Convex quadratic programming

NATCOR

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http://people.brunel.ac.uk/~mastjjb/jeb/jeb.html

All of the slides below are available from
http://people.brunel.ac.uk/~mastjjb/jeb/natcor.html

PORTFOLIO OPTIMISATION

We are going to be dealing with quadratic programs. The archetypical example of such a program is the portfolio optimisation problem, as originally proposed by Markowitz, in a mean-variance framework.

Colloquially we have a universe of potential assets in which we can invest and the decision problem is:

> *How can we split our investment between these assets in an appropriate way?*

To proceed with Markowitz mean-variance portfolio optimisation we need some notation, let:

N be the number of assets (e.g. stocks) available

μ_i be the expected return of asset i

ρ_{ij} be the correlation between the **returns** for assets i and j ($-1 \leq \rho_{ij} \leq +1$)

s_i be the standard deviation in **return** for asset i

Then the decision variables are:

w_i the proportion of the total investment associated with (invested in) asset i ($0 \leq w_i \leq 1$)

Reflect for a moment – are you surprised that correlation makes an appearance here?

Note here that we have used the word "asset" above. The framework we use is completely general – provided we have a price history for an asset it can be included, so we could consider making up a portfolio from stocks, commodities (e.g. oil, metals), and bonds.

SIMPLE EXAMPLE

I will confess here that I do not always do this myself, but you may want to consider just taking a very small example of the problem under consideration and play around with it. This may give you insight that you did not have before.

Suppose N=2, so we have two assets available in which we can invest. Then the Markowitz approach says that the return we get from investing a proportion w_1 of our wealth in asset 1 and a proportion w_2 of our wealth in asset 2 is

$$\sum_{i=1}^{N} w_i\mu_i = w_1\mu_1 + w_2\mu_2$$

where it must be true that

$$\sum_{i=1}^{N} w_i = w_1 + w_2 = 1$$

which states that we invest all of the money we have available.

The **risk (variance)** associated with this investment is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j$$

$$\begin{aligned}
&= w_1 w_1 \rho_{11} s_1 s_1 + w_1 w_2 \rho_{12} s_1 s_2 + \\
&\quad w_2 w_1 \rho_{21} s_2 s_1 + w_2 w_2 \rho_{22} s_2 s_2 \\
&= w_1 w_1 s_1 s_1 + 2 w_1 w_2 \rho_{12} s_1 s_2 + w_2 w_2 s_2 s_2 \\
&= (w_1)^2 (s_1)^2 + 2 w_1 w_2 \rho_{12} s_1 s_2 + (w_2)^2 (s_2)^2
\end{aligned}$$

so here all the terms are **quadratic** in the decision variables.

Suppose I take some data for two assets and vary w_1 and w_2 and plot the return that I get from my portfolio (y-axis) against the risk (variance) associated with that portfolio (x-axis) what do you think the plot will look like?

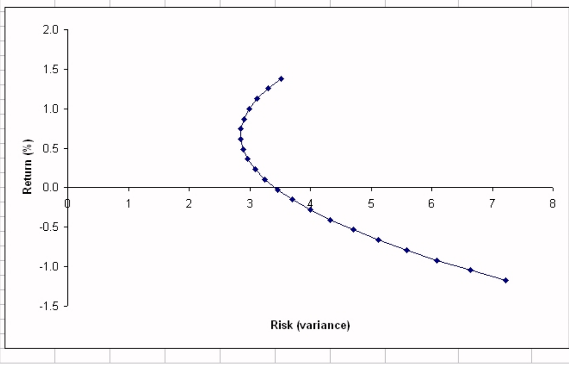
The standard presentation in terms of risk and return in portfolio optimisation is that return is plotted on the vertical axis and risk on the horizontal axis.

Below we show the spreadsheet considered in class.

