X_{k-1} \oplus_2^q (X_k \oplus_2^p Y_2 \oplus_2^p Y_3). \end{aligned}$$

As X_k/q is connected and $(X_k \oplus_2^p Y_2 \oplus_2^p Y_3)/q = (X_k/q \oplus_2^p Y_2 \oplus_2^p Y_3)$ by Lemma 4.8, it follows from Lemma 4.6(b) that $(X_k \oplus_2^p Y_2 \oplus_2^p Y_3)/q$ is connected. Thus k is odd, by (c), and $k \geqslant 3$. Now $M = X_1 \oplus_2^q X_2 \oplus_2^q (X_3 \oplus_2^q \cdots \oplus_2^q X_{k-1} \oplus_2^q X_k \oplus_2^p Y_2 \oplus_2^p Y_3)$. Since $X_1 \oplus_2^q X_2$ is contained in Y_1, this contradicts the minimality of $|Y_1 \oplus_2^p Y_2|$. Hence Y_1 is 3-connected. By a similar argument, Y_2 is also 3-connected.

Because M is a 2-sum, $|Y_1|, |Y_2| \geqslant 3$. Also, every element of $Y_1 - p$ is in M, so we can choose $f \in M \cap Y_1$. But, by Lemma 2.6, provided $|Y_1| \geqslant 4$, $Y_1 - f$ is connected and hence $M - f$ is connected, contradicting (b). Hence $|Y_1| = 3$, and by a similar argument, $|Y_2| = 3$. Since M has no parallel elements, it follows that Y_1 and Y_2 are 3-circuits, so completing the proof of Claim 2. □

Claim 3. *M is a 2-sum of matroids M_1' and M_2', with basepoint p such that $M_1' = M_\triangle$ and so $M_1 = M_1' - p$ is a generalized p-coloop.*

Proof. Let Y_1, Y_2 and Y_3 be as in Claim 2. Let M_1' be the parallel connection of Y_1 and Y_2 and $M_2' = Y_3$. Then $M = M_1' \oplus_2^p M_2'$ and $M_1' = M_\triangle$. Hence $M_1 = M_1' - p$ is a generalized p-coloop. □

We now return to the proof of Lemma 5.2. Let p, M_1' and M_2' be defined as in Claim 3. Thus M_1 is a generalized p-coloop.

Consider M_2'. We will show that it satisfies the hypotheses of Lemma 5.2.

(a) Because M is connected, by Lemma 4.6(b), M_2' is also connected.

(b) Choose $e \in M_2'$. If $e \neq p$, then $e \in S_M$. Applying (b) to M, we deduce that $M - e$ is disconnected with exactly two components. Since $M - e = M_\triangle \oplus_2^p (M_2' - e)$, it follows from Lemma 4.6(b) that $M_2' - e$ is disconnected with exactly two components. If $e = p$, then $M_2' - e = M_2$, which has exactly two components since M satisfies (d).

(c) Choose matroids $N_i'(1 \leqslant i \leqslant n)$, with basepoint q, such that N_i'/q is connected for all i, $1 \leqslant i \leqslant n$ and M_2' is their 2-sum. Hence $q \notin M_2'$, so $q \neq p$. Assume without loss of generality that $p \in N_1'$. Because M_1' and N_1' are both connected, $L_1' = M_1' \oplus_2^p N_1'$ is connected and $q \in L_1'$. Similarly, $L_1'/q = M_1' \oplus_2^p N_1'/q$ is connected. But $M = L_1' \oplus_2^q N_2' \oplus_2^q \cdots \oplus_2^q N_n'$, and because M satisfies (c), n is odd. Thus M_2' also satisfies (c).

(d) Let M_2' be the 2-sum of matroids N_1' and N_2' with basepoint q such that $N_1 = N_1' - q$ is a generalized q-coloop. We need to show that $N_2 = N_2' - q$ has exactly two components. Because M satisfies (d) we know $M_2 = M_2' - p$ has exactly two components and $p \in M_2'$, so $p \neq q$. There are two cases to consider.