	A	B	C	D	E	F
1	Stock prices					
2	A	B	C	D	E	Period
3	916	630.5	440	642	740	0
4	932	639.5	440.5	642	755	1
5	910.5	644.5	443	634	761	2
6	872	626.5	446	642.5	712	3
7	874	637	465	617.5	675	T=4

	H	I	J	K
1	Stock 1	Stock 2	Return 1 (%)	Return 2 (%)
2	***A***	***C***	A	C
3	***916***	***440***		
4	***932***	***440.5***	1.73	0.11
5	***910.5***	***443***	-2.33	0.57
6	***872***	***446***	-4.32	0.67
7	***874***	***465***	0.23	4.17
8		Average	-1.17	1.38
10	Standard deviation STDEV (2dp) RETURNS			
11	2.69	1.88		
13	Correlation CORREL (2dp) RETURNS			
14	A	C		
15	1	0.22	A	
16	0.22	1	C	

	M	N	O	P
1	Stock 1 weight	Stock 2 weight	Return	Risk
2	0	1	1.38	3.52
3	0.05	0.95	1.25	3.30
4	0.1	0.9	1.13	3.13
5	0.15	0.85	1.00	2.99
6	0.2	0.8	0.87	2.90
7	0.25	0.75	0.74	2.86
8	0.3	0.7	0.62	2.85
9	0.35	0.65	0.49	2.89
10	0.4	0.6	0.36	2.97
11	0.45	0.55	0.23	3.09
12	0.5	0.5	0.10	3.25
13	0.55	0.45	-0.02	3.46
14	0.6	0.4	-0.15	3.71
15	0.65	0.35	-0.28	4.00
16	0.7	0.3	-0.41	4.33
17	0.75	0.25	-0.53	4.71
18	0.8	0.2	-0.66	5.12
19	0.85	0.15	-0.79	5.58
20	0.9	0.1	-0.92	6.09
21	0.95	0.05	-1.05	6.63
22	1	0	-1.17	7.22



Note that some points on the trade-off curve between risk (variance) and return above are **efficient**, some are not. Points on this curve which are inefficient are **dominated** by other points.

OPTIMISATION

If we had just N=2 assets as above then it is a simple matter to consider possible investment portfolios simply by enumerating choices for w_1 and w_2 (where $w_2=w_1-1$ in the two asset case).

Of course we almost always have many more than two assets in which we could invest and so the approach considered above becomes infeasible. We need to move from enumerating choices to making a choice via **optimisation**.

Let:

R be the desired expected return from the portfolio chosen

Using the standard **Markowitz mean-variance approach** we have that the unconstrained portfolio optimisation problem is:

$$\text{minimise} \quad \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \rho_{ij} s_i s_j \tag{1}$$

subject to

$$\sum_{i=1}^{N} w_i \mu_i = R \tag{2}$$

$$\sum_{i=1}^{N} w_i = 1 \tag{3}$$

$$w_i \geq 0 \qquad i=1,\ldots,N \tag{4}$$

Equation (1) minimises the total variance (**risk**) associated with the portfolio whilst equation (2) ensures that the portfolio has an expected **return** of R. Equation (3) ensures that the proportions add to one and equation (4) is the non-negativity constraint.

Some formulations replace the equality in equation (2) by an inequality (≥).

This formulation (equations (1)-(4)) is a simple nonlinear programming problem.

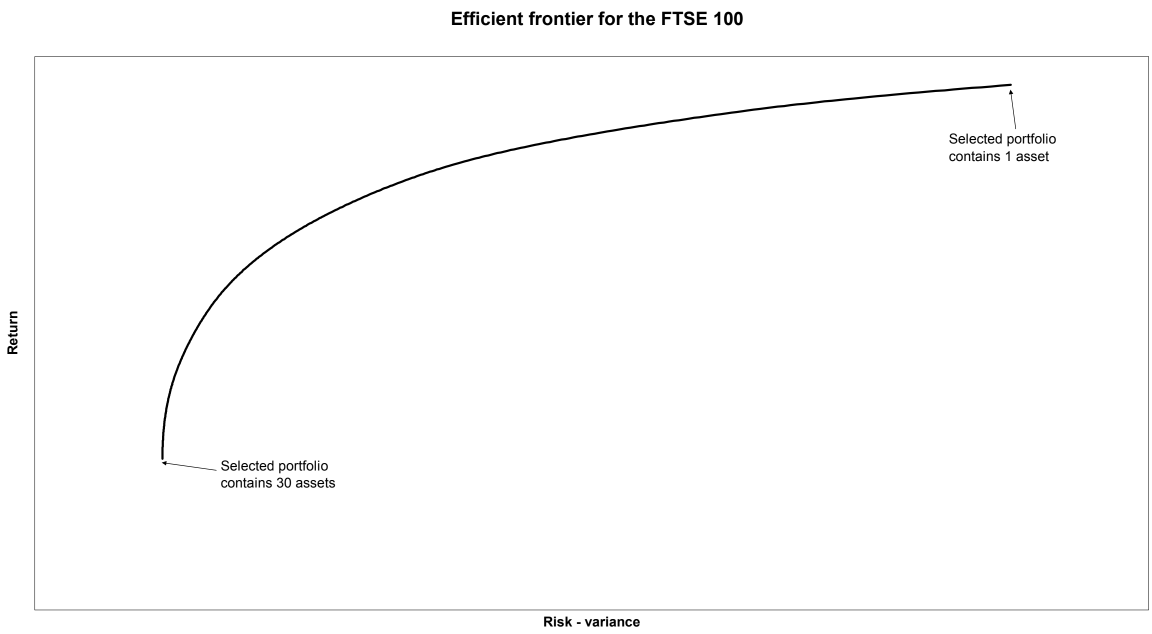
Shorting can be accommodated by allowing w_i to be negative

Usually nonlinear problems are difficult to solve but in this case because the objective is **quadratic** (and the constraints are linear), computationally effective algorithms exist so that there is (in practice) little difficulty in calculating the optimal solution for any particular data set.

Note here that the above formulation (equations (1)-(4)) can be expressed in terms of σ_{ij} the **covariance between the returns** associated with assets i and j since $\sigma_{ij}=\rho_{ij}s_i s_j$.

The point of the above optimisation problem is to construct an ***efficient frontier***, (**unconstrained efficient frontier, UEF**) a smooth non-decreasing curve that gives the best possible tradeoff of **risk** against **return**, i.e. the curve represents the set of **Pareto-optimal** (**non-dominated**) portfolios.

One such efficient frontier is shown below for assets (shares) drawn from the UK FTSE (Financial Times Stock Exchange) index of 100 top companies.



SOLUTION

So far we have considered an example quadratic program, as a motivation as to why we might be interested (at all) in such problems.

Obviously, in practice, we might just take the problem and "MIP it" throw the problem to a standard mathematical programming package (perhaps using AMPL to ease the interface with the package).

Here we shall proceed, by taking one example, to illustrate how we can solve a quadratic program (QP). The example used is taken, with acknowledgment, from Hillier and Lieberman (Introduction to Operations Research, Chapter 13).

Consider the following example QP:

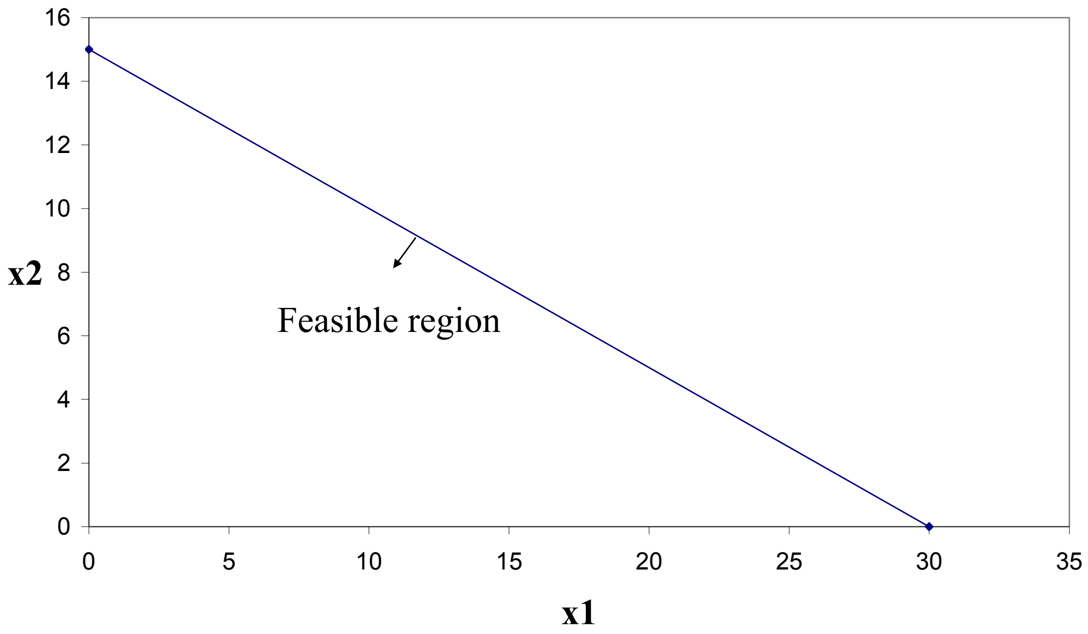
maximise

$$15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

subject to

$$x_1 + 2x_2 \leq 30$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

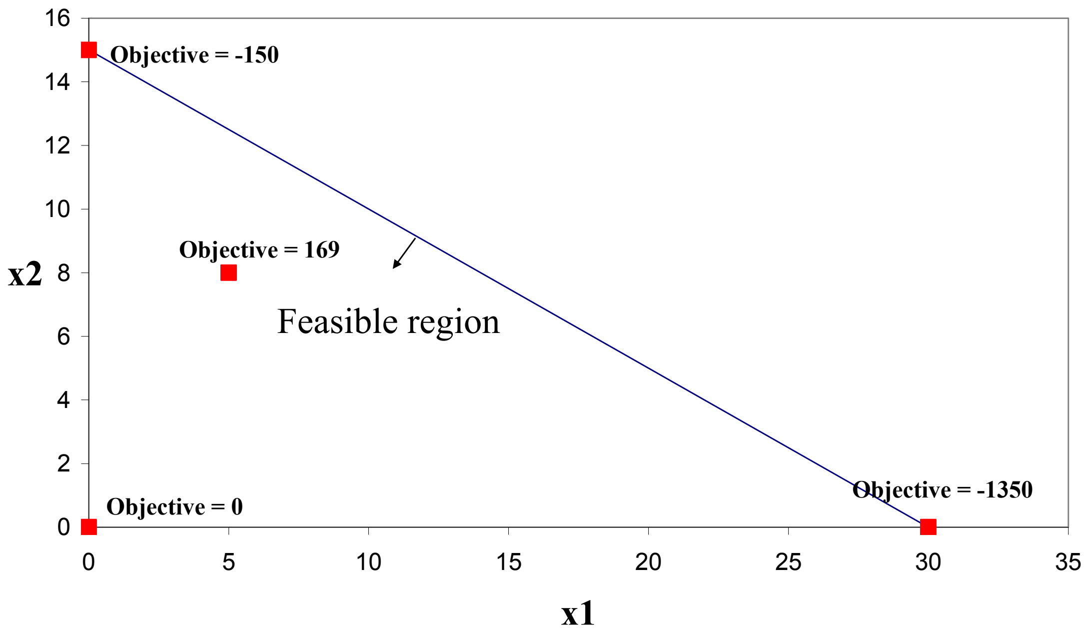
First let us investigate this QP. The feasible region is shown below. This is clearly a convex region (the line segment joining any two points within (or on) the boundary of the region always lies within the region).