- Suppose $p \in N_2'$. Then $M = N_1' \oplus_2^q L_2'$, where $L_2' = N_2' \oplus_2^p M_\triangle$. Since N_1 is a generalized q-coloop we may apply (d) to M and deduce that $L_2 = L_2' - q$ has exactly two components. But $L_2 = N_2 \oplus_2^p M_\triangle$, so it follows from Lemma 4.6(b) that N_2 also has exactly two components.
- Suppose $p \in N_1'$. Then $M = L_1' \oplus_2^q N_2'$, where $L_1' = N_1' \oplus_2^p M_\triangle$. Because N_1 is a generalized q-coloop so is L_1. Applying (d) to M we deduce that N_2 has exactly two components.

Since M_2' satisfies all four conditions and has fewer elements than M, it follows that M_2' is a generalized 3-circuit. Since $M = M_2' \oplus_2^p M_\triangle$, we deduce that M is also a generalized 3-circuit. This contradicts the choice of M and completes the proof of Lemma 5.2. □

6. Zeros of characteristic polynomials

Theorem 6.1. *Let M be a loopless matroid with rank r and characteristic polynomial $P(M,t)$. Then*

(a) $P(M,t) = t^r - |M|\,t^{r-1} + k_{r-2}t^{r-2} - \cdots + (-1)^r k_0$ *where $k_0, \ldots, k_{r-2}$ are positive integers*
(b) $(-1)^r P(M,t) \geqslant (1-t)^r$ *for $t \in (-\infty, 1)$*
(c) $P(M,1) = 0$.

Proof. We use induction on $|M|$. If M is a single element matroid then $P(M,t) = t-1$, which satisfies (a),(b) and (c). Using Lemma 3.5 we may assume that M has no parallel elements. When $|M| \geqslant 2$ we choose $e \in S$ and use Lemmas 2.1, 2.3, 3.2 and the inductive hypothesis on $M-e$ and M/e if e is not a coloop, and use Lemmas 2.1, 3.3 and the inductive hypothesis on $M-e$ if e is a coloop. □

Theorem 6.2. *Let M be a loopless matroid. Then the multiplicity of* 1 *as a zero of $P(M,t)$ is equal to the number of components of M.*

Proof. We adopt the proof technique of Woodall [12] for chromatic polynomials of graphs. If M is a single element matroid, $P(M,t) = t-1$, and the theorem holds. Hence, suppose $|M| \geqslant 2$. Using Lemma 3.5 we may assume that M has no parallel elements.

Suppose M is connected. We show that $\frac{d}{dt}P(M,t)$ is nonzero at $t=1$. Choose $e \in S$. By Lemma 3.2,

$$\frac{d}{dt}P(M,t) = \frac{d}{dt}P(M-e,t) - \frac{d}{dt}P(M/e,t). \tag{6.1}$$

Using Theorem 6.1(b) and (c) and Lemma 2.1(c), $\frac{d}{dt}P(M-e,t)$ has sign $(-1)^{r(M)-1}$ or is zero when $t=1$ and $\frac{d}{dt}P(M/e,t)$ has sign $(-1)^{r(M)}$ or is zero when $t=1$. Thus $\frac{d}{dt}P(M-e,t)$ and $\frac{d}{dt}P(M/e,t)$ are either zero or have opposite signs at $t=1$. By Lemma 2.5, either

$M - e$ or M/e is connected. Thus, by induction, either $\frac{d}{dt}P(M-e,t)$ or $\frac{d}{dt}P(M/e,t)$ is nonzero at $t=1$. Hence, by (6.1), we have $\frac{d}{dt}P(M,t)$ is nonzero at $t=1$.

Finally, if M is not connected then we apply the inductive hypothesis to each connected component of M and use Lemma 3.4. □

Theorem 6.3. *Let M be a loopless matroid with rank $r(M)$, and $c(M)$ components. Then*

$$(-1)^{r(M)+c(M)}P(M,t) \geqslant (t-1)^{r(M)} \tag{6.2}$$

for $t \in (1, \frac{32}{27}]$.

Proof. Suppose Theorem 6.3 is false. Let M be a matroid and $t \in (1, \frac{32}{27}]$ such that $P(M,t)$ does not satisfy (6.2), and assume that $|M|$ is as small as possible. Clearly $|M| \geqslant 2$. Using Lemma 3.5 we may deduce that M has no parallel elements. We shall show that M satisfies the hypotheses of Lemma 5.2 and hence is a generalized 3-circuit.