Now if we had a linear objective the maximum value of the objective would be achieved on the boundary of the feasible region, indeed at a vertex (a standard result from linear programming).

But we actually had a quadratic objective. So does this make a difference?

The plot below shows the feasible region, but where now we have included the value of the objective at four points, the three vertices of the feasible region, plus one point ($x_1=5,x_2=8$) inside the feasible region.



So the situation is distinctly different from the linear objective case. Clearly here the optimal solution (where we maximise the objective) cannot be at a vertex of the feasible region – since we have a point inside the feasible region which has a better objective function value than any of the vertices.

For QPs it is helpful to have a standard notation and representation. If we use matrix notation the standard representation of a QP is:

maximise $cx - \frac{1}{2}x^TQx$
subject to $Ax \leq b$
$x \geq 0$

Here symbols in italics are vectors/matrices, so c is a row vector, x and b are column vectors, Q and A are matrices, and the superscript T denotes the transpose of a vector/matrix.

The elements of Q (q_{ij}) are given constants such that $q_{ij}=q_{ji}$ (so the matrix Q is symmetric, which is the reason for the involvement of the ½ factor in the objective function).

To illustrate this notation, consider our example QP:

maximise

$$15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

subject to

$$x_1 + 2x_2 \leq 30$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

To put the constraints in our standard form of $Ax \leq b$ we have

$$A = [1, 2]$$

$$x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$$b = [30]$$

so $$[1, 2] \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \leq [30]$$

To put the objective

$$15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

in our standard form of cx - ½x^TQx we have:

$$c = [15, 30]$$

$$Q = \begin{vmatrix} 4 & -4 \\ -4 & 8 \end{vmatrix}$$

Here to get Q we double the coefficients on the nonlinear terms and change the signs (note the - ½ in - ½x^TQx).

$$x^TQx = [x_1, x_2] \begin{vmatrix} 4 & -4 \\ -4 & 8 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

$$= [x_1, x_2] \begin{vmatrix} 4x_1 - 4x_2 \\ -4x_1 + 8x_2 \end{vmatrix}$$

$$= 4(x_1)^2 - 4x_1x_2 - 4x_1x_2 + 8(x_2)^2$$

$$= 4(x_1)^2 - 8x_1x_2 + 8(x_2)^2$$

so $cx - \frac{1}{2}x^T Qx =$

$$[15, 30] \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} - \tfrac{1}{2}(4(x_1)^2 - 8x_1x_2 + 8(x_2)^2)$$

$$= 15x_1 + 30x_2 - \tfrac{1}{2}(4(x_1)^2 - 8x_1x_2 + 8(x_2)^2)$$

$$= 15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

as required

So we have the problem expressed in standard form, where note that Q is symmetric.

So how are we to solve our QP? Recall we saw above that the solution is not at a vertex of the feasible region.

Well the answer is that if Q has a certain property we have theory/algorithms available that will **guarantee** to find the optimal solution to the QP.

This property is that:

$$x^{T}Qx \geq 0 \quad \forall x$$

which states that Q is a positive semidefinite matrix.

This property is equivalent to saying that all the eigenvalues of our symmetric matrix Q are real and nonnegative.

For our example problem we can show that Q is positive semidefinite by checking x^TQx directly.

We have from the algebra above that
$x^TQx = 4(x_1)^2 - 8x_1x_2 + 8(x_2)^2$

so we need to show that

$$4(x_1)^2 - 8x_1x_2 + 8(x_2)^2 \geq 0 \qquad \forall x_1,x_2$$

i.e. $(x_1)^2 - 2x_1x_2 + 2(x_2)^2 \geq 0 \qquad \forall x_1,x_2$

i.e. $(x_1 - x_2)^2 + (x_2)^2 \geq 0 \qquad \forall x_1,x_2$

which must be true as it is the sum of two nonnegative terms

To solve our QP we have the Karush-Kuhn-Tucker (KKT) conditions. If $f(x)$ is our objective in standard form (with Q being positive semidefinite) with $g_1(x) \leq b_1$, $g_2(x) \leq b_2$, …, $g_m(x) \leq b_m$ being linear constraints then a solution $X=[X_j]$ is optimal if and only if there exist m numbers $u_1, u_2, \ldots, u_m$ such that:

$$[\partial f/\partial x_j - \sum_{i=1}^{m} u_i(\partial g_i/\partial x_j)] \leq 0 \qquad \text{at } x_j=X_j \ \forall j$$

$$X_j[\partial f/\partial x_j - \sum_{i=1}^{m} u_i(\partial g_i/\partial x_j)] = 0 \qquad \text{at } x_j=X_j \ \forall j$$

$$[g_i(X) - b_i] \leq 0 \qquad \forall i$$

$$u_i[g_i(X) - b_i] = 0 \qquad \forall i$$

$$X_j \geq 0 \qquad \forall j$$

$$u_i \geq 0 \qquad \forall i$$

Two of these conditions relate to the product of two terms always being zero, i.e. either one of these terms is zero, or the other, or both.

For our QP we have

$f = 15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$

with just one constraint (so m=1), namely

$g_1 = x_1 + 2x_2 \le b_1 = 30$

$\partial f/\partial x_j - \sum_{i=1}^{m} u_i(\partial g_i/\partial x_j) \le 0$ at $x_j=X_j$ $\forall j$

becomes

for j=1 $\quad 15 + 4X_2 - 4X_1 - u_1 \le 0$

for j=2 $\quad 30 + 4X_1 - 8X_2 - 2u_1 \le 0$

$X_j[\partial f/\partial x_j - \sum_{i=1}^{m} u_i(\partial g_i/\partial x_j)] = 0$ at $x_j=X_j$ $\forall j$

becomes

for j=1 $\quad X_1[15 + 4X_2 - 4X_1 - u_1] = 0$

for j=2 $\quad X_2[30 + 4X_1 - 8X_2 - 2u_1] = 0$

$g_i(X) - b_i \le 0$ $\forall i$ becomes

$X_1 + 2X_2 - 30 \le 0$

$u_i[g_i(X) - b_i] = 0$ $\forall i$ becomes

$u_1(X_1 + 2X_2 - 30) = 0$

where $X_1, X_2, u_1 \ge 0$

To get a clearer picture of these constraints move any constant terms to the right-hand side and for the three inequality constraints above add nonnegative slack variables (y_1, y_2 and v_1) so that for these equations we get

$4X_2 - 4X_1 - u_1 + y_1 = -15$
$4X_1 - 8X_2 - 2u_1 + y_2 = -30$
$X_1 + 2X_2 + v_1 = 30$

Now consider the three constraints that require a product to be zero, these are

$$X_1[15 + 4X_2 - 4X_1 - u_1] = 0$$
$$X_2[30 + 4X_1 - 8X_2 - 2u_1] = 0$$
$$u_1(X_1 + 2X_2 - 30) = 0$$

which using the above becomes

$$X_1[-y_1] = 0$$
$$X_2[-y_2] = 0$$
$$u_1(-v_1) = 0$$

or equivalently

$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$

(since all variables are ≥ 0)

Here we have the sum of three product terms has to be zero. Each of these variable pairs (X_1,y_1), (X_2,y_2) and (u_1,v_1) are called **complementary** variables (since only one of the two variables can be nonzero, equivalently at least one of the two variables must be zero).

The single constraint

$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$

is known as a **complementarity** constraint.

Our complete set of KKT constraints now is:

$$4X_2 - 4X_1 - u_1 + y_1 = -15$$
$$4X_1 - 8X_2 - 2u_1 + y_2 = -30$$
$$X_1 + 2X_2 + v_1 = 30$$
$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$
$$\text{all variables} \geq 0$$

Multiplying the first two equations by -1 to get a positive right-hand side we have

$$-4X_2 + 4X_1 + u_1 - y_1 = 15$$
$$-4X_1 + 8X_2 + 2u_1 - y_2 = 30$$
$$X_1 + 2X_2 + v_1 = 30$$
$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$
all variables ≥ 0

If we did not have the complementarity constraint present then we would have a set of linear equations. It is the (nonlinear) complementarity constraint that makes arriving at variables values that satisfy these constraints difficult.

Note here that these equations define a **feasibility problem**, we need a feasible solution. The key point is that the KKT conditions tell us that if we solve this feasibility problem we also solve the optimisation problem we originally started out with (and which was our focus of attention).

For simplicity above we have developed the KKT conditions using a specific example. As you might suspect the KKT constraints can be written in a general form, and hence applied for any QP problem.

For our standard QP

$$\begin{array}{ll} \text{maximise} & cx - \tfrac{1}{2}x^{T}Qx \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

the general form of the KKT constraints is:

$$\begin{aligned} &Qx + A^{T}u - y = c^{T} \\ &Ax + v = b \\ &x^{T}y + u^{T}v = 0 \\ &x,y,u,v \geq 0 \end{aligned}$$

Returning to our KKT constraints we have

$$-4X_2 + 4X_1 + u_1 - y_1 = 15$$
$$-4X_1 + 8X_2 + 2u_1 - y_2 = 30$$
$$X_1 + 2X_2 + v_1 = 30$$
$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$
all variables ≥ 0

These can be solved by a modification of the simplex method for linear programming. Paradoxically we now return to an optimisation problem (albeit a linear optimisation problem, whereas we originally started with a quadratic optimisation problem).

First see if taking three variables from our complementarity constraint (one from each product pair) and setting them equal to zero yields a feasible solution to all the constraints (if it does we are done).

Setting $X_1=X_2=u_1=0$ leads to $y_1=-15$, $y_2=-30$ and $v_1=30$. Here the values for y_1 and y_2 violate the constraint that all variables have to be ≥ 0.

As we have violated constraints associated with y_1 and y_2 for the constraints involving y_1 and y_2 introduce artificial variables z_1 and z_2 (both ≥ 0) to get

$$-4X_2 + 4X_1 + u_1 - y_1 + z_1 = 15$$

$$-4X_1 + 8X_2 + 2u_1 - y_2 + z_2 = 30$$

so now the solution $X_1=X_2=u_1=0$ and $v_1=30$ (as before) plus:

- $y_1=0$, $z_1=15$
- $y_2=0$, $z_2=30$

satisfies all of our constraints.

If we could somehow find a solution satisfying all of our constraints, but with $z_1=z_2=0$ then we would have solved our original feasibility problem. This can be achieved via the linear optimisation problem:

minimise $z_1 + z_2$
subject to

$$-4X_2 + 4X_1 + u_1 - y_1 + z_1 = 15$$
$$-4X_1 + 8X_2 + 2u_1 - y_2 + z_2 = 30$$
$$X_1 + 2X_2 + v_1 = 30$$
$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$
$$\text{all variables} \geq 0$$

This is a linear program (LP) except for the complementarity constraint.

To solve this problem we use the simplex method but with a restricted-entry rule:

When you are choosing an entering basic variable, exclude from consideration any nonbasic variable whose complementary variable is already a basic variable; the choice should be made from the other nonbasic variables according to the usual criterion for the simplex method. This rule keeps the complementarity constraint satisfied throughout the course of the algorithm.

Once a solution with an objective function value of zero is obtained then we have a solution to the original feasibility problem.

Here we (to accord with the Hiller and Lieberman treatment) change our objective by substituting for $z_1 + z_2$
using

$$-4X_2 + 4X_1 + u_1 - y_1 + z_1 = 15$$
$$-4X_1 + 8X_2 + 2u_1 - y_2 + z_2 = 30$$

to get

$$z_1 + z_2 = 45 - 4X_2 - 3u_1 + y_1 + y_2$$

so we want to

minimise $z_1 + z_2$

which is the same as

minimise $45 - 4X_2 - 3u_1 + y_1 + y_2$

To carry out the modified simplex algorithm we need to set up the initial tableau as below.