Claim 4. *M is connected.*

Proof. We proceed by contradiction. Suppose $M = M_1 \oplus M_2$. By Lemma 3.4, $P(M,t) = P(M_1,t)P(M_2,t)$. As M is the smallest counterexample, M_1 and M_2 satisfy (6.2). Thus

$$\begin{aligned}(-1)^{r(M)+c(M)}P(M,t) &= (-1)^{r(M_1)+c(M_1)}P(M_1,t)(-1)^{r(M_2)+c(M_2)}P(M_2,t)\\ &\geqslant (t-1)^{r(M_1)+r(M_2)}\\ &= (t-1)^{r(M)}.\end{aligned}$$

This gives the required contradiction. □

Claim 5. *M/e is connected for all $e \in S$.*

Proof. Suppose M/e is not connected. By Lemmas 4.3 and Claim 4, M is the parallel connection of two connected matroids M_1 and M_2, about basepoint e. So, by Lemma 4.2, we have $r(M) = r(M_1) + r(M_2) - 1$ and

$$P(M,t) = P(M_1,t)P(M_2,t)/(t-1).$$

Applying (6.2) to M_1 and M_2 we obtain

$$\begin{aligned}(-1)^{r(M)+1}P(M,t) &= (-1)^{r(M_1)+1}P(M_1,t)(-1)^{r(M_2)+1}P(M_2,t)/(t-1)\\ &\geqslant (t-1)^{r(M_1)+r(M_2)}/(t-1)\\ &= (t-1)^{r(M)}.\end{aligned}$$

This gives the required contradiction. □

Claim 6. *$M - e$ has exactly two components for all $e \in S$.*

Proof. The proof will be in two stages. We first show that $M - e$ has an even number of components for all $e \in S$, and hence $M - e$ is disconnected.

Using Lemmas 2.1(c), 2.3(b), 3.2 and Claim 4 we deduce that

$$(-1)^{r(M)+1}P(M,t) = (-1)^{r(M-e)+1}P(M-e,t) + (-1)^{r(M/e)+1}P(M/e,t).$$

Thus, if $c(M-e)$ is odd we have

$$(-1)^{r(M)+1}P(M,t) = (-1)^{r(M-e)+c(M-e)}P(M-e,t) + (-1)^{r(M/e)+1}P(M/e,t).$$

Using Claim 5 and applying (6.2) to $M-e$ and M/e gives

$$\begin{aligned}(-1)^{r(M)+1}P(M,t) &\geqslant (t-1)^{r(M-e)} + (t-1)^{r(M/e)} \\ &= (t-1)^{r(M)} + (t-1)^{r(M)-1}.\end{aligned}$$

This contradicts the choice of M and t and hence $c(M-e)$ is even.

Let $M-e = M_1 \oplus \cdots \oplus M_{2r}$, $X_1 = M_1 \oplus \cdots \oplus M_r$, $X_2 = M_{r+1} \oplus \cdots \oplus M_{2r}$. Then $M-e = X_1 \oplus X_2$. By Lemmas 4.5 and 4.4(a),(b), $M = N_1 *^e N_2$, where $N_1 = M/S_{X_1}$, and $N_1 = M/S_{X_1}$, $N_2 = M/S_{X_2}$ are connected, and $r(M) = r(N_1) + r(N_2)$. By Lemma 4.4(d),

$$\begin{aligned}(-1)^{r(M)+1}P(M,t) &= -\left(\frac{t-2}{t-1}\right)(-1)^{r(N_1)+1}P(N_1,t)(-1)^{r(N_2)+1}P(N_2,t) \\ &\quad + (-1)^{r(N_1)+1}P(N_1,t)(-1)^{r(N_2/e)+1}P(N_2/e,t) \\ &\quad + (-1)^{r(N_1/e)+1}P(N_1/e,t)(-1)^{r(N_2)+1}P(N_2,t).\end{aligned}$$

If N_1/e and N_2/e are both connected then, applying (6.2), we obtain

$$\begin{aligned}(-1)^{r(M)+1}P(M,t) &\geqslant \left(\frac{2-t}{t-1}\right)(t-1)^{r(N_1)+r(N_2)} \\ &\quad + (t-1)^{r(N_1)+r(N_2/e)} + (t-1)^{r(N_1/e)+r(N_2)} \\ &= (4-t)(t-1)^{r(M)-1} \\ &> (t-1)^{r(M)}\end{aligned}$$

since $t \in (1, \frac{32}{27}]$. This contradiction implies that at least one of N_1/e and N_2/e is disconnected. Without loss of generality, N_1/e is disconnected. Then, by Lemma 2.5, $N_1 - e$ is connected. However, using Lemma 2.4, we have

$$N_1 - e = M/S_{X_1} - e = (M-e)/S_{X_1} = (X_1 \oplus X_2)/S_{X_1} = X_2.$$

Thus, $X_2 = M_{r+1} \oplus \cdots \oplus M_{2r}$ is connected. Hence $r = 1$ and $M - e$ has exactly two components. □

Claim 7. *Suppose M is the 2-sum of n connected matroids M_i, with basepoint p, such that M_i/p is connected for all i, $1 \leqslant i \leqslant n$. Then n must be odd.*

Proof. Let M' denote the parallel connection of $M_1, M_2, \ldots, M_n$. Hence $M = M' - p$, and by Lemma 4.2(c), $M'/p = M_1/p \oplus M_2/p \oplus \cdots \oplus M_n/p$. By Lemmas 3.2, 4.2(d), 3.4,

$$P(M,t) = P(M',t) + P(M'/p,t),$$

$$P(M',t) = (t-1)^{-n+1}\prod_{i=1}^{n} P(M_i,t)$$

and

$$P(M'/p,t) = \prod_{i=1}^{n} P(M_i/p,t).$$

Hence

$$P(M,t) = (t-1)^{-n+1}\prod_{i=1}^{n} P(M_i,t) + \prod_{i=1}^{n} P(M_i/p,t).$$

Using Lemmas 4.2(a) and 2.3(b) gives

$$\begin{aligned}(-1)^{r(M)+1}P(M,t) &= (t-1)^{-n+1}\prod_{i=1}^{n}(-1)^{r(M_i)+1}P(M_i,t)\\ &\quad + (-1)^{n}\prod_{i=1}^{n}(-1)^{r(M_i/p)+1}P(M_i/p,t).\end{aligned}$$

If n is even then we may apply (6.2) to $P(M_i,t)$ and $P(M_i/p,t)$ for $1 \leqslant i \leqslant n$, to give

$$\begin{aligned}(-1)^{r(M)+1}P(M,t) &\geqslant (t-1)^{-n+1}\prod_{i=1}^{n}(t-1)^{r(M_i)} + \prod_{i=1}^{n}(t-1)^{r(M_i/p)}\\ &= (t-1)^{r(M)} + (t-1)^{r(M)-1}.\end{aligned}$$

This contradicts the choice of M and hence n must be odd. □

Claim 8. *Suppose $M = M_1 \oplus_2^q M_2$ where $M_1 - q$ is a generalized q-coloop. Then $M_2 - q$ has exactly two components.*

Proof. Suppose $|M_2| \leqslant 3$. Then $|M_2| = 3$ by the definition of 2-sum and, since M_2 is connected and has no parallel elements, $M_2 = U_{2,3}$. Thus the claim holds. Henceforth we will assume that $|M_2| > 3$.

Claim 8(a). $(t-1)^{-1}P(M_1,t) - P(M_1/q,t) > 0$.

Proof. Let $N = M_1 \oplus_2^q U_{2,3}$. By Lemma 4.6(b), N is connected. Since $M_1 - q$ is a generalized q-coloop we have that $r(M_1 - q)$ is odd. By Lemma 2.1(c), $r(M_1) = r(M_1 - q)$. Using Lemma 4.6(a) and the fact that $r(U_{2,3}) = 2$, we have $r(N) = r(M_1) + 1$. Thus $r(N)$ is even.