Note that we use as our starting basis the solution z_1=15, z_2=30 and v_1=30 (all other variables zero) we considered before that satisfies all of our constraints (including the complementarity constraint).

The initial simplex tableau is:

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
z_1	4	-4	1	-1			1		15
z_2	-4	8	2		-1			1	30
v_1	1	2				1			30
Obj		-4	-3	1	1				-45

Select the variable with the most negative objective coefficient to enter the basis, here X_2. Note that its complementary variable y_2 is nonbasic, so selecting X_2 is allowed under the restricted-entry rule.

The ratios for variables with positive coefficients in the X_2 column of the tableau are 30/8 for z_2 and 30/2 for v_1. The minimum value is 30/8 for z_2 so z_2 leaves the basis.

Summarising, the pivot row is the z_2 row; the pivot element is 8; the pivot column is the X_2 column.

Conduct a pivot operation:

- divide the pivot row by the pivot element
- add/subtract multiples of the pivot row to all the other rows to get zeros in the pivot column

The new simplex tableau is

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
z_1	2		2	-1	-0.5		1	0.5	30
X_2	-0.5	1	0.25		-0.125			0.125	3.75
v_1	2		-0.5		0.25	1		-0.25	22.5
Obj	-2		-2	1	0.5			0.5	-30

Select the variable with the most negative coefficient to enter the basis, here X_1 or u_1. Note under the restricted-entry rule:

- complementary variable (y_1) for X_1 is nonbasic, so selecting X_1 is allowed
- complementary variable (v_1) for u_1 is basic, so selecting u_1 is not allowed

The ratios for variables with positive coefficients in the X_1 column of the tableau are 30/2 for z_1 and 22.5/2 for v_1. The minimum value is 22.5/2 for v_1 so v_1 leaves the basis.

Doing the pivot operation we get (to 3 decimal places)

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
z_1			2.5	-1	-0.75	-1	1	0.75	7.5
X_2		1	0.125		-0.063	0.25		0.063	9.375
X_1	1		-0.25		0.125	0.5		-0.125	11.25
Obj			-2.5	1	0.75	1		0.25	-7.5

Select the variable with the most negative coefficient to enter the basis, here u_1.
Note under the restricted-entry rule:

- complementary variable (v_1) for u_1 is nonbasic, so selecting u_1 is allowed

The ratios for variables with positive coefficients in the u_1 column of the tableau are 7.5/2.5 for z_1 and 9.375/0.125 for X_2. The minimum value is 7.5/2.5 for z_1 so z_1 leaves the basis.

Doing the pivot operation we get (to 3 decimal places)

Basis	X_1	X_2	u_1	y_1	y_2	v_1	z_1	z_2	RHS
u_1			1	-0.4	-0.3	-0.4	0.4	0.3	3
X_2		1		0.05	-0.025	0.3	-0.05	0.025	9
X_1	1			-0.1	0.05	0.4	0.1	-0.05	12
Obj							1	1	0

Here we are done, as the objective has value zero. Hence we have a solution u_1=3, X_2=9 and X_1=12, all other variables zero.

It is easy to confirm that this solution satisfies:

$$-4X_2 + 4X_1 + u_1 - y_1 + z_1 = 15$$
$$-4X_1 + 8X_2 + 2u_1 - y_2 + z_2 = 30$$
$$X_1 + 2X_2 + v_1 = 30$$
$$X_1y_1 + X_2y_2 + u_1v_1 = 0$$
$$\text{all variables} \geq 0$$

As we have a solution satisfying our KKT constraints this must be the optimal solution to our original QP

maximise

$$15x_1 + 30x_2 + 4x_1x_2 - 2(x_1)^2 - 4(x_2)^2$$

subject to

$$x_1 + 2x_2 \leq 30$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

i.e. the optimal solution to this problem is $x_1=12$ and $x_2=9$, for which the associated objective function value is 270.