Applying (6.2) inductively to N we deduce that

$$P(N,t) < 0. \tag{6.3}$$

Applying Lemma 4.6(c) to N with $N = M_1 \oplus_2^q U_{2,3}$, we obtain

$$\begin{aligned}P(N,t) &= P(M_1/q,t)(t-1)^2\\ &\quad + (t-1)(t-2)[(t-1)^{-1}P(M_1,t) - P(M_1/q,t)].\end{aligned} \tag{6.4}$$

Since $r(M_1)$ is odd, it follows from Lemma 2.3(b) that $r(M_1/q)$ is even. Since $M_1 - q$ is a generalized q-coloop, by Lemma 5.1, M_1/q is disconnected with exactly two components. Applying (6.2) to M_1/q we deduce that

$$P(M_1/q,t) > 0. \tag{6.5}$$

Since $t \in (1, \frac{32}{27}]$, it follows from (6.3) that $(t-1)P(N,t) < 0$, from (6.5) that $(t-1)^2 P(M_1/q, t) > 0$, and that $(t-1)(t-2) < 0$. Hence, from (6.4), we obtain

$$(t-1)^{-1} P(M_1, t) - P(M_1/q, t) > 0. \qquad \square$$

Claim 8(b). *$c(M_2 - q)$ is even.*

Proof. By Lemma 4.6 and Claim 4, M_1 and M_2 are connected, and

$$(-1)^{r(M)+1} P(M,t) = (-1)^{r(M_1/q)+1} P(M_1/q, t)(-1)^{r(M_2-q)} P(M_2 - q, t) + \alpha\beta,$$

where $\alpha = (-1)^{r(M_2)+1} P(M_2, t)$ and

$$\beta = (t-1)^{-1}(-1)^{r(M_1)+1} P(M_1, t) - (-1)^{r(M_1/q)} P(M_1/q, t).$$

Applying (6.2) to M_2 gives $\alpha > 0$. Using Claim 8(a) and the fact that $r(M_1)$ is odd we may deduce that $\beta > 0$. Thus

$$(-1)^{r(M)+1} P(M,t) \quad \geqslant \quad (-1)^{r(M_1/q)} P(M_1/q, t)(-1)^{r(M_2-q)+1} P(M_2 - q, t).$$

If $c(M_2 - q)$ is odd then, since $c(M_1)/q$ has exactly two components, we may apply (6.2) to deduce that

$$\begin{aligned} (-1)^{r(M)+1} P(M,t) &\geqslant (t-1)^{r(M_1/q) + r(M_2 - q)} \\ &= (t-1)^{r(M)}. \end{aligned}$$

This contradiction implies that $c(M_2 - q)$ is even. $\square$

Let $M_2 - q = X_1 \oplus \cdots \oplus X_{2r}$, $Y_1 = X_1 \oplus \cdots \oplus X_r$ and $Y_2 = X_{r+1} \oplus \cdots \oplus X_{2r}$. Then $M_2 - q = Y_1 \oplus Y_2$ so, by Lemma 4.5, $M_2 = N_1 *^q N_2$ where $N_1 = M_2/S_{Y_2}$ and $N_2 = M_2/S_{Y_1}$. Thus $M = M_1 \oplus_2^q (N_1 *^q N_2)$. Since M is connected, it follows from Lemmas 4.6(b) and 4.4(b) that M_1, N_1 and N_2 are connected. By Lemma 4.9, $M = (M_1 *^q N_1) \oplus_2^q N_2$.

Suppose $r \geqslant 2$. Then, since $N_1 - q = Y_1$, $N_1 - q$ is disconnected so, by Lemma 2.5, N_1/q is connected. Since M_1 and N_1 are connected, it follows from Lemma 4.6(b) that $(M_1 *^q N_1)/q = M_1 \oplus_2^q N_1$ is connected. Now the fact that $M = (M_1 *^q N_1) \oplus_2^q N_2$ contradicts Claim 7. Thus $r = 1$. $\square$

Using Claims 4, 6, 7 and 8 it follows that M satisfies the hypotheses of Lemma 5.2 and hence M is a generalized 3-circuit. Thus M is graphic. But, by [4, Theorem 5] and [8, Theorem 2.4], Theorem 6.3 is known to be true for graphic matroids. This contradicts the choice of M as a counterexample, and completes the proof of Theorem 6.3. $\square$

Lemma 6.1. ([8], Theorem 2.5) *Let $t_0 > \frac{32}{27}$, $\epsilon > 0$. Then there exists a graph G such that $P(G,t)$ has a zero in $(t_0 - \epsilon, t_0 + \epsilon)$.*

Corollary 6.1. *The maximal zero-free intervals for characteristic polynomials of loopless matroids are precisely $(-\infty, 1)$ and $(1, \frac{32}{27}]$.*

Proof. The result follows directly from Theorem 6.1, Theorem 6.3 and Lemma 6.1. $\square$

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